FINDING GEOMETRIC PATTERNS AS A GAME OF DYNAMIC EXPLORATIONS

Toni Chehlarova, Evgenia Sendova

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia 1113, Acad. G. Bonchev Str., Block 8, Bulgaria toni.chehlarova@gmail.com jenny@math.bas.bg

Abstract

The paper deals with the inquiry based style of learning as applied to traditional and open geometry problems by means of dynamic geometry software. The so called what-if strategy *(i.e. exploring what happens if the formulation of the original problem varies) is demonstrated in the context of a well know problem from the geometry textbook – to find the locus of the midpoints of the segments joining a fixed point within a circle with the points of that circle. After making a dynamic construction for the locus the students are offered additional tools in support of the rigorous proof. The exploration game continues with varying the initial conditions of the problem (e.g. replacing the circle by other figures and the midpoint with a point dividing the segment in a fixed ratio). Then the well known problem of finding the locus of the centers of the equilateral triangles inscribed in an equilateral triangle is considered together with its ambitious generalization, viz. to find the locus of the centers of the regular m-gons inscribed in a regular n-gon ($m \leq n$). The process of generalization leading to open problems is considered together with the construction of appropriate dynamic tools for explorations. It is the very process rather than the description of the results which is of primary interest since it illustrates how the atmosphere around the working mathematicians could be transferred into a class setting. The expectation is that some teachers and students would gain motivation in attacking the considered open problems themselves.*

Keywords

Inquiry based learning, dynamic geometry software, loci related problems, what-if strategy

INTRODUCTION

Many interesting geometric problems deal with finding a locus ─ the set of points satisfying a particular condition. The traditional problems on loci are limited to finding simple curves. Language based computers environments allow for much more sophisticated explorations (Sendov, Sendova, 1995). While the computer language offers a vast spectrum of expressive means, enabling the user to enlighten the finest details of his thought, it is often found to be a great obstacle for the math teachers. Thus, the inquiry based learning in mathematics has been recently promoted within a number of European educational projects (DALEST, *Meeting in Mathematics*, *Math2Earth*, *InnoMathEd*, *Fibonacci*, DynaMAT) by means of dynamic geometry software offering direct manipulation of geometric objects (Christou et al., 2007, Georgiev et al., 2008, Bianco and Ulm, 2010, Baptist and Raab, 2013, Andersen et al., 2010).

In this paper we shall demonstrate how the inquiry based style of learning could be applied in the context of traditional and open geometry problems.

LOOKING AT THE CLASSICS WITH A DYNAMIC EYE

A very important component of the inqury-based mathematics learning is the *what-if* strategy, i.e. to explore what will happen if we vary the formulation of the original problem. Let us illustrate this strategy in the context of a well know problem from the geometry textbooks:

A traditional geometry problem: *What is the locus of the midpoints* ***M*** *of the segments joining a fixed point* ***P*** *within a circle with the points of that circle?*

To solve this problem by means of dynamic geometry software (say *GeoGebra*) the students study the behaviour of the midpoint under question while moving the endpoint of the segment on the circle along it (Fig. 1a):

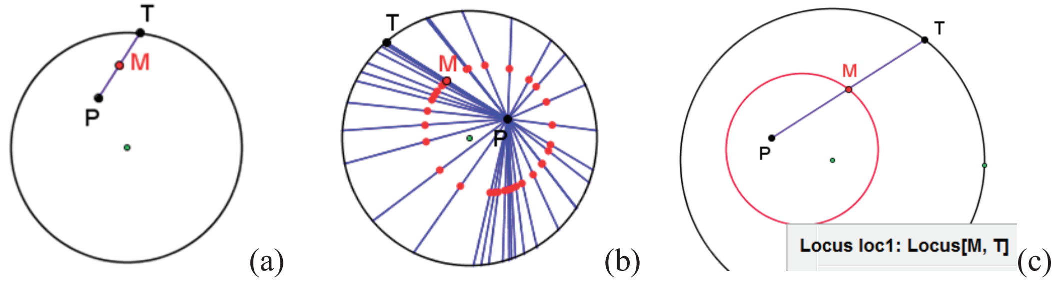

Fig. 1 The first steps of the exploration

They could strengthen their conjectures about the shape of locus by following the the trace of the midpoint's path (possibly with the segment) (Fig. 1b) and finaly chek experimentally their conjectures by constructing the locus of the midpoint by the inbuilt tool (Fig. 1c).

The game is not over, however. It is time to ask some *What-if* questions, e.g. *What if **M** is not the midpoint, but divides the segment at a fixed ratio? What if **P** is outside of the circle?.*

The typical conjecture of the students is that in this case the locus would look like a more general curve of a second degree, e.g. an ellipse. The teacher guides the explorations by suggesting to make the ratio a variable **e** in which **M** is dividing the segment (i.e. to create a slider in our case) (Fig. 2a and 2b):

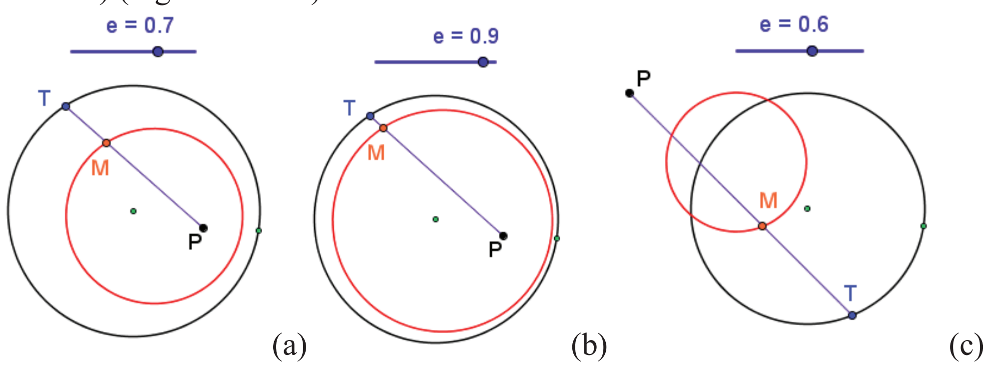

Fig. 2 Changing the ratio of the division

The students are genuinely surprised to find that the locus remains a circle. A further idea arises — to explore the situation when the point **P** is outside of the circle (Fig. 2c) — a circular shape once again! Then a new idea is suggested bringing an interesting effect — to trace the segment for a point outside the circle:

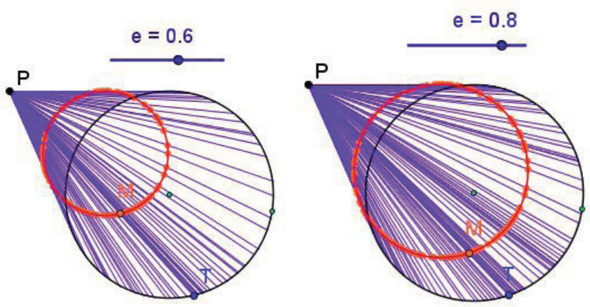

Fig. 3 Changing the position of point P

Now is the time for the teacher to raise students' suspition - could they be absolutely sure that **M** describes a circle? Couldn't it be in fact an ellipse which is very close to a circle…

One way to verify their conjecture (still experimentally) is to construct 3 points (**H**, **I**, **J**) on the locus, pass a circle through them and check if this circle concides with the locus. Another way which could help them prove the conjecture rigorously is to observe some interesting properties of the construction enriched with some auxiliary elements (Fig.4):

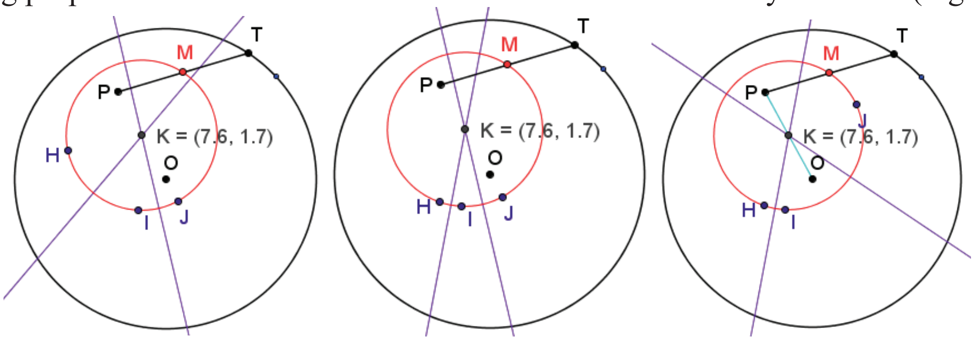

Fig. 4 Auxiliary construction in support of the rigorous proof

It is easy for the students to see that that the center **K** of the locus keeps its coordinates the same. In addition the segments **KJ**, **KI** and **KH** have equal legth which shows experimentally (but **with a greater degree of conviction**) that the locus is a circle. Furthermore, they would notice that **K** is the midpoint of the segment **PO**, where **O** is the center of the original circle. Now they are ready to prove rigorously that the locus is a circle with a center the midpoint **K** of the segment **PO** and a radius ─ half of the radius of the given circle.

The exploration game can continue with replacing the circle by a square, a triangle, an arbitrary regular polygon, a curve of their own choice.

If the students have studied *dilation* (in the Bulgarian curriculum it is introduced a year after the first occurrence of *loci*) they could use it to solve the problems but it is very appropriate for them to get used to generalizing their findings. Applying the *What-if* strategy could cultivate an exploratory spirit in mathematics classes - the students are encouraged to explore interesting partial cases, to generalize relatively simple problems in various directions, and even to attack and generalize challenging problems of Olympic level (Atanasova, 2011).

FROM A WELL KNOWN PROBLEM TO AN OPEN ONE

Here we demonstrate a process which is typical for the working mathematicians – we generalise a well-known problem, then we attack it with tools we believe are the most appropriate for the purpose (in our case with dynamic constructions we have specially designed in a *step-by-step refinement and enrichment* spirit). We systemize our explorations and reflect on the ideas we get. It is the very process that will be of our primary interest rather than the description of the results. In addition, we expect some teachers and students to get motivated in attacking some of the open problems themselves.

A well-known problem:
Find the locus of the centers of the equilateral triangles inscribed in an equilateral triangle.

An ambitions generalization of this problem could be formulated as follows.

An ambitious generalization:
Find the locus of the centers of the regular $\boldsymbol{m}$*-gons inscribed in a regular* $\boldsymbol{n}$*-gon,* $\boldsymbol{m} \leq \boldsymbol{n}$.

Further below we shall write $(\boldsymbol{m};\boldsymbol{n})$ to denote the construction of a regular $\boldsymbol{m}$-gon inscribed in a regular $\boldsymbol{n}$-gon. Note that we are not even sure for which $\boldsymbol{m}$ and $\boldsymbol{n}$ the $(\boldsymbol{m};\boldsymbol{n})$

constructions are possible. Let us start our *attack* with a more modest problem, dealing with the case **(3;*n*)** for ***n*** = 3, 4, …

The first attack – the (3;*n*) case:
*Find the locus of the centers of the equilateral triangles inscribed in a regular **n**-gon.*

A primitive (hand-made) dynamic model

We construct an equilateral triangle two of whose vertices are on the ***n*-**gon and move the third one so as to get an inscribed triangle**.** To get the flavor of the dynamic construction to be then generalized it is natural to start with the simplest case (***n*=3)**, and proceed in what could be called a *hand-made* model (Fig.5):

- We select two arbitrary points **M** and **N** on different sides of the given (the *blue*) triangle.
- Then we construct an equilateral (*red*) triangle with a side **MN.**
- Next we move **N** (keeping **M** at irs current position) so that the red triangle becomes inscribed in the blue one. The center of the red triangle is a point of the locus sought.
- Now we repeat the above proces for a new position of **M.**

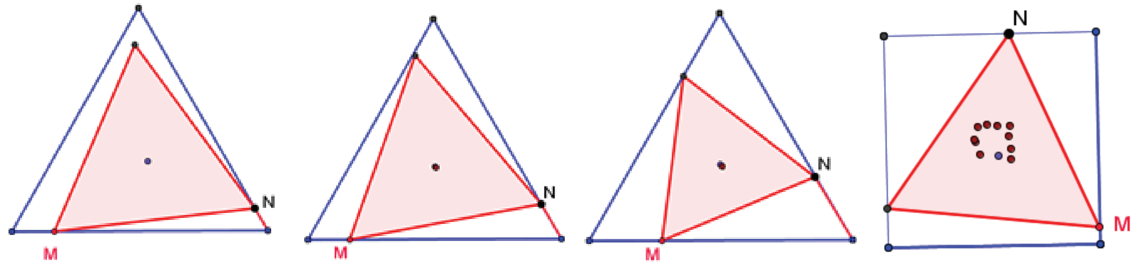

Fig. 5 The hand made **(3;3)** and **(3;4)** dynamic models

Thus, using consecutive positions of point **M** we get an approximate idea about the locus **—** in the **(3;3)** case the centers seem to coincide (or are at least close enough)… If we apply a similar procedure for the **(3;4)** case the centers appear to be on a square. But inscribing the triangle *by hand* is a time-consuming method (still better than constructing on a paper and considering just one possibly misleading case due to imprecision (Pehova, 2011).

To automize the construction let us take a better look at the **(3;3)** construction. It is natural to conjecture that in this case the locus is a single point coinciding with the center of the given triangle (Fig.6).

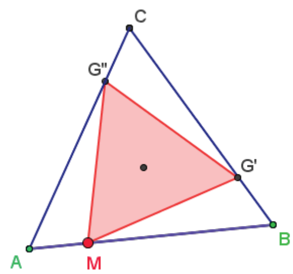

Fig. 6 The **(3;3)** dynamic construction

The congruence of the triangles **AMG"** and **BG'M** implies **AM**=**BG'**. Therefore, we can use in this particular case a dynamic construction based on the congruence.

An automized dynamic model for (3;n) constructions

There are various methods of creating automized models for the **(3;3)** constructions. Here is one of them:

- We construct a point **M** on the contour of a regular **3**-gon`(the triangle **ABC**)

- We construct the image **G'** of **M** under a rotation of 120° about the center of **ABC**
- We construct the image **G''** of **G'** under a rotation of 120° about the center of **ABC**
- We connect the points **M, G'** and **G''** in a triangle.

For **n**>3 we can proceed as follows:

- We construct a point **M** on the contour of a regular **n**-gon.
- Then we construct the image of the **n**-gon under a rotation **ρ** of 60° about **M**.
- We construct their interestion point **F.** (It will be another vertex of the equilateral triangle whose first vertex is **M**, and which is inscribed in the ***n***-gon.)
- Then we construct the thrid vertex as the pre-image **F'** of **F**.
- We connect **M**, **F'** and **F** to get the equilateral triangle inscribed in the ***n***-gon.

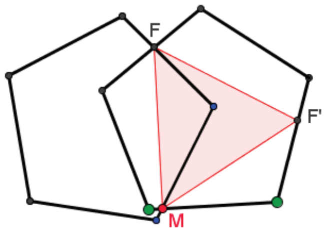

Fig. 7 Constructing a **(3;5)** dynamic model

Here are some snap-shots of the trace the triangle's center in the **(3;4)** construction leaves during the movement of the inscribed triangle:

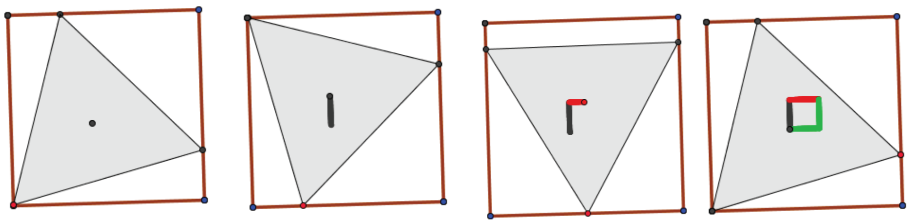

Fig. 8 The **(3;4)** dynamic construction

When we move the red point (**M**) until the next vertex of the triangle coincides with a vertex of the square (i.e. takes its initial position) we observe the trace becoming a shape which looks as a half of square. By analogy, when moving the point **M** along the rest of the sides of the square the center of the triangle will leave a trace which completes a square-like shape and after which it will start repeating the trace (three times). If the considered locus of the **(3;4)** construction is a square indeed could we conjecture that the corresponding locus of the **(3;5)** construction would be a regular pentagon? In the latter case it is sufficient to observe the effect of the movement of the red point on a part of the pentagon only.

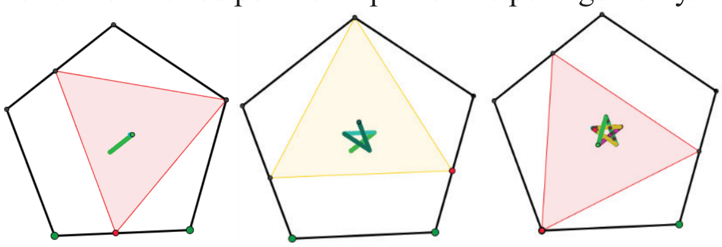

Fig. 9 The **(3;5)** dynamic construction

A-a-ah! Still 5 sides but it does not look like a pentagon – rather like a pentagram! Then what we suspected to be a square could be considered maybe as a „4-side star"...

Again, the center of the triangle describes the locus three times while the red point makes a full round along the original pentagon.

In the **(3;6)** case the locus appears to be a single point:

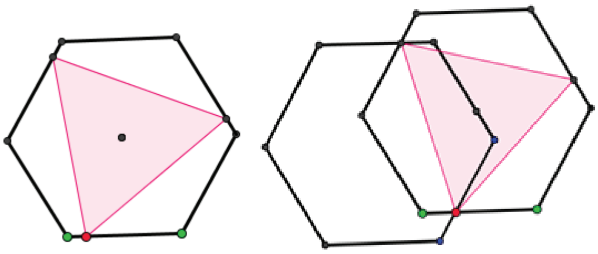

Fig. 10 The **(3;6)** dynamic construction

Such was the locus in the **(3;3)** case. By analogy we could conjecture that the same would hold for **(3;9),** and more general – for **(3;3*k*).** We could make separate construction for the **(*m*;*km*)**.

Further explorations providing insight

The (m; km) model

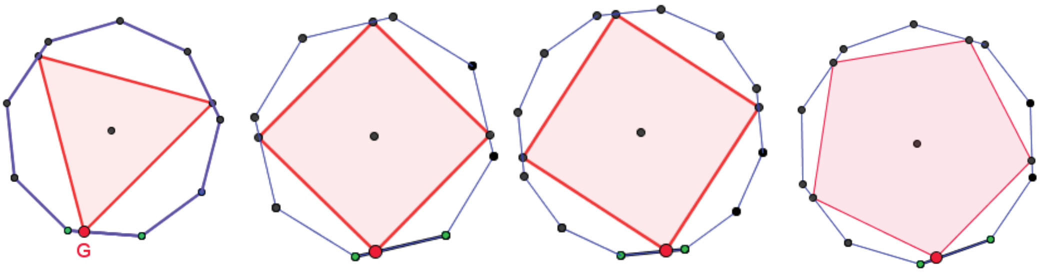

Fig. 11 The **(*m*;*km*)** dynamic constructions

The **(*m*;*km*)** constructions could be also achieved by analogy of the methods in Fig.6. The general conjecture we could draw after exploring the **(*m*;*km*)** model is that *for every point **G** on the **n**-gon (**n**=**km**) there exists an inscribed **m**-gon with a vertex **G** and the locus under consideration is a single point coinciding with the center of the **n**-gon.*

Let us continue our explorations with the (**3;*n***) model. In the case of **(3;7)** for instance we are expecting a star with its generating module emerging when going along one of the heptagon's sides. Indeed (Fig. 12)!

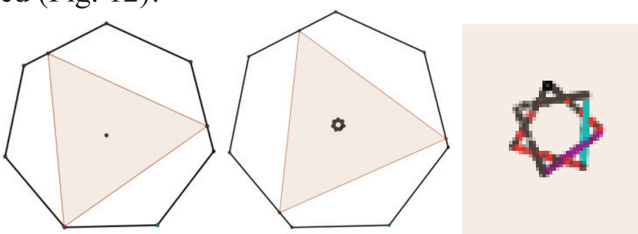

Fig. 12 The **(3;7)** dynamic model

Exploring further the (**3;*n***) model leads us to the conjecture that it is possible to inscribe an equilateral triangle in every regular ***n***-gon, i.e. (**3;*n***) *is always a possible construction*.

It is interesting to see what is the situation in the case of the (**4;*n***) model (Fig. 13).

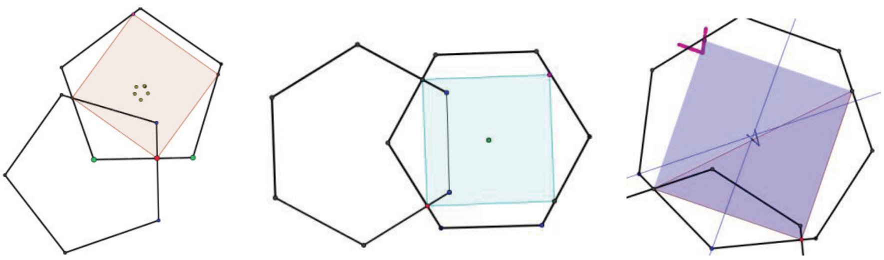

Fig. 13 The **(4;5), (4;6)** and **(4;7)** dynamic models

For a number of specific cases for $m > 4$, it is easy to make the conjecture that the construction is not always possible. In some cases additional means are needed for the inquiry. For example, in the **(5;6)** model it appears at first glance that the fifth vertex is on the hexagon (Fig.14 *a*). But a more careful exploration (Fig. 14 *b* and 14 *c*) shows that this is not so.

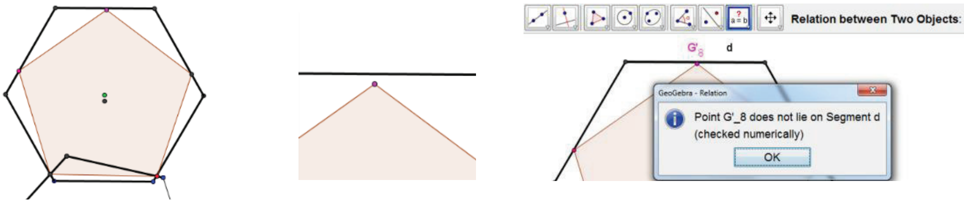

Fig. 14 The **(5;6)** dynamic model

At this point it is a good idea to stop and take a look around – *what is known in relation to our explorations?* We entered the magic phrase ***a regular m-gon inscribed in a regular n-gon*** and here it appeared (Dilworth, Mane, 2010)! Almost the same title and the same denotation showing how natural it is in its simplicity and conciseness when exploring various cases and describing the conjectures and results. Dilworth and Mane present there the necessary and sufficient conditions on ***m*** and ***n*** for inscribing a regular ***m***-gon in a regular ***n***-gon. It is interesting to note that *naively* (their own phrasing) they expected *this problem to be solved in the time of Euclid, but it seems to be not completely solved.*

Here is what Dilworth and Mane prove in (Dilworth and Mane, 2010) by means of complex numbers:

Theorem. *Suppose that m, n* ≥ 3. *A regular m-gon can be inscribed in a regular n-gon if and only if one of the following mutually exclusive conditions is satisfied:*
(a) $m = 3$;
(b) $m = 4$;
(c) $m \geq 5$ *and m divides n;*
(d) $m \geq 6$ *is even and n is an odd multiple of m/*2. *(Note that this includes the case n = m/*2.*)*
In (c) and (d) the polygons are necessarily concentric and in (d) they share a common axis of symmetry. In case (d) we insist that n be an odd multiple of m/2 because if n is an even multiple of m/2, then n is a multiple of m, which is already covered in case (c).

Thus it follows from the Theorem that the locus we are interested is a single point in the cases (c) and (d). The last examples of our explorations belong to (d).

Had we seen this article before attacking it with dynamic means we would feel very reluctant to offer it to students (even if they were very motivated to explore new mathematical territories). However, the explorations themselves harnessed mathematical skills accessible to students knowing about geometric transformations. Furthermore, the patterns and the relationships observed during these explorations gave rise to other interesting questions.

What really matters for us in relation to this problem is not even the solution itself but the whole process of creating a good platform for explorations, enhancing our intuition and understanding about some patterns among the constructions, designing a more systematic approach of explorations, realizing that not all combinations of inscribing a regular ***m***-gon in a regular ***n***-gon are possible, and finally – the belief in teachers' ability to promote the inquiry-based learning of mathematics. In a nut shell, to illustrate the „grook“ (*Grooks*, 2013) of the great Danish mathematician, architect and poet Piet Hein: *Problems worthy of attack, prove their worth by hitting back.*

ACKNOWLEDGEMENT

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