ON THE ACTION OF ELIMINATION IN THE ACTIVITY OF PROBLEM SOLVING

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ABSTRACT

The action of elimination is considered in the paper as a consisting part of the process of problem solving for a definite type of mathematical problems. An automation of elimination is possible in some cases which reduces it to an operation. The last is a result of lasting preparation and solving of suitable mathematical problems. The paper proposes a series of such problems, which describes different situations of elimination and aims at acquainting and learning of the action of elimination.

1. INTRODUCTION

Many practical mathematical problems are connected with objects that are obtained from other objects using reconstruction, decomposition, deformation, changing, transformation and s.o. under definite conditions. A choice of a corresponding action or a sequence of actions should be carried out for the purpose. Possible actions are addition, elimination or displacement of a certain element, replacement of elements or their positions, etc. In a series of papers the authors intend to study the listed actions and to examine their relation to problem solving. The first paper of the series is [1], while the present one is the second. As pointed out in [2], "the basic "consisting components" of human activities are performing actions… The action itself is a process which is subjected to a conscious aim." We are interested in solving of particular classes of problems, in which the corresponding activity includes a definite action. We call this definite action to be "elimination". Some examples are considered in the sequel.

2. SOME EXAMPLES OF ELIMINATION

In the proof of inequalities an often used approach is to eliminate a positive (or negative) expression by which the inequality keeps its validity or becomes stronger. Take the Bernoulli inequality:

If $x > -1$ *is real and* n *is a positive integer, then* $(1+x)^n \ge 1+nx$.

The first step of the proof is to verify the case $n=1$. Then, an inductive assumption is done, i.e. we assume that $(1+x)^n \ge 1+nx$ is true and the problem is reduced to the proof of the inequality $(1+x)^{n+1} \ge 1+(n+1)x$. Now, we have

$(1+x)^{n+1} = (1+x)^n(1+x) \ge (1+nx)(1+x) = 1+(n+1)x+nx^2$. Since $nx^2 \ge 0$, this term could be *eliminated* and we get $1+(n+1)x+nx^2 > 1+(n+1)x$. Thus, $(1+x)^{n+1} \ge 1+(n+1)x$.

In the above example the action of elimination is applied to the term nx^2 and as a result the inequality is proved. Another example is connected with the comparison of the fractions $\frac{45}{46}$ and $\frac{46}{47}$. A possible approach is to use their differences to the unity, which should be eliminated after, aiming at keeping of the "equilibrium", i.e.

$$\frac{45}{46} = \frac{45}{46} + \frac{1}{46} - \frac{1}{46} = 1 - \frac{1}{46} \text{ and } \frac{46}{47} = \frac{46}{47} + \frac{1}{47} - \frac{1}{47} = 1 - \frac{1}{47}.$$

The inequality $\frac{1}{46} > \frac{1}{47}$ implies that $\frac{45}{46} < \frac{46}{47}$.

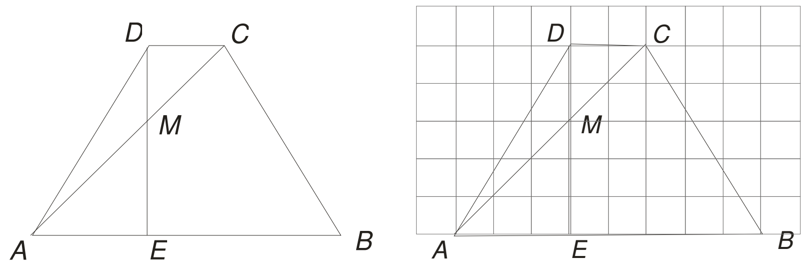
A more complicated example is considered in [1, p.7]. An important remark is that the actions "addition" and "elimination" are opposite. Usually, when one of them is applied the other should be applied after too.

Elimination is basic in the similarity method. Take a construction of a plane figure, which is similar to a given one. In such a case we eliminate one of the elements of the figure (the metric one) and use the remaining ones only. The eliminated element defines a similarity, which transforms the obtained figure to the one we are looking for. For example, if a triangle *ABC* should be constructed when two of its angles α, β and the altitude h_c from the vertex *C* are given, then at a first step the metric element (h_c) is eliminated and we take a similar triangle with the given α and β.

The same idea works in problems for calculation in the case when only one of the given elements is metric. Very often, the metric element is eliminated and a new "suitable" element is chosen instead. Then, the calculations are carried out for the new object. Afterwards, the similarity coefficient helps to find the value of the quantity under consideration. We take the following example:

Given is an isosceles trapezoid ABCD, for which the sum of the bases AB and CD is equal to 40 cm and the altitude DE $(E \in AB)$ *interests the diagonal AC in the point M. Find the length of the smaller base CD if* $AM : MC = 3:2$.

After eliminating the condition $AB + CD = 40\,cm$ and establishing the relation $AM : MC = AE : DC = 3:2$, we conclude that if $DC = 2\ cm$ for example (see the figure), then $AE = 3\ cm$ and $AB = 8\ cm$.



A similar idea lies in the so called "lying rule", which has been used in the Antiquity. In Rhind's Papyrus [3] one could find the following problem:

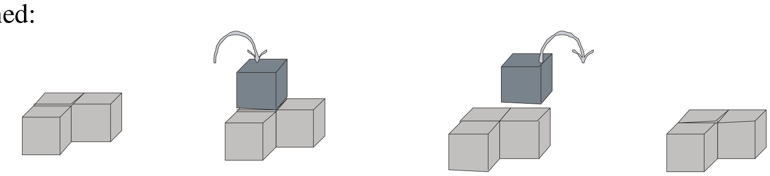
One hundred units of bread should be distributed among 5 men in such a way, that the second man should receive as much more than the first one as the third man should receive more than the second one, the fourth man – more than the third one and the fifth man – more than the fourth one. Also, the first two men should receive 7 times less than the remaining three. How much should receive each of the 5 men.

According to James R. Newman [3] the solution in Ancient Egypt undergoes the following steps: Since the distribution is a 5-term arithmetic progression, take its difference to be $5\frac{1}{2}$. If the first man gets 1, the second one gets $6\frac{1}{2}$, the third – 12, the fourth – $17\frac{1}{2}$ and the fifth – 23. The total sum is 60. The key in the solution is the difference of $5\frac{1}{2}$. To guess this number, reason in this way: take the difference to be 1 instead of $5\frac{1}{2}$; then the distribution is 1; 2; 3; 4; 5; the sum of the first two terms is 3 and $\frac{1}{7}$ part of the sum of the remaining three terms is equal to $\frac{1}{7}(3+5+7) = 1\frac{5}{7}$; unfortunately $3 \neq 1\frac{5}{7}$; notice that $3 - 1\frac{5}{7} = 1\frac{2}{7}$; now take the difference of the progression to be 2; the corresponding distribution is 1; 3; 5; 7; 9.; we have $1+3=4$ and $\frac{1}{7}(5+7+9) = 3$; again $4 \neq 3$ but in this case $4-3=1$ and the last difference is less than the corresponding one in the first case by $1\frac{2}{7} - 1 = \frac{2}{7}$; the observation shows that when the difference of the progression grows by 1, then we approach the necessary result by $\frac{2}{7}$; thus, the difference 1 of the first progression should be increased as many times as $1\frac{2}{7}$ is greater than $\frac{2}{7}$, i.e. $1\frac{2}{7}:\frac{2}{7}=4\frac{1}{2}$; for this reason it is enough to take an arithmetic progression with a difference $1+4\frac{1}{2}=5\frac{1}{2}$. In this way we get the distribution 1, $6\frac{1}{2}$, 12, $17\frac{1}{2}$, 23. The sum of this progression is 60 and in order to obtain the final answer of the problem it is enough to multiply the terms of the progression by the number $\frac{100}{60}=\frac{5}{3}$.

In the above problem the action of elimination concerns the condition for the hundred units of bread which are taken into consideration at the very final step. Many other examples of elimination are possible too. Actions of elimination are also carried out in finding the defining domains of expressions, equations and inequalities when values of variables are eliminated ensuring a sense for the corresponding objects.

3. A DIDACTIAL SYSTEM OF PROBLEMS

The examples from chapter 2 show that elimination is an action in the problem solving activity which consists in disregarding of an element or a group of elements of a given object under suitable conditions. Addition and elimination could be considered as opposite actions. A possible visualization of both the actions is by using of the following cube constructions, when new constructions are obtained:



In the sequel, a didactical set of problems is proposed to train the action of elimination. In some cases a corresponding visualization is proposed too.

Problem 1. Eliminate one of the numbers from the given number sequence to obtain an arithmetic progression:

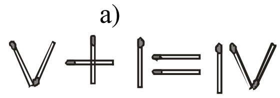
a) 2; 8; 11; 14; 17; b) - 1; 1; 3; 7; c) - 5; 5; 15; 25; 35.

In a) the sequence of the consecutive differences is 6; 3; 3; 3. Thus, eliminating the first term of the initial sequence we obtain an arithmetic progression. In c) the sequence of the consecutive differences is 10; 10; 10; 10 and the only possibilities are to eliminate the first or the last term of the initial sequence.

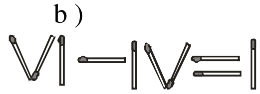
Answers: a) 8; 11; 14; 17; b) - 1; 3; 7; or - 1; 1; 3; c) 5; 15; 25; 35 or - 5; 5; 15; 25.

Problem 2. Eliminate only one of the safety matches to obtain a true equality:

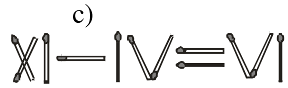
a)



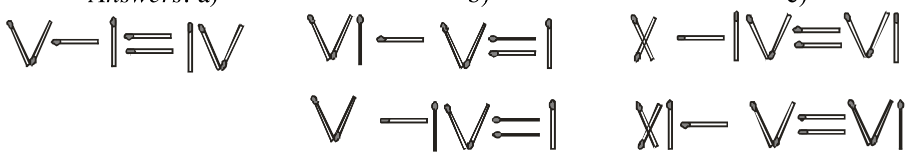
b)



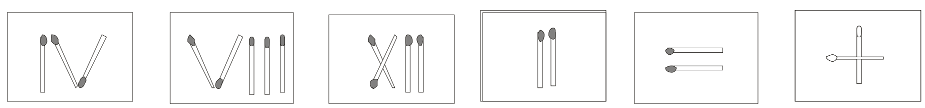
c)



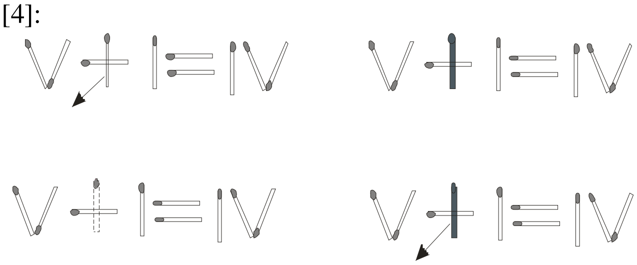
Answers: a) b) c)



The unit element in the above eliminations is the safety match. It is a model of a roman digit, a part of a roman digit, the sign “–“ or a part of the signs “+” and “=”. In training it is useful to comment what kind of mathematical objects could be obtained by eliminating of one or several safety matches from the following configurations:

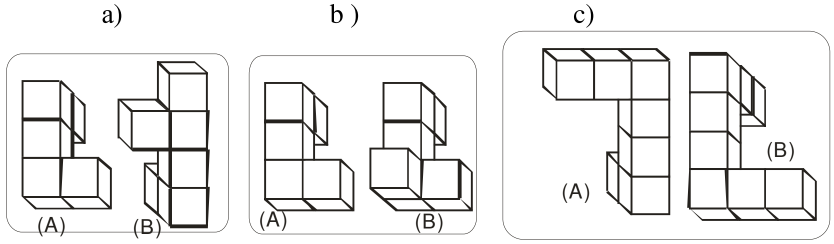


A suitable visualization is by an arrow, showing the element to be eliminated. Different coloring, dotting lines or combinations of different means are possible too [4]:

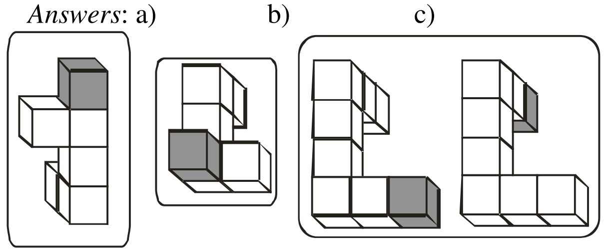


Problem 3. Eliminate a unit cube from the construction (B) in a way that the new construction could take the position of (A) in the space:

a) b) c)



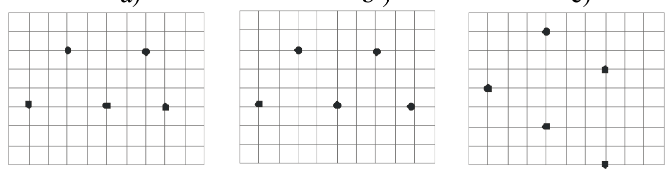
Answers: a) b) c)



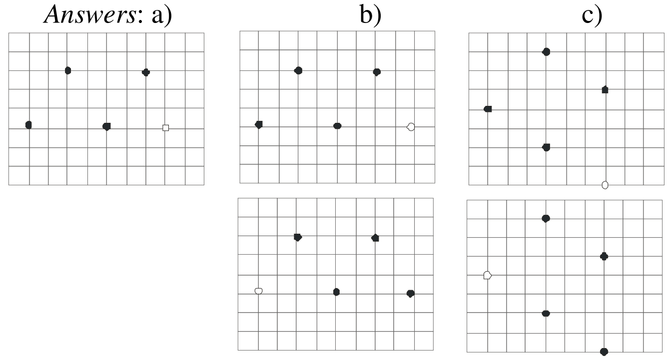
Instead of different coloring as it is done in the solution of the last problem one could use enumeration of the unit cubes in the construction or suitable arrows too.

Problem 4. Eliminate one of the black points from the figure in a way that the remaining four points should define a parallelogram:

a) b) c)

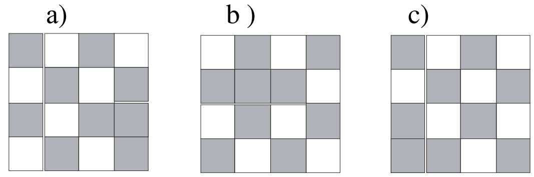


Answers: a) b) c)

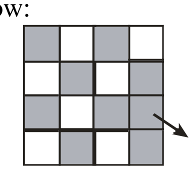


Problem 5. Eliminate one of the black squares to obtain a chess-board arrangement:

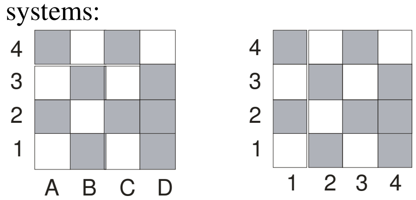
a) b) c)



The visualization is by an arrow:



or by using coordinate systems:



Answers: a) D2; b) B3; c) A1.

Problem 6. Eliminate one of the digits to obtain possibly the greatest 4-digit number:

a) 19348; b) 32671; c) 76343; d) 65551.

Answers: a) **1**9348; b) 3**2**671; c) 76**3**43; d) 6555**1**.

Problem 7. Eliminate one of the digits from the set {9; 5; 4; 3; 0} in such a way that using the remaining digits it should become impossible to obtain an integer which is divisible by 2 and 5 simultaneously.

Answer: 0.

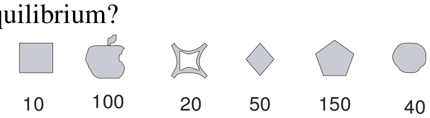
Problem 8. Eliminating one of the symbols you should obtain:

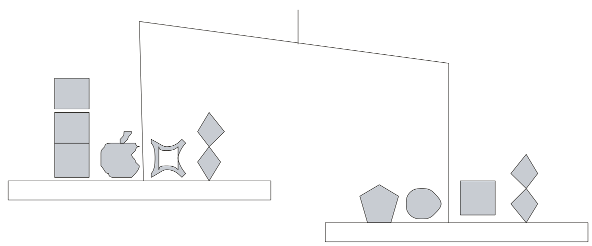
a) an ordinary regular fraction from $\frac{27}{4}$; b) a divisible fraction from $\frac{54}{12}$;

c) a true equality from 22 - 21 = 21; d) a true equality from 4,2 + 8 = 50.

Answers: a) $\frac{2}{4}$; b) $\frac{4}{12}$; $\frac{54}{2}$; c) 22 - 21 = 1; 22 - 1 = 21 ; d) 42 + 8 = 50.

Problem 9. Which of the figures (shown by their weights) should be eliminated to obtain an equilibrium?





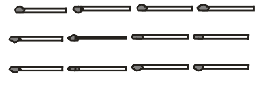
Answer:  .

Problem 10. Eliminate two of the terms from the given number sequence to obtain an arithmetic progression:

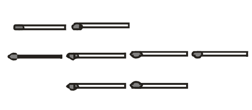
a) 1; 4; 7; 10; 14; 17; b) 0; 1; 3; 5; 7; 9; c) 0; 2; 3; 4; 6.

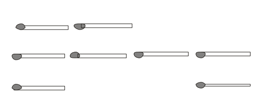
Answers: a) 1; 4; 7; 10; b) 3; 5; 7; 9 or 1; 3; 5; 7; c) 2; 4; 6 or 0; 3; 6 or 0; 2; 4 or 2; 3; 4.

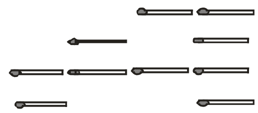
Problem 11. Sixteen safety matches are arranged as shown in the figure. Eliminate six of them to obtain an even number of matches in each row and each column:



Three possibilities are shown below:

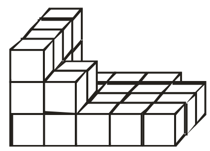




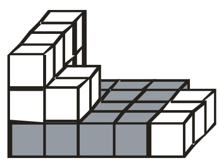


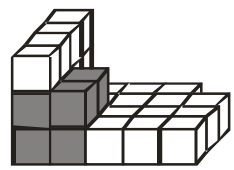
Problem 12. Given is a cube construction consisted of 29 unit cubes. What is the minimal number of unit cubes that should be eliminated to obtain:

a) a rectangular parallelepiped; b) a cube?

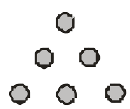


Answers: a) 13; b) 19.





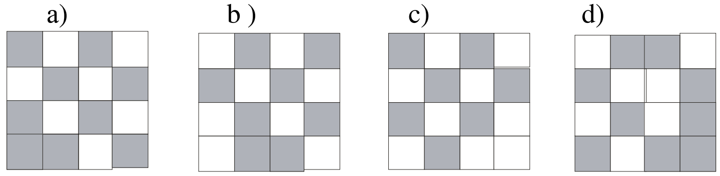
Problem 13. What is the minimal number of points to be eliminated from the figure that any three of the remaining points should not form an equilateral triangle?



Answer: 2

Problem 14. Is it possible to obtain a chess-board arrangement by eliminating a single black square?

a) b) c) d)



Answers: a) yes; b) no; c) no; d) no.

To solve b) and c) one could use the actual number of the black squares in the figure. For example, in c) the number of the black squares is 7 which is less than the half of all squares and hence the answer is negative.

In order to give a correct meaning of the actions of addition and elimination we propose the following inverse problems in which the initial object should be found:

Problem 15. A term is eliminated from an arithmetic progression and the following sequence is obtained. Find the arithmetic progression.

a) 1; 4; 10; 13; 16; b) 2; 4; 6; 8; 10.

The action of elimination is included in the problem and an action is necessary for the solution.

Answers: a) 1; 4; **7**; 10; 13; 16; b) **0**; 2; 4; 6; 8; 10; or 2; 4; 6; 8; 10; **12**.

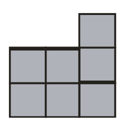
Problem 16. A term is added to an arithmetic progression. Find the arithmetic progression if the new sequence is the following:

a) 3; 6; 7; 9; 12; b) 2; 4; 6; 8; 10; 12.

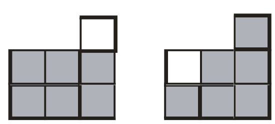
Answers: a) 3; 6; 9; 12; b) 2; 4; 6; 8; 10 or 4; 6; 8; 10; 12.

Some of the considered problems could be used in introducing of new concepts and forming of a new knowledge. We propose several examples for the concept of symmetry.

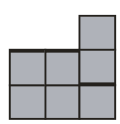
Problem 17. Eliminate one of the unit squares to obtain a figure with an axis of symmetry:



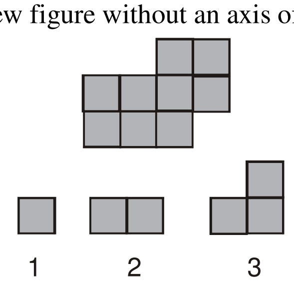
Answers:



Problem 18. Eliminate two of the unit squares to obtain a figure with an axis of symmetry:

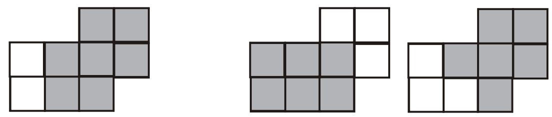


Problem 19. Which of the consisting parts 1, 2 and 3 should be eliminated from the given figure to obtain a new figure without an axis of symmetry?

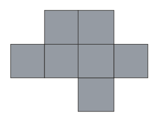


Answer: 1.

By eliminating the corresponding part from the given figure a new figure with an axis of symmetry is obtained.

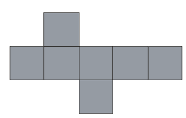


Problem 20. Is it possible to obtain a figure with an axis of symmetry by eliminating of two unit squares?



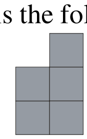
Answer: Yes.

Problem 21. Find the minimal number of unit squares to be eliminated in order to obtain a figure with an axis of symmetry.

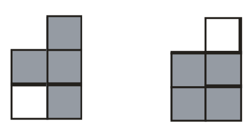


Answer: 1.

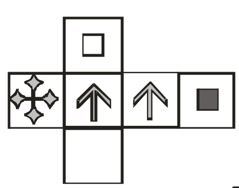
Problem 22. A unit square is added to a figure with an axis of symmetry. Find the initial figure if the new one is the following:

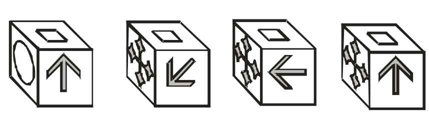


Answer:



Problem 23. A cube development is given. To which of the cubes does it belong?



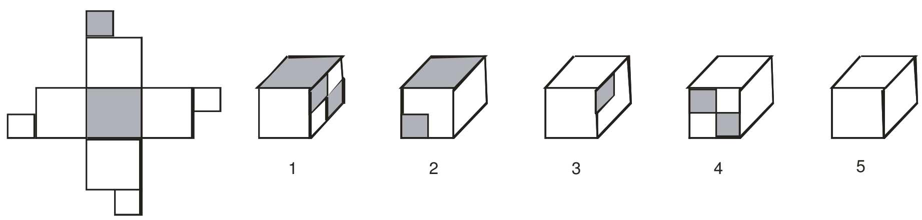


1 2 3 4

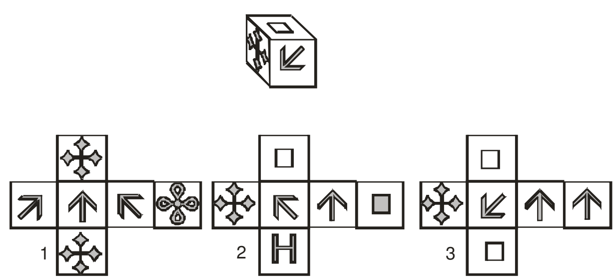
Answer: 4.

A possible reasoning in this problem is the following: There is no side of the cube development which contains a circle and hence the first cube is not the correct answer. Also, there is no side of the cube development which contains an arrow directed to a vertex of this side and consequently the second cube is not the correct answer. Finally, there is no side of the cube development which contains an arrow directed to a “cross”. Hence, the third cube is not the correct answer. The only candidate for the correct answer is the fourth cube and now verification is needed only. Similar reasoning is possible for the next problems too.

Problem 24. Which of the five cubes could be obtained from the piece of paper by suitable folding and gluing?



Problem 25. One of the three developments belongs to the given cube. Find it.



4. FINAL REMARKS

It is not possible to consider other interesting examples of the action of elimination because of the limited volume of the present paper. The set of eliminations is considerably rich and an appropriate classification is needed. What is needed in addition is to transform elimination into regularity in the process of teaching and learning. As stated in the Introduction, other actions are possible in the problem solving activity. Some of them are the subject of another investigation.

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