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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

THE INDECOMPOSABILITY OF A CERTAIN KIND OF SEMI-NORMS

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To the memory
of my teacher Y. Tagamlitzki

1. Introduction. About thirty years ago, the author of this paper (who was a student at that time) participated in the famous Student Scientific Circle at the Chair of Differential and Integral Calculus in Sofia University. The head of the chair and of the circle, Professor Y. Tagamlitzki, had posed a number of problems of the following sort: a certain cone is given and its indecomposable elements have to be found. In one of these problems, the cone under consideration consisted of all semi-norms on a given linear space. In 1955, this problem was solved in the case when the space is two-dimensional. Namely, the late Dimiter M. Dobrev, who also was a member of the circle at the time, proved that in this case each indecomposable semi-norm is the absolute value of some fixed linear functional (the result is reflected implicitly in [1]). Dobrev's result gave rise to the hypothesis that the same is true also in the general case. But in 1957 Dobrev showed that his hypothesis fails even in the three-dimensional case. His proof was by *reductio ad absurdum* and it yielded no explicit counter-example. Such examples were found a little later by the present author (in the corresponding proof was used the fact that certain compact convex sets of linear functionals have sufficiently many apohedral elements in the sense of [2]; instead of apohedral elements, exposed elements in the sense of [3] can be used in the same proof). Dobrev's and our disproofs of the hypothesis were reported at a session of the circle in November 1957. The author's result was presented also at a student scientific conference in Sofia University, held in April 1958. Two years later, when the author was already an assistant professor at Professor Tagamlitzki's chair, a generalization of this result was presented also at an All-Bulgarian Conference of the Scientific Circles (in the proof of this generalization extreme elements were used, instead of apohedral ones). However, no publication of the mentioned results of the author appeared because it was clear to us that the specified classes of counter-examples do not exhaust the class of all possible counter-examples and, on the other hand, it would be not quite in the style of Tagamlitzki's scientific group such not final results to be published.

What is the state of the considered problem now, in 1986? As far as we know, no satisfactory characterization of the indecomposable semi-norms is found yet even in the three-dimensional case. The author has no information about somebody trying to solve the problem in the three- or more dimensional case. Hence, the not final character of author's results seems not to be such a decisive factor now, if the question about publication is concerned. Having this in mind, the author decided after some hesitation to publish his results from 1960. Maybe this publication could stimulate the interest in the history of

Tagamlitzki's Student Scientific Circle and, as an effect, some other unpublished results from that time may become accessible.

2. Formulation of the main result. We shall consider linear spaces over the field of the real numbers. As usual, a semi-norm on such a space Z is a non-negative real-valued function P defined on Z and satisfying the conditions that $P(\alpha z)=|\alpha|P(z)$ and $P(z_1+z_2)\leq P(z_1)+P(z_2)$ for all real numbers α and all z, z_1, z_2 in Z. A semi-norm P on the linear space Z is called indecomposable, if the following holds: P is not identically equal to 0 and, whenever P_1 and P_2 are semi-norms on Z such that

$$P(z)=P_1(z)+P_2(z) \tag{1}$$

for all z in Z, then there are (non-negative) real numbers α_1 and α_2 such that $P_1(z)=\alpha_1 P(z)$, $P_2(z)=\alpha_2 P(z)$ for all z in Z.

It is not difficult to prove that for each linear functional f on Z, if f is not identically equal to 0 then the function

$$P(z)=|f(z)| \tag{2}$$

is an indecomposable semi-norm on Z. Dobrev's result from 1955 was that in the case when Z is two-dimensional, the semi-norms of this form are the only indecomposable semi-norms on Z, and his result from 1957 was that it is not so when Z has more than two dimensions.

The main result in this paper is expressed by the following

Theorem. *Let Q and R be semi-norms on the linear spaces X and Y respectively, and let the following conditions be satisfied:* (i) *neither Q nor R is identically equal to 0;* (ii) *at least one of the semi-norms Q and R is not the absolute value of some fixed linear functional on the corresponding linear space. Then the function*

$$P(x, y)=\max\{Q(x), R(y)\} \tag{3}$$

is an indecomposable semi-norm on the linear space $X\times Y$.

The proof of this theorem will be given in Section 4, after some preparatory work in Section 3.

Remark 1. The result stated in the above theorem is the author's result of 1960. His result of 1957 differs from it in the presence of the additional assumption that X and Y are finite-dimensional.

Remark 2. The formulated theorem will be no longer true, if we omit some of the assumptions (i) and (ii). For example, if Q is identically equal to 0, then P can be indecomposable only in the case when R is indecomposable. If condition (ii) is not satisfied, then P can be indecomposable only in the case when condition (i) is also violated. Indeed, if $Q(x)=|g(x)|$ for all x in X and $R(y)=|h(y)|$ for all y in Y, g and h being linear functionals on X and on Y respectively, then the identity $P(x,y)=\frac{1}{2}|g(x)+h(y)|+\frac{1}{2}|g(x)-h(y)|$ holds; assuming that P is indecomposable and using this identity, we easily conclude that some of the semi-norms g and h is identically equal to 0.

3. On the linear functionals majorized by a semi-norm. If Z is a linear space, then Z^* will be the linear space consisting of all linear functionals on Z and supplied with the topology of pointwise convergence (i. e. with the topology which is induced by the Tychonoff product topology in $\mathbf{R}^Z$, where $\mathbf{R}$ is the real line). If P is a semi-rorm on Z, then we shall denote by P' the set of all elements f of Z^* such that $f(z)\leq P(z)$ for all z in Z. It is well known that P' is a compact convex subset of Z^*. Let $P^{:}$ be the set of the extreme points of P'. By the Krein-Milman Theorem, $P^{:}$ is non-empty. By the Hahn-Banach and Krein-Milman Theorems, for all z in Z the equality

$$P(z) = \sup\{f(z) \mid f \in P^{:}\} \tag{4}$$

holds (as a matter of fact, the "sup" may be replaced here by "max", but this will be not essential for what follows).

If the semi-norm P is not identically equal to 0, then there are non-zero elements of P' and this, together with the symmetry of P', implies that the zero of Z^* does not belong to $P^{:}$. Due to the symmetry of $P^{:}$, there will be at least two different elements of $P^{:}$ in this case, which will be opposite each other. If every two elements of $P^{:}$ turn out to be linearly dependent, then the equality (4) and the symmetry of $P^{:}$ imply that P has the form (2), where $f \in Z^*$. Consequently, if P has not this form, then there are some two elements of $P^{:}$ which are linearly independent. On the other hand, if P has the above form, then $P^{:} = \{f, -f\}$.

For each f in P' let $P^*(f) = \sup\{f(z) \mid P(z) \leq 1\}$. Then $0 \leq P^*(f) \leq 1$ and $f(z) \leq P^*(f)P(z)$ for all f in P' and all z in Z. It is clear that $P^*(\lambda f) = \lambda P^*(f)$ and $P^*(\lambda g + (1-\lambda)h) \leq \lambda P^*(g) + (1-\lambda)P^*(h)$ for each real number λ with $0 \leq \lambda \leq 1$ and all f, g, h in P'. For each non-zero element f of P' the inequality $P^*(f) > 0$ holds and the element $f/P^*(f)$ also belongs to P'. Using this fact, it is easy to see that in the case when P is not identically equal to 0, for all elements f of $P^{:}$ the equality $P^*(f) = 1$ holds.

From now on, we shall assume that a linear space Z is given, together with three semi-norms P, P_1 and P_2 on it such that the equality (1) holds for all z in Z.

The following proposition and its proof are known to the author from a lecture of Professor Y. Tagamlitzki delivered at the mentioned Student Scientific Circle in 1957:

Lemma 1. *The equality* $P' = \{f_1 + f_2 \mid f_1 \in P_1', f_2 \in P_2'\}$ *holds.*

Proof. Let $\mathcal{S}$ be the set on the right-hand side of the above equality. This set is obviously a convex subset of P'. Since P_1' and P_2' are compact and the addition in Z^* is a continuous operation, the set $\mathcal{S}$ is also compact and, hence, it is closed. Consider now an arbitrary element f_0 of P'. We have to prove that $f_0 \in \mathcal{S}$. Suppose $f_0 \notin \mathcal{S}$. Then, by the separation theorem and the duality between Z^* and Z, there are an element z_0 of Z and a real number γ such that $f_0(z_0) > \gamma$ and $f(z_0) \leq \gamma$ for all f in $\mathcal{S}$. By the Hahn-Banach Theorem, there are $f_1 \in P_1'$ and $f_2 \in P_2'$ such that $f_1(z_0) = P_1(z_0)$ and $f_2(z_0) = P_2(z_0)$. Then $f_1 + f_2 \in \mathcal{S}$, and we get a contradiction in the following way:

$$f_0(z_0) \leq P(z_0) = P_1(z_0) + P_2(z_0) = f_1(z_0) + f_2(z_0) = (f_1 + f_2)(z_0) \leq \gamma.$$

For each two (not necessarily different) elements g and h of Z^*, let $[g, h]$ denote the closed segment with endpoints g and h, i. e. $[g, h] = \{\alpha g + (1-\alpha)h \mid 0 \leq \alpha \leq 1\}$. The following lemma gives some additional information connected with Lemma 1:

Lemma 2. *Let g and h be such elements of Z^* that $[g, h]$ is an extreme subset of P'. Let $g = g_1 + g_2$, $h = h_1 + h_2$, where $g_1, h_1 \in P_1'$, $g_2, h_2 \in P_2'$. Then there is a real number α with $0 \leq \alpha \leq 1$ such that $g_1 - h_1 = \alpha(g - h)$ (and, consequently, $g_2 - h_2 = (1-\alpha)(g-h)$).*

Proof. Obviously, $\frac{1}{2}g + \frac{1}{2}h = \frac{1}{2}(g_1 + h_2) + \frac{1}{2}(g_2 + h_1)$, and the elements $g_1 + h_2$, $g_2 + h_1$ belong to P'. Since $\frac{1}{2}g + \frac{1}{2}h \in [g, h]$, this implies that

g_1+h_2 also belongs to $[g, h]$. Hence, there is a real number α with $0 \leqq \alpha \leqq 1$ such that $g_1+h_2=\alpha g+(1-\alpha)h$. From here we get

$$g_1-h_1=(g_1+h_2)-(h_1+h_2)=\alpha g+(1-\alpha)h-h=\alpha(g-h).$$

Corollary. *Each element of $P^{:}$ has a unique representation in the form f_1+f_2 with $f_1 \in P_1'$, $f_2 \in P_2'$.*

Proof. If $f \in P^{:}$, then $[f, f]$ is an extreme subset of P', and we can apply Lemma 2 for $g=h=f$.

Using the above corollary, we define a mapping C of $P^{:}$ into P_1' by the condition that $f-C(f) \in P_2'$ for all f in $P^{:}$. Then an essential part of the content of Lemma 2 can be formulated as

Lemma 3. *If g and h are such elements of $P^{:}$ that $[g, h]$ is an extreme subset of P', then $C(g)-C(h)=\alpha(g-h)$ for some real number α with $0 \leqq \alpha \leqq 1$.*

We need some more properties of the mapping C. We note two of them.

Lemma 4. *For each f in $P^{:}$ the element $-f$ also belongs to $P^{:}$ and the equality $C(-f)=-C(f)$ holds.*

Proof. We use the symmetry of each of the sets P', P_1' and P_2'.

Lemma 5. *For all z in Z, the equality $P_1(z)=\sup\{C(f)(z) \mid f \in P^{:}\}$ holds.*

Proof. Let $z \in Z$. Set $C_1(f)=C(f)$, $C_2(f)=f-C(f)$ for all f in $P^{:}$. It is clear that for $i=1, 2$ the number $P_i(z)$ is an upper bound of the set $\{C_i(f)(z) \mid f \in P^{:}\}$ (since $C_i(f) \in P_i'$ for all f in $P^{:}$). On the other hand, for each f in $P^{:}$ the equality

$$P(z)-f(z)=(P_1(z)-C_1(f)(z))+(P_2(z)-C_2(f)(z))$$

holds, which implies the inequality $P_1(z)-C_1(f)(z) \leqq P(z)-f(z)$. This inequality together with the equality (4) shows that there are elements of the set $\{C_1(f)(z) \mid f \in P^{:}\}$ which are arbitrarily close to $P_1(z)$.

4. The proof of the main result. From now on, in addition to the assumptions from Section 3, we shall suppose the assumptions of the theorem from Section 2 to be fulfilled, Z to be the space $X \times Y$ and the function P to be defined by means of (3). It is clear that P is a semi-norm on Z and P is not identically equal to 0.

For each u from X^* let $u^{(1)}$ be the element of Z^* defined by the equality $u^{(1)}(x, y)=u(x)$. Similarly, for each v from Y^* let $v^{(2)}$ be the element of Z^* defined by $v^{(2)}(x, y)=v(y)$. The mappings $u \longrightarrow u^{(1)}$ and $v \longrightarrow v^{(2)}$ are injective linear mappings of X^* and Y^* into Z^*. Obviously, each element f of Z^* has a unique representation in the form $f=u^{(1)}+v^{(2)}$, where $u \in X^*$ and $v \in Y^*$, namely that one where u and v are defined by

$$u(x)=f(x, 0), \quad v(y)=f(0, y). \tag{5}$$

Lemma 6. *Let $f \in P'$. Then f can be represented in the form $f=u^{(1)}+v^{(2)}$, where $u \in Q'$, $v \in R'$ and $Q^*(u)+R^*(v)=P^*(f)$.*

Proof. Let u and v be defined by means of (5). Then obviously $u \in Q'$, $v \in R'$. Consider an arbitrary element $z=(x, y)$ of Z with $P(z) \leqq 1$. Then $Q(x) \leqq 1$, $R(y) \leqq 1$ and, consequently,

$$f(z)=u(x)+v(y) \leqq Q^*(u)+R^*(v).$$

Hence, the real number $Q^*(u)+R^*(v)$ is an upper bound of the set $\{f(z) \mid P(z) \leqq 1\}$. Since $u(x)$ and $v(y)$ can be arbitrarily close to $Q^*(u)$ and $R^*(v)$ respectively, it is clear that $f(z)$ can be arbitrarily close to $Q^*(u)+R^*(v)$. Thus

$$Q^*(u)+R^*(v)=\sup\{f(z)\mid P(z)\leq 1\}=P^*(f).$$

Corollary. *Each element of* $P^{:}$ *has the form* $u^{(1)}$, *where* $u\in Q^{:}$, *or the form* $v^{(2)}$, *where* $v\in R^{:}$.

Proof. Let $f\in P^{:}$. Represent f as in Lemma 6. We shall prove first that $u=0$ or $v=0$. Suppose it is not so. Then we have

$$Q^*(u)>0,\ R^*(v)>0,\ f=Q^*(u)\cdot\frac{u^{(1)}}{Q^*(u)}+R^*(v)\cdot\frac{v^{(2)}}{R^*(v)},$$

$$Q^*(u)+R^*(v)=1,\ \frac{u^{(1)}}{Q^*(u)}\in P',\ \frac{v^{(2)}}{R^*(v)}\in P',$$

and from here we get the impossible equality $\frac{u^{(1)}}{Q^*(u)}=\frac{v^{(2)}}{R^*(v)}$. Consider now the case when $u=0$. Then $f=v^{(2)}$ and we easily conclude that $v\in R^{:}$. In the case when $v=0$, the situation is similar.

Lemma 7. *If* $u\in Q^{:}$ *and* $v\in R^{:}$, *then* $[u^{(1)}, v^{(2)}]$ *is an extreme subset of* P'.

Proof. Let $u\in Q^{:}$, $v\in R^{:}$. Then $u^{(1)}$ and $v^{(2)}$ belong to P' and, therefore, $[u^{(1)}, v^{(2)}]\subseteq P'$. Since Q and R are not identically equal to 0, we have $Q^*(u)=R^*(v)=1$. Assume that $f\in[u^{(1)}, v^{(2)}]$, $f=\lambda g+(1-\lambda)h$, where $0<\lambda<1$, $g\in P'$, $h\in P'$. We shall prove that g and h also belong to $[u^{(1)}, v^{(2)}]$. The assumption that $f\in[u^{(1)}, v^{(2)}]$ means that

$$f=\alpha u^{(1)}+(1-\alpha)v^{(2)}=(\alpha u)^{(1)}+((1-\alpha)v)^{(2)} \tag{6}$$

for some real number α with $0\leq\alpha\leq 1$. By Lemma 6, we have

$$P^*(f)=Q^*(\alpha u)+R^*((1-\alpha)v)=\alpha Q^*(u)+(1-\alpha)R^*(v)=\alpha+(1-\alpha)=1.$$

We have also

$$P^*(f)\leq\lambda P^*(g)+(1-\lambda)P^*(h),\quad \lambda P^*(g)\leq\lambda,\quad (1-\lambda)P^*(h)\leq 1-\lambda,$$

hence, $\lambda P^*(g)=\lambda$, $(1-\lambda)P^*(h)=1-\lambda$, and consequently, $P^*(g)=P^*(h)=1$. Again by Lemma 6, there are q, s from Q' and r, t from R' such that

$$g=q^{(1)}+r^{(2)},\quad h=s^{(1)}+t^{(2)}, \tag{7}$$

$$P^*(g)=Q^*(q)+R^*(r),\quad P^*(h)=Q^*(s)+R^*(t).$$

Then

$$Q^*(q)+R^*(r)=Q^*(s)+R^*(t)=1, \tag{8}$$

$$f=(\lambda q^{(1)}+(1-\lambda)s^{(1)})+(\lambda r^{(2)}+(1-\lambda)t^{(2)})=(\lambda q+(1-\lambda)s)^{(1)}+(\lambda r+(1-\lambda)t)^{(2)}. \tag{9}$$

After comparing both obtained decompositions (6) and (9) of f, we get

$$\alpha u=\lambda q+(1-\lambda)s,\ (1-\alpha)v=\lambda r+(1-\lambda)t.$$

From here, we have

$$\alpha=Q^*(\alpha u)\leq\lambda Q^*(q)+(1-\lambda)Q^*(s),\ 1-\alpha=R^*((1-\alpha)v)\leq\lambda R^*(r)+(1-\lambda)R^*(t).$$

But

$$(\lambda Q^*(q)+(1-\lambda)\ Q^*(s))+(\lambda R^*(r)+(1-\lambda)R^*(t))$$
$$=\lambda(Q^*(q)+R^*(r))+(1-\lambda)(Q^*(s)+R^*(t))=\lambda+(1-\lambda)=1=\alpha+(1-\alpha).$$

Hence,

$$\alpha=\lambda Q^*(q)+(1-\lambda)Q^*(s),\quad 1-\alpha=\lambda R^*(r)+(1-\lambda)R^*(t). \tag{10}$$

If none of q and s is identically equal to 0, then $Q^*(q)>0$, $Q^*(s)>0$ and $\alpha>0$; so we have

$$u=\frac{\lambda Q^*(q)}{\alpha}\frac{q}{Q^*(q)}+\frac{(1-\lambda)Q^*(s)}{\alpha}\frac{s}{Q^*(s)}$$

and using (10) and the fact that $u\in Q^{:}$, we conclude that

$$u=\frac{q}{Q^*(q)}=\frac{s}{Q^*(s)}.$$

Consequently,

$$q=Q^*(q)u,\quad s=Q^*(s)u.$$

These equalities are easily seen to be true also in the case when q or s is identically equal to 0. In a similar way, we prove that

$$r=R^*(r)v,\quad t=R^*(t)v.$$

Then the equalities (7) give

$$g=Q^*(q)u^{(1)}+R^*(r)v^{(2)},\quad h=Q^*(s)u^{(1)}+R^*(t)v^{(2)}$$

and using (8), we conclude that $g, h\in[u^{(1)}, v^{(2)}]$.

Corollary. *All elements of the form $u^{(1)}$, where $u\in Q^{:}$ and all elements of the form $v^{(2)}$, where $v\in R^{:}$, belong to $P^{:}$.*

Proof. From the fact that $[u^{(1)}, v^{(2)}]$ is an extreme subset of P', it follows that $u^{(1)}$ and $v^{(2)}$ are extreme elements of P'.

From the above corollary and Lemmas 3 and 7, we obtain immediately

Lemma 8. *For each u in $Q^{:}$ and each v in $R^{:}$ there is a (uniquely determined) real number $\alpha_{u,v}$ with $0\leq\alpha_{u,v}\leq 1$ such that*

$$C(u^{(1)})-C(v^{(2)})=\alpha_{uv}(u^{(1)}-v^{(2)}).$$

In the following lemmas, we shall use the denotation $\alpha_{u,v}$ without further explanations.

Lemma 9. *If u_1, u_2 are linearly independent elements of $Q^{:}$ and v is an arbitrary element of $R^{:}$, then $\alpha_{u_1,v}=\alpha_{u_2,v}=\alpha_{u_1,-v}=\alpha_{u_2,-v}$.* Similarly, if u is an arbitrary element of $Q^{:}$ and v_1, v_2 are linearly independent elements of $R^{:}$, then

$$\alpha_{u,v_1}=\alpha_{u,v_2}=\alpha_{-u,v_1}=\alpha_{-u,v_2}.$$

Proof. We shall give the proof only of the first statement of the lemma. Let u_1, u_2 be linearly independent elements of $Q^{:}$ and v be an arbitrary element of $R^{:}$. We have (by Lemmas 8 and 4):

$$C(u_1^{(1)})-C(v^{(2)})=\alpha_{u_1,v}(u_1^{(1)}-v^{(2)}),$$

$$C(u_2^{(1)})-C(v^{(2)})=\alpha_{u_2,v}(u_2^{(1)}-v^{(2)}),$$

$$C(u_1^{(1)})+C(v^{(2)})=\alpha_{u_1,-v}(u_1^{(1)}+v^{(2)}),$$

$$C(u_2^{(1)})+C(v^{(2)})=\alpha_{u_2,-v}(u_2^{(1)}+v^{(2)}).$$

From here we get

$$0=\alpha_{u_1,v}(v_1^{(1)}-v^{(2)})-\alpha_{u_2,v}(u_2^{(1)}-v^{(2)})-\alpha_{u_1,-v}(u_1^{(1)}+v^{(2)})+\alpha_{u_2,-v}(u_2^{(1)}+v^{(2)})$$

$$=(\alpha_{u_1,v}-\alpha_{u_1,-v})u_1^{(1)}+(\alpha_{u_2,-v}-\alpha_{u_2,v})u_2^{(1)}+(\alpha_{u_2,v}+\alpha_{u_2,-v}-\alpha_{u_1,v}-\alpha_{u_1,-v})v^{(2)}.$$

Now, it is sufficient to notice that $u_1^{(1)}$, $u_2^{(1)}$, $v^{(2)}$ are linearly independent elements of Z^*.

Lemma 10. *The number $\alpha_{u,v}$ does not depend on the choice of the functionals u and v.*

Proof. Since at least one of the semi-norms Q and R is not the absolute value of some linear functional, at least one of the sets $Q^{:}$ and $R^{:}$ contains two linearly independent elements. Let, for example, $Q^{:}$ contain such two elements. Then each element of $Q^{:}$ will be linearly independent with some of these two elements, and, using the first statement of Lemma 9, it is easy to see that the number $\alpha_{u,v}$ does not depend on u, when v is fixed, and this number is an even function of v, when u is fixed. To prove Lemma 10, it is sufficient to show also that $\alpha_{u,v}$ does not depend on v, when u is fixed. This kind of independence follows immediately from what was just said, if each two elements of $R^{:}$ are linearly dependent, since then $R^{:}$ consists of two elements which are opposite to each other. On the other hand, if $R^{:}$ contains some two linearly independent elements, then we can prove the needed independence of $\alpha_{u,v}$ by applying the second statement of Lemma 9 (in such a way as we applied its first statement above).

Now we are ready to prove the theorem formulated in Section 2. Let α be the value of $\alpha_{u,v}$ for arbitrary u in $Q^{:}$ and arbitrary v in $R^{:}$. Consider such u and v. We have

$$C(u^{(1)})-C(v^{(2)})=\alpha(u^{(1)}-v^{(2)}),\quad C(u^{(1)})+C(v^{(2)})=\alpha(u^{(1)}+v^{(2)}).$$

From here we get $C(u^{(1)})=\alpha u^{(1)}$, $C(v^{(2)})=\alpha v^{(2)}$. Thus, $C(f)=\alpha f$ for all f in $P^{:}$ (by the corollary of Lemma 6). Then (4) and Lemma 5 give that

$$P_1(z)=\alpha P(z) \tag{11}$$

for all z in Z. Of course, this equality, together with (1), implies

$$P_2(z)=(1-\alpha)\,P(z). \tag{12}$$

5. The existence of other indecomposable semi-norms. We shall briefly describe how the existence of other indecomposable semi-norms can be proven, which are essentially different from the semi-norms of the form (2) and the semi-norms of the form (3). We shall restrict ourselves to the case of semi-norms on a three-dimensional linear spase Z. If P is a semi-norm on Z having the form (2) or the form (3), then it is easy to see the existence of a straight line and of a plane in Z^*, which contain the zero element of Z^* and all elements of $P^{:}$ (in the case when P has the form (3), we have to use the corollary of Lemma 6). Now we shall show how to construct an indecomposable semi-norm P such that the elements of $P^{:}$ cannot be distributed in such a way. Consider an arbitrary symmetric convex polyhedron whose faces are triangles and whose vertices cannot be distributed in the considered way (for example, consider a regular icosahedron). It is not difficult to see the existence of a semi-norm P on Z such that P' is isomorphic to the considered polyhedron (provided this polyhedron is considered as a closed set). For proving the indecomposability of P, suppose the equality (1) holds identically for some semi-norms P_1 and P_2 on Z. If f_1, f_2, f_3 are the vertices of an arbitrary face of P', then Lemma 3 shows that $C(f_1)$, $C(f_2)$, $C(f_3)$ are the vertices of some (possibly degenerated) triangle homothetic to the given face. By considering chains, whose consecutive members are neighbouring faces of P', we prove that the scale factor of the mentioned homothetic transformation does not depend on the choice of the face whose vertices are f_1, f_2, f_3.

From here, the equalities (11) and (12) can be obtained in a way quite similar to what has been done at the end of Section 4.

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