Long report

for the award of the title Doctor of Sciences at the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences in the area of science 4. Natural sciences, mathematics and informatics, professional field 4.5 Mathematics, prepared by Prof. Dr. Sc. Mladen Svetoslavov Savov, "Probability, Operations Research and Statistics", Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski" and "Operations research, Probability and Statistics", Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, appointed member of the scientific panel by order No. 457/03.12.2024 by the director of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences issued by the director of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences upon a decision of the Scientific council of Institute of Mathematics and Informatics, Bulgarian Academy of Sciences contained in its protocol No. 15/29.11.2024 and writing a long report according to the decision made by the scientific panel during its first meeting on 09.12.2024.

This long report was prepared in compliance with order No. 457/03.12.2024 by the director of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences issued on the basis of the decision by the Scientific council of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences contained in protocol No. 15/29.11.2024 and in accordance with the requirements of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences for the award of the title Doctor of Sciences in mathematics which exceed in quantitative value the minimal national requirements.

As a member of the scientific jury I have received all the documents submitted by the candidate Tsvetelin Stefanov Zaevski (Institute of Mathematics and Informatics, Bulgarian Academy of Sciences).

1. Biographical data about the candidate

Tsvetelin Zaevski was born in 1974. He earned his master's degree in Applied Mathematics from the Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski" in 1999. In 2013, he defended his PhD thesis in probability under the supervision of Prof. Racho Denchev. Tsvetelin Zaevski was appointed as Ass. Professor at the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences in 2014 and he was promoted to Assoc. Professor in 2021. He is a highly productive mathematician, as evidenced by his CV, and actively contributes to academic administration. His responsibilities include serving as Head of Department, being a member of the Scientific Council of IMI-BAS and taking on other leadership roles. Currently, Tsvetelin Zaevski supervises a promising PhD student whom he has attracted from the business circles.

2. Fulfillment of the minimal requirements

Associate Professor Zaevski has submitted 13 papers for this procedure. Twelve of these publications are in journals with impact factor, distributed as follows: 7 in Q1, 1 in Q2, 1 in Q3, and 3 in Q4. He has also provided 35 citations, 29 of which are indexed in WoS and SCOPUS. All minimal requirements in every category have been exceeded, and all additional institutional requirements have been met.

3. Scientific contributions of the thesis

The thesis systematically develops the foundation for pricing American-style derivatives based on the classical Black-Scholes model. Specifically, under the assumption that an asset price follows a geometric Brownian motion, the work introduces a unified methodology for pricing various American-style derivatives with either finite or infinite maturities. Recognizing that such pricing problems typically lead to free boundary problems—a situation where typically no analytical solution is available—the candidate dedicates a significant portion of the research to devising and implementing a fast and efficient numerical approach to solve these free boundary problems. The flow of the thesis clearly demonstrates a gradual development of this investigation as evidenced by the interdependent chapters.

The candidate examines through this unified approach several specific derivatives in a comprehensive manner. It covers both put and call options, addresses derivatives with finite and infinite maturities, and considers the impact of the various rights and obligations imposed on both the writer and the holder. Consequently, one might expect this work to be influential in the literature, with some of its achievements eventually adopted by practitioners.

The derivatives that are studied have a unifying feature in the exhibition of both one-sided and two-sided early exercise regions. This property is initially demonstrated for a specific payment structure and then extended to a much more general framework. When the maturity is infinite, both the pricing and the determination of the early exercise region can usually be explicitly obtained via a unified methodology based on $\sup\inf$ problems and the martingale approach. For finite maturity, analytic results are understandably more scarce and typically provide information about some general properties of the curves that separate the exercise and continuation regions, e.g. monotonicity. To price the respective derivatives with finite maturity the candidate employs a piece-wise linear curve exponent to approximate the early exercise boundaries, a method motivated by the explicit theoretical formulas for the geometric Brownian motion hitting such boundaries, and then proceeds to solve numerically the respective PDE. All of this is supplemented by extensive particular numerical studies.

The candidate demonstrates a very high level of technical knowledge in modern mathematical finance and explains the main ideas and results with clarity. However, the thesis contains some repetitive, non-essential mistakes (in this case), which may reflect a tendency in applied work to overlook certain technical details. I have no doubt regarding the veracity of the main results and the validity of the proposed methodology.

Next, I will consider separately each chapter of the thesis.

3.1. **Chapter 2. First hitting time properties.** This chapter presents results on the form and asymptotic behaviour of Laplace transforms for the hitting time of standard Brownian motion to linear and piece-wise linear boundaries. The new information on these quantities serves as a basis for some of the other results in this thesis, particularly in the context of approximating the early exercise curve for the respective region. The results are derived using well-known expressions from the literature; however, some of the computations are technical, which is why they have been published.

Let me summarize the results. Set $b := b(\cdot)$ to be a linear or piece-wise linear curve and τ to be the first hitting time to b. Then in general the candidate investigates the form and the asymptotic, as $T \to \infty$, of the Laplace transforms $\mathbb{E}\left[e^{-\theta\tau} 1_{\tau \le T}\right], \mathbb{E}\left[e^{\theta B_T} 1_{\tau > T, B_T > y}\right], \theta \in \mathbb{R}, y \in \mathbb{R}$. Here, I should recommend that z is not used for real variable throughout Chapter 2 and to preserve y for this purpose.

When b is a linear curve Theorem 2.1, Proposition 2.2 and Proposition 2.5 furnish an expression for $\mathbb{E}\left[e^{-\theta\tau} 1_{\tau \le T}\right]$ in terms of the c.d.f of the standard normal distribution. This expression encompasses the cases where $b(0)$ and θ are either positive or negative. The results build on existing findings regarding the density of the first hitting time. The derivations are straightforward yet involve technical computations. It is important to note that, although $\mathbb{E}\left[e^{-\theta\tau} 1_{\tau \le T}\right]$ is analytic in θ, the expression in Theorem 2.1 includes an apparently removable pole, which is why Proposition 2.2 reformulates it. In the proof of Proposition 2.2, the bounds of the integral involving complex variables ($y \in [0, \sqrt{T}] \mapsto icy - b_2 y^{-1}$) must be clearly specified by defining the contour of integration and the signs should be carefully manipulated (also in Remark 2.2).

Theorems 2.2,2.4 and 2.5 deal with $\mathbb{E}\left[e^{-\theta\tau} 1_{\tau \le T}\right], \mathbb{E}\left[e^{\theta B_T} 1_{\tau > T, B_T > y}\right]$ for linear and piece-wise linear b. These theorems provide formulas for the Laplace transform of these quantities, expressed in terms of the cumulative distribution function N of the standard normal distribution (note that the claim of 2.4 contains $b(T)$ whereas the proof - $b_n(T)$). The results are based on available expression for $\mathbb{P}(\tau > T | B_T = y)$ but require careful integration, manipulation and application of the basic properties of the Brownian motion. There is one unspecified quantity $\mathbb{V}$ in the proofs and it should be addressed. Φ_T appears in Theorem 2.5 and it confuses the matter. I do not doubt the validity of the claims but the exposition must be improved.

The next set of results deals with the hitting times to strips defined by two linear or piece-wise linear strips. Theorems 2.6-8 present expressions for

$$\mathbb{E}\left[e^{-\theta\tau}1_{\tau\leq T},\ \text{exits from one of the curves}\right], \mathbb{E}\left[e^{\theta B_T}1_{\tau>T,B_T>y}\right].$$

These expressions are quite long but contain standard quantities. The proof of Theorem 2.6 is very economical and should be expanded whilst the proof of Theorem 2.7 is sloppy (there appears $s_0, \sum_{k=0}^{4}, \sum_{k=1^4}$). Again $\mathbb{V}$ appears undefined in the proofs. Since the derivation is based on established facts and basic properties I think it is a matter of clarification and exposition.

Chapter 2 concludes with results on the limit behaviour, as $T \to \infty$, of $e^{kT}\mathbb{E}\left[e^{\theta B_T}1_{\tau>T}\right]$, see Theorem 2.10, and $e^{kT}\mathbb{E}\left[e^{\theta B_T}1_{\tau>T,B_T>y(T)}\right]$, for linear b and suitable linear $y := y(T)$. The claims are exhaustive and long as they cover all possible scenarios for varying k, θ, b, y. The results are proved after several preliminary technical lemmae. The proof of Lemma 2.10 is inaccurate as $\pm\infty$ is subtracted in the limit on p.45. This can be amended by adding and subtracting the terms in the limit itself. Also I_2, I_3 seem to converge to zero and their discussion is then obsolete.

3.2. **Chapter 3. Preliminaries.** This chapter presents the primary approach of the thesis. It introduces the general framework for derivative pricing, including the widely recognized partial integro-differential equations governing the asset price. The Black-Scholes model is addressed separately (noting that t is missing in Section 3.5 and similar instances), with provisions for dividend payments integrated into the discussion. The candidate also introduces an alternative parametrization proposed by Shiryaev and demonstrates that the model with dividends is equivalent to a dividend-free model with modified parameters. It is worth noting that the framework accommodates negative interest rates, a scenario that has recently proven to be plausible.

The proof of Proposition 3.5 is not new or alternative. It is a standard textbook example and the proof lacks a proper justification by the dominated convergence theorem.

Finally, the numerical scheme is presented and important constants are set up.

3.3. **Chapter 4. A new approach for pricing discounted American options.** Chapter 4 commences with a brief motivation in the problem of pricing discounted American options and shows why the dividends payment can be incorporated in such a model, i.e. they can be discarded. The main achievements are also sketched.

In this chapter, the early exercise region (Υ) and the continuation region ($\bar{\Upsilon}$) for call options are formally introduced. The widely recognized result that non-discounted American call options have an empty early exercise region ($\Upsilon = \emptyset$) is rigorously proven.

Furthermore, Proposition 4.2 establishes a monotonicity property for call options: if $(t, x) \in \Upsilon$, then $(t, y) \in \Upsilon$ for $y > x$. Similarly, for put options, the reverse monotonicity property is demonstrated in Proposition 4.3: if $(t, x) \in \Upsilon$, then $(t, y) \in \Upsilon$ for $y < x$. These results provide a deeper structural understanding of the regions associated with the early exercise and continuation of American options. The utilization of Lemma 3.1 should be better explained in the proofs. Also, $c(t)$ is casually introduced in-text whereas it should have separate definition. The value $c(T)$ is evaluated for both put and call options in Propositions 4.5 and 4.4 and this value is important for the subsequent algorithm for evaluating the price of the options. The proofs must be improved, e.g. where the differentiability of N is used, why V has a left-derivative at maturity and $t \mapsto T$ at one or two places.

Next, the candidate introduces the algorithm for pricing call and put options. The idea is as follows: one assumes a strategy whereby the option is exercised provided an exponent of a piece-wise linear curve defined on a partition of $[0, T]$ is hit by the price of the asset. The curve has to satisfy $e^{a_i t_i + b_i} = e^{a_{i+1} t_i + b_{i+1}} = C_i$ at each node. The constant $C_n = c(T)$ is nothing but the already derived value of the exercise curve at maturity. Then going backwards one specifies recursively $C_{n-1}, C_{n-2}, \ldots, C_1$ by solving a particular sequence of optimization problems, which must be better described (many typos in (4.7) and unclear explanations), but which is undoubtedly an original contribution by Ts. Zaevski. Then, two Monte-Carlo schemes and a finite difference scheme are introduced for solving this optimization problem and the results seem to improve significantly the current methods in terms of speed.

Finally, the perpetual options ($T = \infty$) are considered in which case the exercise curve is clearly a constant c. Theorem 4.1 derives the price and the constant c for call options, whereas Theorem 4.2 does so for the put options. The proofs rely on the limits of Laplace transforms of hitting times obtained in Chapter 2 but I reckon that these may not be needed. For example, if in the proof of Theorem 4.1 one considers $\xi \wedge T$ then $S_{\xi \wedge T} \leq c$ and being bounded the limit as $T \to \infty$ passes in easily. However, the proofs are skilfully done and do not need further change.

3.4. **Chapter 5: Pricing discounted American capped options.** Capped options are designed to limit the exposure of the writer to significant fluctuations in the price of the underlying asset. These options include an embedded cap, which is a predetermined price level beyond which the value of the option is not registered. For a capped call option, the embedded cap sets an upper limit. If the price of the underlying asset exceeds this cap, the value of the option is capped, and the writer is not obligated to pay beyond this level. For a capped put option, the cap establishes a lower limit. If the price of the underlying asset falls below this cap, the payout is capped, preventing further losses for the writer.

The candidate has made a thorough investigation of American capped options and he has obtained the following main set of results:

(1) **Exercise Boundaries**:
   - For **put options**, as established in **Theorem 5.1**, the exercise boundary is given by:
$$c^A(t) \wedge L,$$
     where:
     - L is the capped level, and
     - $c^A(t)$ is the exercise boundary of an unrestricted American option.
   - For **call options**, as established in **Theorem 5.3**, the exercise boundary is given by:
$$c^A(t) \vee L.$$
   - In the special case where $T = \infty$ (perpetual options), the boundaries simplify to constants, as detailed in **Corollaries 5.1 and 5.2**.

(2) **Option Pricing**:
   - The **prices** of put and call capped options are derived in **Theorems 5.2 and 5.4**, respectively.
   - The case $T = \infty$ is specifically addressed in **Corollaries 5.1 and 5.2**.

It is important to highlight that the prices of capped options depend on the cap level and the boundary value at maturity of the unrestricted American option. The proofs are both technically intricate and challenging. While the results are accurate, the exposition requires improvement in several areas:

- it is not clearly stated whether $c^A(t) \in (D_2, D_1)$ or $c^A(t) \in (D_1, D_2)$, which primarily follows from the potential monotonicity of the boundary and the fact that D_2 represents the perpetual boundary value;
- the application of Dynkin's formula requires a more detailed and thorough explanation to ensure clarity;
- there needs to be a clear distinction between working with time to maturity and natural time to avoid confusion;
- the computation of V in the most complex scenario requires significantly better explanation and elaboration.

The candidate has developed and implemented numerical schemes for determining the exercise boundary and calculating the price of capped American options. The numerical experiments are highly convincing, effectively making the entire chapter a comprehensive and self-contained study on these derivatives.

3.5. **Chapter 6. On some generalized American style derivatives.** In this chapter the candidate develops the theory of generalized American options whose introduction and purpose is very well motivated. The latter financial instruments have basically more general payoff structure which, however, preserves the existence of exercise region. Typically, twice differentiable functions $G(x)$ and payment structure $N(t,x) = e^{-\lambda t}G(x)$ satisfy this requirement. The special case when $G(x) = Mx^n + C$ is considered in detail. The theoretical framework seems to be solid but there are many unclear points and vague proofs which make the reading very difficult. I will outline my comments on these omissions and ambiguities hoping to get a clear response from the candidate.

(1) Proposition 6.1 is not properly proved. A recurring theme in this dissertation is to have $a_n - b_n > 0 \implies \liminf_{n\to\infty} a_n - b_n > 0$ or similar claims. The fact that the limit in the proof is non-negative does not mean the payment is larger; it can be always smaller but in the limit it may go to zero. At present, it seems that $(t,x) \in \Upsilon \implies \mathcal{B}G(x) \leq 0$. Given the importance of this claim this should be carefully addressed. Is it enough to alter Definition 6.1!? What are the implications in Proposition 6.5?

(2) Proposition 6.3 seems to be correct but its proof must be clarified. It seems that $\bar{xi}$ cannot exceed τ anyway!?

(3) The claim in the proof of Proposition 6.4 that the region is continuous function should be expounded upon.

(4) Proposition 6.6 is stated for global maximum but it is applied for a local one. This claim should be proved. Lemma 6.2 is not well-written; it uses Prop 6.6 but it is never referred to.

The claims for the put case should be checked too.

In the Section on finite maturity combined put-call options are excluded for some reasons but those have never been introduced. In 6.6. K is used as arbitrary constant whereas it is the strike all the time; one should recall p, q.

Overall, I expect this chapter to be properly discussed at the next stage.

3.6. **Chapter 7. American strangle strategies with arbitrary strikes.** This option structure combines elements of both call and put options, offering flexibility to the holder.

Key Features of the Option:

1. **Dual Nature**: The holder can choose to execute the option either as:
    - a put option with strike K_1, providing a payoff of $C_1(K_1 - x)^+$, or
    - a call option with strike K_2, providing a payoff of $C_2(x - K_2)^+$.
2. **Weighting Factors**:
    - C_1: weight or number of shares in the put component;
    - C_2: weight or number of shares in the call component.
3. **Strike Prices**:
    - K_1: strike price for the put component;
    - K_2: strike price for the call component.
4. **Exponential Decay**: The payoff is discounted by $e^{-\lambda t}$, representing a decay factor that accounts for time t and a rate λ.

Payoff Expression:

$$N(t, x) = e^{-\lambda t} \max\{C_1(K_1 - x)^+, C_2(x - K_2)^+\}.$$

If $x = D_0$ which is computed in the thesis then the contribution of each component is the same and the option could be said to be in the balance. Each element put or call has its exercise region denoted by Υ^p, Υ^c respectively.

The first set of results describes these regions with the following outcome:

(1) if $\lambda = 0$ then $\Upsilon^c = \emptyset$, see Proposition 7.1;
(2) if $\lambda > 0$ then $\Upsilon^c \neq \emptyset$ and $(t, x) \in \Upsilon^c \implies (t, y) \in \Upsilon^c$, for $y > x$, see Proposition 7.2;
(3) $\Upsilon^p \neq \emptyset$ and $(t, x) \in \Upsilon^c \implies (t, y) \in \Upsilon^c$, for $y < x$, see Proposition 7.3;
(4) Proposition 7.4 yields monotonicity of the exercise boundaries;
(5) Propositions 7.5-6 yield the value of the boundaries at maturity.

Lemma 3.1 is not clearly used in the proofs it appears in and there is again > 0 rather than ≥ 0 in the proof of Proposition 7.5 (outlined already trend).

The case when the maturity is ∞ is treated separately as in this case Theorems 7.1-2 give the exercise boundaries (constants) when $\lambda > 0$ and $\lambda = 0$. This is achieved via an optimization problem. It has to be noted the curious fact that even when an early exercise for the call combination is not viable the put boundary still depends on C_2 (but not on K_2). I would strongly recommend the candidate to try the following reparametrization which may simplify the matter: $B = A/a; x = yA$; get linear in A function $f(A, a, y)$ which you can maximize as a function of two variables relatively easily.

The proofs for the aforementioned statements seem to be correct but some additional clarity is needed.

The chapter ends with the numerical consideration of the finite maturity case when a lot of experiments have been conducted which confirm the value of these developments.

3.7. **Chapter 8. Quadratic American strangles in the light of two-sided optimal stopping problems.** In this chapter the candidate considers options with payoff structure $N(t, x) = e^{-\lambda t}G(x)$ where G is such that two-sided optimal stopping might arise (Condition 8.1). Usually one demands two derivatives in the space variable. The motivation behind these options is that some investors prefer to hedge strongly against strong fluctuations of the asset price compromising the return when the price is close to the strike.

Propositions 8.1-3 give the structure of the exercise regions and their proofs have some of the drawbacks as mentioned in previous chapters for similar results. Proposition 8.4 gives even the values of the two-sided time-dependent strip at maturity and they coincide with the universal constants $C \leq D$ from Condition 8.1. In the case of perpetuities the exercise boundaries ought to be constants $A \leq C \leq D \leq B$. This involves an optimization problem in A, B which is thoroughly investigated by the candidate in Propositions 8.5-6, which are technical.

For finite maturity the already standard procedure for approximating the exercise boundary and, subsequently, of the option price is conducted. The approach for solving the problem is very well-presented.

The final theoretical part deals with the case when $G(x) = (x - K)^2$ which corresponds to the name of the chapter itself. An extensive analysis is presented which solves the following two problems: depending on the parameters to determine whether the optimal stopping is one- or two-sided and to obtain the price of the options.

This chapter is a clear attestation that the candidate is well-versed with the area and possesses quite good technical skills.

3.8. **Chapter 9. Cancellable call options under perpetual assumptions.** In this chapter the candidate considers the so-called Israeli options which give to the holder and the writer rights to respectively exercise and cancel the option. The writer is usually penalized for cancellation. The general payment structure in this part can be set up as follows:

$$N(\zeta, S_\zeta) = N_1(\zeta_1, S_{\zeta_1})1_{\zeta_1 \leq \zeta_2} + (N_2(\zeta_2, S_{\zeta_1}))1_{\zeta_2 < \zeta_1},$$

where $N_1(t, x) = e^{-\lambda t}(x - K)^+; N_2(t, x) = e^{-\lambda t}((x - K)^+ + \eta)$.

The goal of the presented study is to analytically determine, whenever possible, the exercise region of the holder (Υ^b), the cancellation region of the writer (Υ^s) and the continuation region. Having obtained this goal the pricing of the option is established.

The intricacy here is that we have a game between the parties and for each strategy, i.e. stopping time, ζ there is an optimal strategy $A(\zeta, \cdot)$, respectively $B(\zeta, \cdot)$, for the other party. The optimization seeks for $A(B(\zeta)) = \zeta$ or a fixed point for $A \circ B$.

The first set of results describes the respective regions with the following outcome:

(1) for $\lambda = 0$ then $\Upsilon^b = \emptyset$, see Proposition 9.1;
(2) if $\eta \geq K$ then $\Upsilon^s = \emptyset$, see Proposition 9.2;
(3) if $x < K$ then $x \notin \Upsilon^s$, see Proposition 9.3;
(4) if $x \in \Upsilon^b$ and $y > x$ then $y \in \Upsilon^b$, see Proposition 9.4;
(5) if $x \in \Upsilon^s; K < y < x \implies y \in \Upsilon^s$.

The proofs are relatively technical and reveal the abilities of the candidate. Lemma 3.1 is again used in a vague way. The proof of Proposition 9.6 is a bit messy: in point 2 I do not get why immediately there is $K_1 \in \Upsilon^b$; it does not follow with direct application of Proposition 9.4; perhaps by it is done by contradiction and limit!?

Finally, the central result in this work, i.e. Theorem 9.2, determines the boundaries and the price of the Israeli option. It is excellent that the candidate has managed to obtain such a complete result. I am not sure about $2c$ in (9.9)!? Then the optimization is not as hard as the one from the previous chapter.

3.9. **Chapter 10. Cancellable put options without maturities.** The results presented in this chapter are analogous to those in Chapter 9 but focus specifically on put options. The candidate clearly explains why put options are not simpler in his studies, primarily due to the potential presence of a negative risk-free rate.

The results can be summarized as follows:

(1) $x > k \implies x \notin \Upsilon^s$, see Proposition 10.1;
(2) $x \in \Upsilon^b \implies y \in \Upsilon^b$ if $0 < y < x$, see Proposition 10.2;
(3) $x \in \Upsilon^s \implies y \in \Upsilon^s$ if $x < y < K$, see Proposition 10.3;
(4) $r > 0 \implies \Upsilon^s = \emptyset$ or $\Upsilon^s = \{K\}$, see Proposition 10.4;
(5) Theorem 10.1 offers full results on pricing such options.

The proofs are analogous to those of Chapter 9 with the same recommendations. It is certainly worth checking whether the optimization problem cannot be solved by the parametrization offered before. Numerical study is presented.

3.10. **Chapter 11. Perpetual cancellable options with a proportional penalty.** The results of this chapter are similar to the results of Chapters 9-10 because the main difference is that the penalty for cancellation is not a fixed sum but a proportion of the payment, i.e. $N_2(t,x) = \eta N_1(t,x)$ and it would be good to mention that $\eta \geq 1$. It would be beneficial for the candidate to speculate which version is better. For me it seems natural to have a penalization proportional to the payoff but the market may demand something else.

Regarding the results in brief - the exercise regions are specified both for put and call options; the pricing is done via optimization reminiscent of the one of the previous chapters; the proofs are now fairly standard but the technique by which one uses results for options with zero penalty from the previous chapters is very nice; finite maturities are considered and extensive numerical experiments are conducted.

3.11. **Chapter 12. Perpetual cancellable options with convertible features.** This chapter continues the thorough investigation of financial derivative under the Black-Scholes assumption. The candidate considers perpetual options, cancellable options with penalties which potentially include a mixture of proportion of the payment, shares and fixed penalty. The payment of the writer upon cancellation is therefore

$$N_2(t,x) = \eta_1 N_1(t,x) + \eta_2 x + \eta_3.$$

The problem is clearly set out and some preliminary results are given where the first Lemma should be improved both as a statement and a proof. I shall discuss only the call case, the other being similar in terms of results. I start with the exercise region.

(1) $x < K \implies x \in \bar{\Upsilon}$, see Proposition 12.1;
(2) $\eta_3 \geq K\eta_1 \implies \Upsilon^s = \emptyset$, see Proposition 12.2, which crucially depends on previous results and in whose proof the reference to Theorem 4.1 must be improved;
(3) $\eta K_1 > \eta_3, x > K, x \in \Upsilon^s, K < y < x \implies y \in \Upsilon^s$, see Proposition 12.3, whose proof is not very clear either;
(4) $\eta_1 = 1, \eta_2 = \eta_3 = 0 \implies \bar{\Upsilon} = (0, K)$, see Proposition 12.4;
(5) $x \in \Upsilon^b; y > x \implies y \in \Upsilon^b$, see Proposition 12.5;
(6) $\lambda = 0 \implies \Upsilon^b = \emptyset$, see Proposition 12.6;
(7) $r < 0 \implies \Upsilon^s = \emptyset$ or $\Upsilon^s = \{K\}$.

Theorem 12.1 then gives a full description of the exercise boundaries which are two constants which are found via an optimization procedure already adopted in earlier chapters and it also furnishes the price of the option. It seems that here the optimization is done with slightly different reparametrization which seems to be better. Finally the smooth fit is discussed as well as the case $\lambda = 0$. Extensive numerical results are given.

3.12. **Chapter 13. Pricing cancellable American put options on the finite time horizon.** This chapter is suitable for the last one of the dissertation. It uses results and approaches from the theory developed thus far. Considering finite maturity does not allow for fully explicit results for the cancellable put options but the previous results deliver some basic analytical information which is used for numerical results. Basically, main properties of the optimal region can be obtained as mathematical statements whereas their form is numerically approximated. Overall, the writer's region exhibits quite rich behaviour depending on the strike, the previous perpetual options, etc.

4. Recommendation and overall evaluation

The thesis provides an extensive study of a specialized mathematical subfield, thereby fulfilling the primary criterion for the Doctor of Sciences degree at Institute of Mathematics and Informatics, Bulgarian Academy of Sciences. Additionally, Associate Professor Tsvetelin Zaevski is recognized as a well respected expert in this area, both nationally and internationally, largely due to his high-quality publications and adeptness with advanced mathematical techniques. Furthermore, he meets all the minimal legal requirements for the award of the Doctor of Sciences degree. I wholeheartedly recommend that the scientific panel confer the degree of Doctor of Sciences upon Associate Professor Tsvetelin Zaevski.

I have highlighted the key issues in my earlier discussion. Additionally, it would be beneficial to include a proof of the convergence of the exponential piece-wise linear approximation to the early exercise curve. While the numerical results and underlying principles leave little doubt as to the validity of the approach, a rigorous confirmation is necessary to complete the story. It is also worthwhile to check with results on first passage time to see whether the viable options can be extended should the exercise and continuation regions are guaranteed by these results. Finally, I encourage further international collaboration, and I believe that Associate Professor Zaevski is already on the path toward this goal—an important aspect of a successful academic career.

5. Conclusion

According to the applied documents the candidate Assoc. Prof. Tsvetelin Zaevski satisfies all the minimal requirements set by the Act for the Development of the Academic Staff in the Republic of Bulgaria, the Regulations for its implementation and the Regulations of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, stipulating the specific additional requirements for the award of scientific titles and academic positions. My professional opinion concerning his work is fully confirmed by the applied documents which clearly demonstrate that Assoc. Prof. Tsvetelin Zaevski is a very good specialist in the research area of his dissertation.

Therefore, I give an overall positive evaluation of the dissertation and the work of Assoc. Prof. Tsvetelin Zaevski and I **highly recommend** that the scientific panel awards Assoc. Prof. Tsvetelin Zaevski the academic title Doctor of Sciences in the area of science 4. Natural sciences, mathematics and informatics, professional field 4.5 Mathematics.

. . . .

Prof. Dr.Sc. Mladen Savov

Sofia
19.02.2025