ENHANCING THE ART OF PROBLEM POSING IN A DYNAMIC 3D COMPUTER ENVIRONMENT

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In mathematics the art of asking questions
is more valuable than solving problems
Georg Cantor

Abstract
The paper deals with the experience the authors have gained in stimulating both teachers and students in the posing of problems. Dynamic 3D applications of the Elica computer environment are considered as supporting the visualization, representation, and exploration of 3D geometric constructions - found to be important phases of the problem formulating.

Keywords
Problem solving, problem posing, spatial intelligence, 3D geometry, Elica computer environment

1. Problem posing and spatial intelligence

The mathematics education is traditionally performed in problem solving, the emphasis being on solving problems by students. Still, elements of problem posing are introduced even at primary level since its significance has been established by many great mathematicians and math educators ([1]-[4]). But the students are expected to create analogues of specific models of problems mainly with arithmetic (later algebraic) content. As far as the geometric content is concerned such experience has been very limited (especially in the case of 3D). When stereometry (3D geometry) is introduced in the 5^{th} and 6^{th} grade the focus is on learning the formulae for the surface and volume of solids rather than on stimulating the spatial imagination, on observing and describing properties of 3D constructions, or on formulating problems in such a context [5].

A purposeful enhancing of the so called *spatial intelligence*, defined as *the ability to think in pictures*, *to perceive the visual world accurately, and recreate it in the mind or on paper* [6], is lacking in the traditional mathematics education in the Bulgarian schools (and not only there). One of the main reasons is that the enhancement of skills of mental operations on objects (such as translation, rotation, extension, contraction, removing, adding or replacing) requires the development of a suitable environment and educational activities.

A possible approach to improving the situation has been offered in the frames of the DALEST project (**D**eveloping an **A**ctive **L**earning **E**nvironment for the Learning of **St**ereometry) [7]. Students from 5^{th} and 6^{th} grade from the participating countries (Bulgaria, Cyprus, Greece, Portugal, UK) were provided with appropriately developed computer applications in which they could explore the properties of 3D objects according to concrete educational scenarios, to solve problems and formulate their own problems [8], [9].

In the current paper we shall focus on an educational experiment related to the synthesis among the problem solving and problem posing carried out in one of the DALEST applications (*Cubix Editor*) [10], [11]. The overall goal of the experiment was in harmony with *the two good criteria for what is worth knowing* as formulated by Bruner in [12]:

- *whether the knowledge gives a sense of delight, and*
- *whether it bestows the gift of intellectual travel beyond the information given, in the sense of containing within it the basis for generalization.*

Our experience shows that it is extremely motivating for the students to see their teacher not only as a solver of problems (in textbooks and in math journals) but also as a creator of his/her own problems. And if in addition the teacher is able to demonstrate the process of problem generation this would motivate and help students to try creating problems on their own…

2. Teachers as authors of problems

2.1. The main difficulties

What follows in this section is a description of the experience of the first author in the role of a teacher who has been involved for a long period (more than 20 years) in creating problems stimulating the mathematical and spatial intelligence of students [13]. The main difficulties faced when formulating problems and presenting their solutions have been related to

- the invisibility of solids
- the use of intersections and different views complicating additionally the construction representations
- the impossibility of describing some configurations with the available knowledge of students, etc

When provided with relevant environment (in our case – DALEST applications) the teacher revisited some of her ideas that hadn't been accomplished before. In her recent investigations she developed a specific didactic system of problems (*Virtual constructions and configurations of cubes*) on the basis of *Cubix Editor*. In the process she involved a group of 12-13 year old students in solving problems as a basis for generating their own problems according to the proposed models. A further step of the experiment was to include older students (highly motivated in studying mathematics) so as to compare their approach to creating problems of the kind.

Let us note that we didn't have a preference as for the organizational form of work – both the individual and the team work showed that the participants expressed their wish to be original, to share, to get a feedback.

2.2. The virtual construction tool

The main tool for creating the cubical constructions we used was *Cubix Editor* which allows the construction of 3D unit-sized cube structures by clicking on a board of a chosen size. A very useful characteristic of the application is the rotation of the platform, which enables dynamic visualizations of the front, side and top view (Figure 1). Another important feature for our experiment was the possibility to use different colors for a unit cube

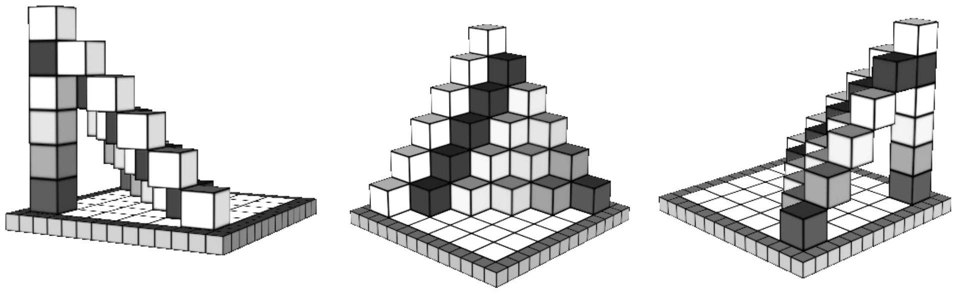


Figure 1: Three views of a structure in the *Cubix* Editor

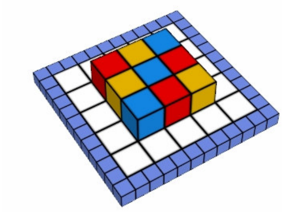
2.3. The advantages of the virtual constructor for creating problems

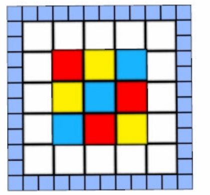
Creating problems with operations on cubic constructions has passed through three main stages – without computer, by means of a graphics editor, and using a virtual constructor. Some options of software packages such as *MSWord* and *CorelDraw* make it possible to build a construction based on ready-made or composed modules. This is the reason for the appearance (in the recent 20 years) of a rich collection of problems aimed at the development of a spatial intelligence. At first they were developed mainly for math competitions, or as a part of psychological investigations. Later on they became part of the curriculum since the difficulties in formulating the problem and explaining the solution had been overcome to a great extent thanks to the graphics software tools. Still, the necessity of a greater precision of the formulation gave birth to new computer applications. The development of virtual constructors was a crucial point in the formulation of problems of the kind. The interplay between ideas and their virtual implementation facilitates the verification of the solution and stimulates the posing of new questions, the modification of the original formulation thus leading to a system of problems. Furthermore, all these activities are not time-consumable. Here follow some examples.

Problem 1. *Construct a* $3\times3\times3$ *cube by means of 9 red, 9 blue and 9 yellow unit cubes so that each its column of size* $1\times1\times3$ *contains a cube of each color.*

Hint 1. *Earlier you solved the problem of filling the unit squares in a 3x3 square with numbers from 1 to 3 so that each number appears just once in a raw or a column. Use the analogy*!

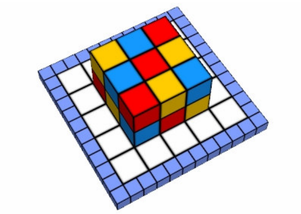
1	3	2
2	1	3
3	2	1





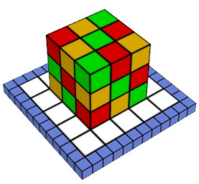
Hint 2. *How many cuboids of size* $1\times1\times3$ *are there in a cube of size* $3\times3\times3$ *?*

Hint 3. *Add a layer to the construction below so as to get a cube* $3\times3\times3$ *satisfying the requirements.*

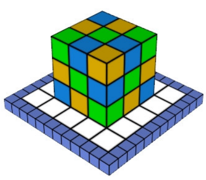


Problem 1.2. *Each row in the three directions of the cube of size* $3\times3\times3$ *contains a cube of each of 3 colors. What is the color of the central unit cube?*

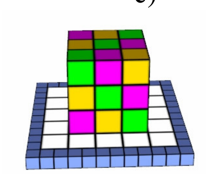
a)



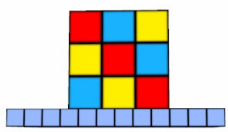
b)



c)



d)



Let us enlist the most important features of this software application which make it suitable for solving and posing problems of the above kind. It enables **the students** to:

- remove unit cubes from a model of the cube under consideration and check immediately their solution;
- work with a model of the part of the cube which is visible under the perspective presented on the sheet of paper;
- fix various hypotheses whose validity is to be checked;
- fix intermediate stages of the reasoning behind checking the validity of a hypothesis;
- pose new problems based on the given one, e.g. *what is the color of the middle unit cube of the bottom layer?*

As for the teacher, s/he could provide the students with an individual approach, a relevant assistance and a visualization of the solution. As an illustration, a model of the part of the cube in c) is given in Figure 2 together with a visualization of the process of checking the case of a purple (here a dark-grey) cube leading to a contradiction.

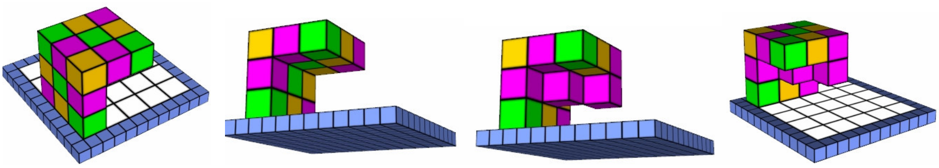
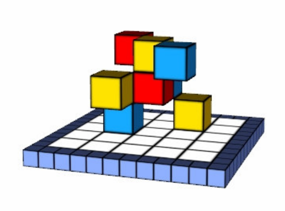


Figure 2: Visualizing the process of checking a specific hypothesis

Many practical and mathematics problems are related to getting an object from another under specific conditions. For this purpose an action (or a sequence of actions) should be performed on the object (e.g. adding, removing, shifting, etc.). Turning such actions in operations, i.e. *automating* them, is a result of a serious preparation and solving of appropriately formulated problems [14], e.g. as the following one:

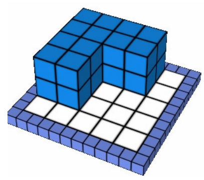
Problem 2. *Add a unit cube in such a way that each layer (in each direction) contains a cube of each of the 3 colors.*



While the solution of Problem 1 could be carried out by means of material cubes, this is not the case with Problem 2. The easy performance of the operations *adding*, *removing*, *shifting* and *exchange* by means of Cubix Editor enabled the students to solve the problem and also to describe their solution. Furthermore, they could formulate their own problems of the kind (as we shall discuss in the next section).

The next problem was directed towards making a formulation rigorous [8].

Problem 3. *Divide the cubic construction in:*



a) 2 equal parts
б) 4 equal parts.

This formulation is ambiguous – it is not clear what the number of the unit cubes is as seen from another perspective (Figure 3):

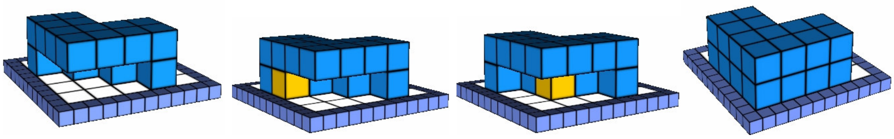


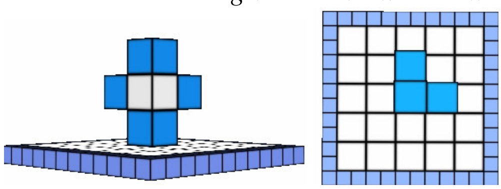
Figure 3. Illustration of the ambiguity of the original formulation of Problem 3

Furthermore, it is not clear what the meaning of “division” and “part” in this case is. The process of editing of the formulation so as to make it precise could be based on formulating variations of the problem with different solutions. Such an activity enables the teacher to show the importance of the details in the formulation and the necessity of investigations in case of ambiguity.

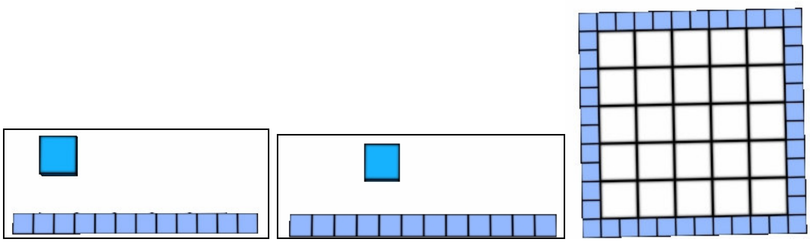
The verbal description of a 3D construction or *vice versa*, making a construction by a verbal description, lead students to becoming aware of the ambiguity or over-determination of a specific description, of the difference between a natural and an algorithmic language, of the necessity of editing a formulation of a problem [9].

There are problems for finding a construction when its views from several different points are given. Usually the projections on the three coordinate planes are used in such problems but within *Cubix Editor* one could create *non-traditional* views, e.g.:

Problem 4. *Recreate the construction given the views below*:



Problem 5 *Two view of a cube built with Cubix Editor are given. Draw a view from above on the board to the right.*



3. … And the students (as authors of problems)

A 10th grade student from the *National High School* of *Mathematics and Science* formulated the following problem for a $n \times n \times n$ cube**:**

The problem of Galin: *Let us color a $n \times n \times n$ cube in n colors so that no unit cubes in a line parallel to the coordinate axes are of the same color.*

He provided the following solution in consecutive steps:

- We choose n colors $x_1, x_2, ..., x_n$.(the favorite of the mathematicians).
- We divide the cube in n walls, each of size $n \times n \times 1$, which will be denoted by $y_1, y_2, ..., y_n$.
- We color y_1 as follows:

x_1	x_2	x_3	...	x_{n-2}	x_{n-1}	x_n
x_2	x_3	x_4	...	x_{n-1}	x_n	x_1
x_3	x_4	x_5	...	x_n	x_1	x_2
$\vdots$						
x_n	x_1	x_2	...	x_{n-3}	x_{n-2}	x_{n-1}

- Similarly, we color y_2 as follows:

x_2	x_3	x_4	...	x_{n-1}	x_n	x_1
x_3	x_4	x_5	...	x_n	x_1	x_2
x_4	x_5	x_6	...	x_1	x_2	x_3
$\vdots$						
x_1	x_2	x_3	...	x_{n-2}	x_{n-1}	x_n

  i.e. **we cut the first column of** y_1 and move it to the right most position.
- Then, we color y_3, by cutting out the first column of y_2 and move it to the right most position.

x_3	x_4	x_5	...	x_n	x_1	x_2
x_4	x_5	x_6	...	x_1	x_2	x_3
x_5	x_6	x_7	...	x_2	x_3	x_4
$\vdots$						
x_2	x_3	x_4	...	x_{n-1}	x_n	x_1

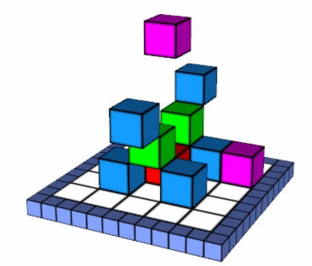
- By following the same mechanism we color the walls $y_4, y_5, ..., y_n$.
- We glue the walls $y_1, y_2, ..., y_n$ in this sequence.

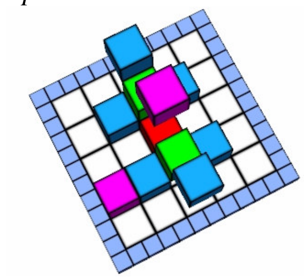
- We enjoy the cube obtained :).

The problem formulation was a challenging but attractive task not only for older students in specialised math schools but also for much younger kids (5-6 graders).

Two problems of Koya

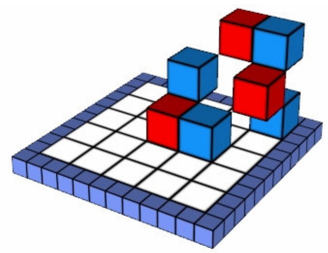
1. *Remove a cube so as to get a symmetric composition.*

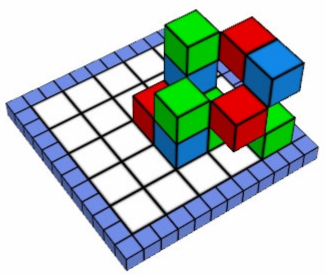




After realising that the first view of the composition was not sufficient for solving her problem Koya added a second one – from above.

2. *Remove a cube so as to get a composition for which each layer (along each direction) contains a cube of each color present in the composition.*





The main steps Koya took in the process of creating a problem were the following:

- Verbalizing the idea to use a two-color composition which will be “the answer” and then adding a cube.
- Reducing the size of the composition to 3x3x3 so as to make it more accessible for the solver
- Using additional (auxiliary) cubes of a different color to facilitate the experiments with their removal. (Calling them “auxiliary composition”.)
- Looking for an appropriate place to put the additional cube by means of trials. Determining that it shouldn’t be a neighbour of a cube of the same color.
- Generalizing the problem - after formulating the problem about two colors, using it as a basis for the next problem – dealing with three colors.

- Coming back to the text of the formulation – looking for a more concise form.
- Tuning the formulation to a language closer to the every-day life, e.g. "remove" – "delete".

Similar steps were also observed in the work of other kids when posing problems in this environment. The common effect for all the young authors of problems was that they were eager to pose them to peers, to observe their reaction, their interest, the solving process. They were interested if the text was sufficiently clear. They shared the way of composing the problem. In the process of sharing some errors were discussed, the original composition was modified, and new problems appeared as a result of collective efforts.

The question arose if similar problems could be formulated without using the *Cubix Editor* The first idea was the Rubik's cube but soon it became clear that it has serious limitations. The invisibility and the flying cubes were the main problems when lacking a virtual constructor.

In the process of discussing the concepts appearing in the problem posing the students felt the need of distinguishing among two notions – a *construction* and *composition* of cubes. It was decided that a *construction* should denote a set of cubes which could be constructed physically (possibly with glue or a magnet), more formally if a construction is made of n unit cubes, a new n+1 cube construction is obtained by adding a cube which has a common side with at least one of the cubes of the n-cube construction. For its part, a *composition* of cubes is a set of cubes which are not necessarily a construction of cubes, i.e. there might be cubes in isolation (not touching others by site).

4. Conclusion

Even the first results of our experiment with students in solving and composing problems with a virtual constructor make us optimist in terms of motivation. It is also worth mentioning that such activities of the students enable the harmonization of the work of the two brain hemispheres by means of specific combination of visualizations and logics. Furthermore, the on-going matching between what is expected and what is built by means of real experiments plays a crucial role in creating and enhancing

3. spatial intelligence
4. skills for problem posing

- inquiry skills
- skills for predicting the outcome of a concrete action
- skills for formulating (prognosticating) the action needed for achieving a concrete result
- skills for describing and representing the thinking process by various means
- awareness of the potential of analogy as a tool for solving and posing problems

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