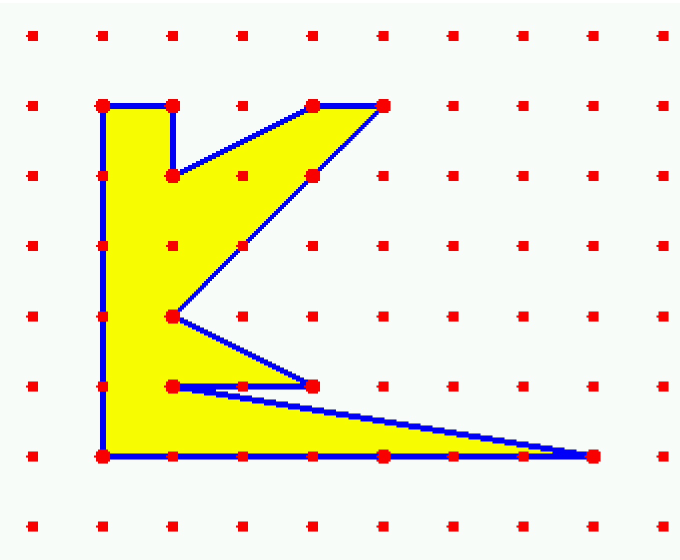
# Pick's Theorem

## Introduction

*You will rediscover an interesting formula in the sequel expressing the area of a polygon with vertices in the knots of a square grid. You may use the software GeoGebra in your research.*

Find the area of the figure if the length of the unit square is 1.



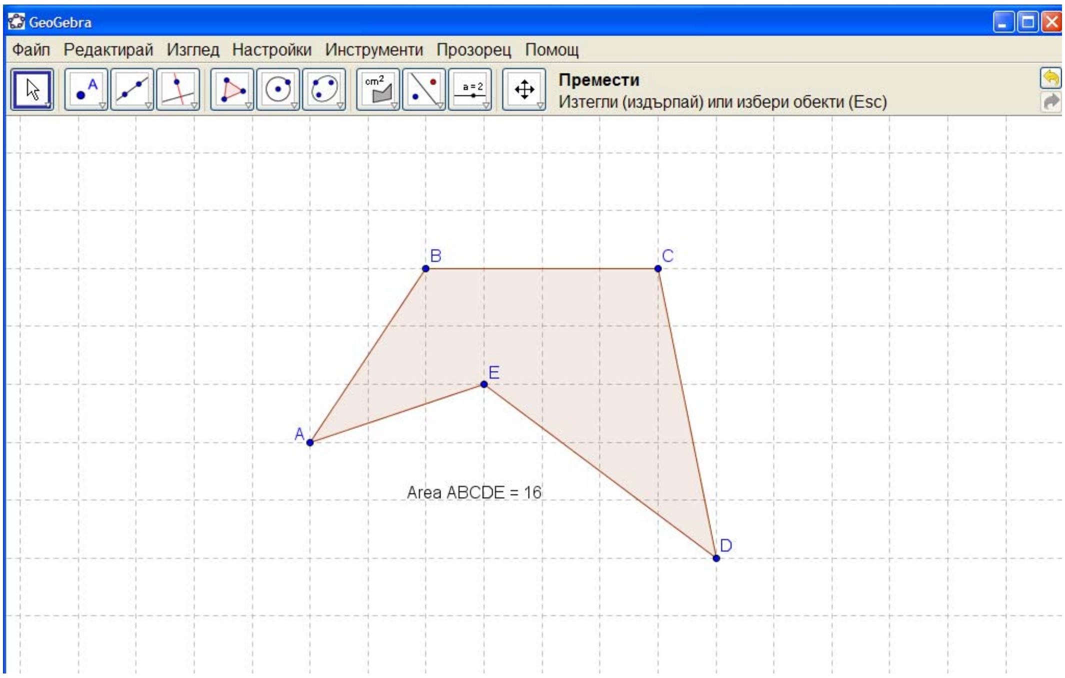
# Explorations ...

You are familiar with several ways to find the area of a polygon:

- By dividing the figure into several parts with areas which are easy to be found;
- By “packing” the figure in a rectangle and expressing its area using the area of the rectangle and the areas of figures which are easy to be found.

In 1899 the Austrian mathematician **Georg Alexander Pick** (1859 – 1942) published a formula for the area of a polygon with vertices in the knots of a square grid. Try to rediscover the formula. This will help you to solve the problem easily and quickly.

For the purpose it would be useful to apply *GeoGebra* in the construction of polygons with vertices in the knots of a virtual square grid . In this way the areas could be calculated automatically.



# Explorations ...

Fill in the columns A1 to D12 of the tables below. Using the table data, formulate a hypothesis for the relation including the number of the grid points along the contour **k**, the number of the grid points within the polygon **v** and the area **S**.

| | *A1* | *A2* | *A3* | *A4* | *A5* | *A6* | *A7* | *A8* | *A9* | *A10* | *A11* | *A12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **v** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| **k** | | | | | | | | | | | | |
| **S** | | | | | | | | | | | | |

| | *B1* | *B2* | *B3* | *B4* | *B5* | *B6* | *B7* | *B8* | *B9* | *B10* | *B11* | *B12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **v** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **k** | | | | | | | | | | | | |
| **S** | | | | | | | | | | | | |

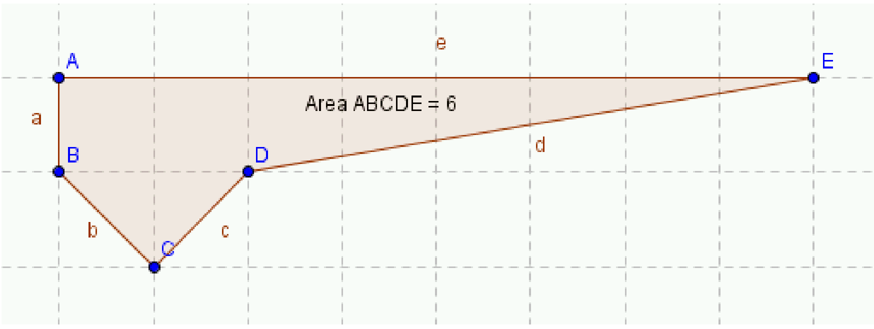
| | *C1* | *C2* | *C3* | *C4* | *C5* | *C6* | *C7* | *C8* | *C9* | *C10* | *C11* | *C12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **v** | | | | | | | | | | | | |
| **k** | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| **S** | | | | | | | | | | | | |

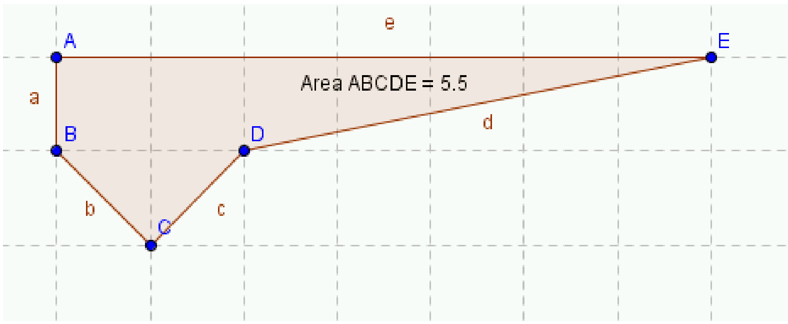
| | *D1* | *D2* | *D3* | *D4* | *D5* | *D6* | *D7* | *D8* | *D9* | *D10* | *D11* | *D12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **v** | | | | | | | | | | | | |
| **k** | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| **S** | | | | | | | | | | | | |

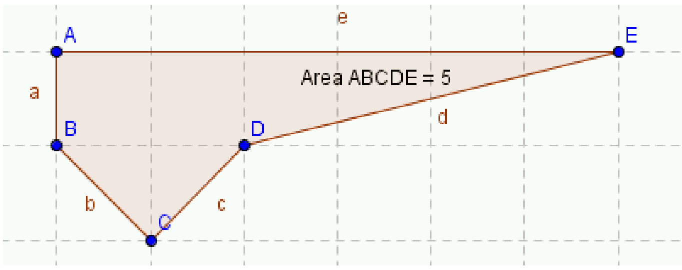
Use GeoGebra.
For equal values of **v** propose appropriate values of **k**, that fit to the relationship to be found easily.

# Explorations ...

It is appropriate to use a given figure and to change it by moving its vertices. Here is an example of a figure with 1 internal grid knot and 12 contour grid knots. By moving one of the vertices, obtain figures with 1 internal grid knot and 11, 10, ... or 3 contour grid knots.




| | *B*1 | *B*2 | *B*3 | *B*4 | *B*5 | *B*6 | *B*7 | *B*8 | *B*9 | *B*10 | *B*11 | *B*12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **v** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| **k** | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |
| **S** | | 6,5 | 6 | 5,5 | 5 | 4,5 | 4 | 3,5 | 3 | | | |

Note that the areas of the constructed polygons are either integers or decimals with 5 tenths. Try to establish when the areas are integers and when they are decimals.

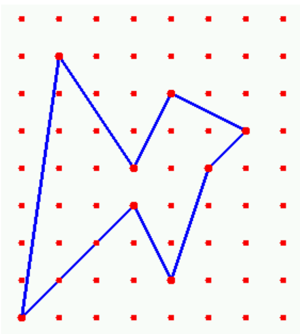
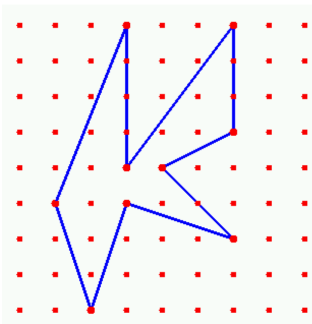
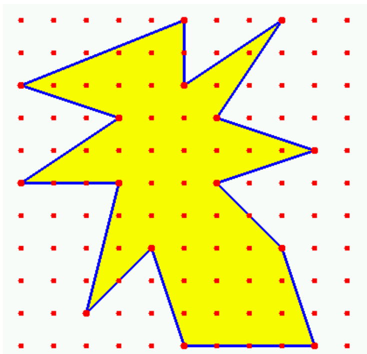
**Georg Alexander Pick**

$$S = v + \frac{k}{2} - 1$$

This formula allows to find the area **S** of a polygon with vertices in the knots of a square grid, where **v** is the number of the grid knots within the polygon and **k** is the number of the grid knots along its contour, including the polygon vertices.

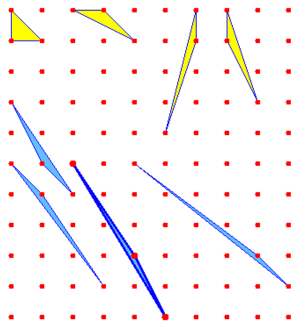
Using this formula, solve the problem on Page 1.
Note that the unit square side is of length 1 which is used in the next problems too.

**Problem.** Find the area of the figure.

**Problem.** In a square grid with unit square side of length 1 construct triangles with area equal to 0.5 sq. cm.

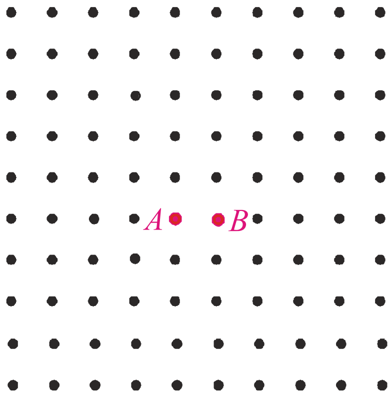
From the Pick's formula it follows that such triangles do not have any internal grid knots and the only grid knots along the contour are the triangle vertices. Here are some examples.



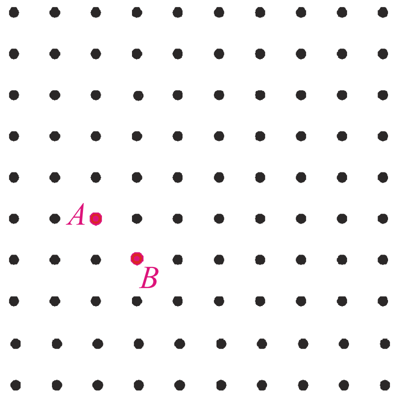
## Pushing up the limits

**Problem.** Given are two vertices $A$ and $B$ of a triangle with area 0.5. Find the set of grid knots which includes the third vertex of the triangle.

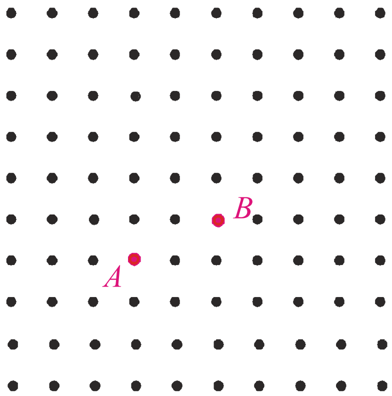
a)



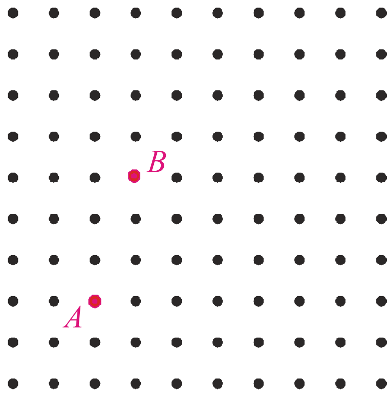

b)




c)

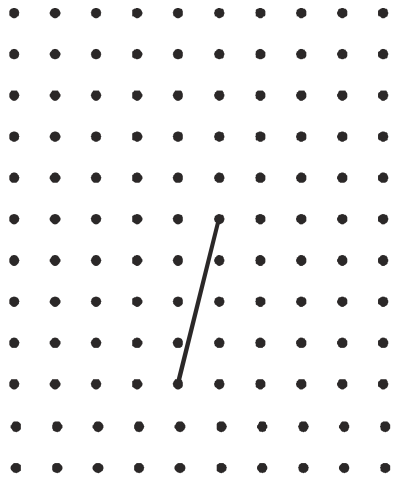



d)

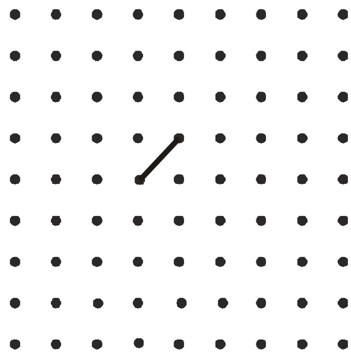


## Pushing up the limits

**Problem.** Given is one of the sides of a triangle *ABC* with area equal to 1.5 square units. How many of the given 120 points could be the third vertex of the triangle?



**Problem.** Given is one of the sides of a trapezoid with area equal to 1.5 square units. Draw the other three sides of the trapezoid if its vertices are among the given 81 points. How many different options are there?



Formulate a problem that could be solved easily by using the Pick's formula.