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FINANCIAL DECISIONS VIA METHODS OF GUARANTEED CONTROL THEORY *

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The paper deals with some problems of financial mathematics that can be studied with the help of the theory of guaranteed control under uncertainty. From this viewpoint the dynamic portfolio selection problem and the option pricing models are considered, and the links between guaranteed and stochastic approaches in financial mathematics are discussed.

Keywords: guaranteed control, financial modelling.

AMS subject classification: 93C95, 90A09.

Introduction

The problems of financial modelling are commonly treated by the use of methods and tools of the probability theory and stochastic calculus. It doesn't seem to be surprising because of the stochastic nature of the most processes in finance. But among the problem in question there are those that could be formulated and studied in terms and with the help of the theory of guaranteed control under uncertainty with unknown but bounded disturbances [1, 2, 3, 4]. In the paper we consider the portfolio selection problem [5, 10] in dynamics, when parameters of the model vary with time. Methods of the guaranteed control theory are used to determine a strategy of portfolio management. We also concern the option pricing theory in which the guaranteed approach seems to play a crucial role despite the stochastic problem formulation. In particular, from this viewpoint CRR – [6] and Black-Scholes models [7] are considered, and the links between guaranteed and stochastic approaches in financial mathematics are discussed.

In the sequel we use the notations: $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^N$, (x, y) – a scalar product in finite dimensional vector space, $\|x\| = \sqrt{(x,x)}$. Vectors are treated as columns, x^T

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– transposed row vector, $\rho(l|Z) = \max\{(l,z)|z \in Z\}$ stands for the support function for set Z.

1 Dynamic portfolio selection

The classic version of the portfolio selection problem has been formulated and solved in [5, 10]. It is assumed that there are N risky assets with given returns $r_i (i = 1, \ldots, N)$ treated as stochastic variables. We denote expected values of them and covariance matrix by x_i and $V = \{\sigma_{ij}\}$ respectively. Along with r_i a risk-free investments are available, the rate of interest for the latter we denote by x_0. A portfolio is determined by the vector $\begin{pmatrix} y_0 \\ y \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{pmatrix} \in \mathbb{R}^{N+1}$ each coordinate of which stands for the share of capital invested in corresponding asset. The values y_i are not necessarily non-negative, that means that borrowing and lending are available, $y_0 = 1 - (e, y)$.

The portfolio return is then determined as $\mu(\hat{x}, \hat{y}) = y_0 x_0 + (y, x)$, and corresponding risk is defined by $\sigma(y) = (y^T V y)^{1/2}$. Here and in the following $\hat{x} = \begin{pmatrix} x_0 \\ x \end{pmatrix}, \hat{y} = \begin{pmatrix} y_0 \\ y \end{pmatrix}$.

The characteristics $\mu = \mu(\hat{x}, \hat{y})$ and $\sigma = \sigma(y)$ of the effective (nondominated) portfolios are determined by the following relations:

$$\mu = x_0 + g\sigma, \quad \mu \geq x_0 \tag{1.1}$$

$$g = \sqrt{(x - x_0 e)^T V^{-1} (x - x_0 e)} \tag{1.2}$$

In the classic setting x_i and σ_{ij} are assumed to be given, and the values of these variables are constant.

There are many generalizations of this problem for the case when dynamics of current asset prices and their characteristics are taken into account (see e.g. [11]).

We consider the following model. Assume that the values x_i vary with time according to the differential inclusion

$$\frac{d\hat{x}}{dt} \in A(t)\hat{x} + Q(t), \; t_0 \leq t \leq \Theta; \tag{1.3}$$

$$\hat{x}(t_0) = \hat{x}^0. \tag{1.4}$$

The multivalued function $Q(t)$ describes the uncertainty in the asset expected return evolution. Remark that no statistical description for this type of uncertainty is given. $Q(t)$ is assumed to be convex, compact and continuous in t, $Q(t) \ni 0$.

Let us suppose that one can change the portfolio structure at each moment $t \in [t_0, \Theta]$ with bounded velocity. It can be written in the form:

$$\frac{dy}{dt} = u, \quad y_0 = 1 - (e, y),$$

where the control function u is restricted by inclusion

$$u \in \mathcal{P}(t).$$

Here $\mathcal{P}(t)$ has the properties similar to those of $Q(t)$.

The system (1.1) – (1.2) describes the efficient portfolios on the risk-return plane. The problem under consideration is to ensure the efficiency of the portfolio despite the expected returns $\hat{x}$, and along with them, the line (1.1) vary with time. In other words, one must specify a feasible control u to reach and then to follow the moving line on the plane (μ, σ) described by (1.1) – (1.4).

We define a feasible control strategy as a multivalued map $U = U(t, \hat{x}, \hat{y})$ measurable in t, upper semi-continuous in $\hat{x}, \hat{y}$ with convex compact values $U(t, \hat{x}, \hat{y}) \subseteq \mathcal{P}(t)$.

Under the above assumptions the inclusions (1.3) and

$$\frac{d\hat{y}}{dt} \in U(t, \hat{x}, \hat{y}) \tag{1.5}$$

have an absolutely continuous solution $\hat{x}(t), \hat{y}(t), t_0 \le t \le \Theta$ for any initial condition (1.4) and

$$\hat{y}(t_0) = \hat{y}^0. \tag{1.6}$$

For each solution $\hat{x}(t) = \begin{pmatrix} x_0(t) \\ x(t) \end{pmatrix}$ $\hat{y}(t) = \begin{pmatrix} y_0(t) \\ y(t) \end{pmatrix}$ one can consider the evolution of portfolio risk-return characteristics:

$$\mu[t] = y_0(t)x_0(t) + (y(t), x(t)) \text{ and } \sigma[t] = (y^T(t)Vy(t))^{1/2}.$$

In order to be efficient the portfolio must have the variables $\mu = \mu[t]$ and $\sigma = \sigma[t]$ that satisfy the equality (1.1) with $g = g[t]$ obtained by substitution $\hat{x}(t)$ into (1.2).

Let $\hat{y}^0$ be an efficient portfolio for $\hat{x}^0$ with the expected return $\mu^0 = \mu(\hat{x}^0, \hat{y}^0)$ and risk $\sigma^0 = \sigma(y^0)$. One can pose the following problems.

Problem 1.1 *Specify a feasible control strategy* $U = U(t, \hat{x}, \hat{y})$ *that guarantees the efficiency of the portfolio* $\hat{y}[t]$ *for* $\hat{x}[t]$ $(t_0 \le t \le \Theta)$ *whatever solutions* $\hat{y}[t], \hat{x}[t]$ *to* (1.3) – (1.4), (1.5) – (1.6) *are taken.*

Problem 1.2 *Specify a feasible control strategy that solves the Problem* 1.1 *and ensure*

a) a prescribed level of risk:

$$|\sigma(\hat{x}[t], \hat{y}[t]) - \sigma^0| \le \alpha,$$

b) a prescribed return:

$$|\mu(\hat{x}[t], \hat{y}[t]) - \mu^0| \le \alpha;$$

c) a prescribed risk premium:

$$\beta \le \mu(\hat{x}[t], \hat{y}[t]) - x_0[t] \le \alpha,$$

where the numbers $\alpha \ge \beta \ge 0$ *are given.*

If $\hat{y}^0$ is not an efficient portfolio for $\hat{x}^0$, then we come to the problem of steering the portfolio to efficient set.

Problem 1.3 *Specify a feasible control strategy* $U = U(t, \hat{x}, \hat{y})$ *that guarantees the efficiency of the portfolio* $\hat{y}[t]$ *for* $\hat{x}[t]$ *at the prescribed moment* $t = t_1 < \Theta$ *and further for* $t \in [t_1, \Theta]$.

Additional conditions similar to those indicated in Problem 1.2 can also be included.

The Problems 1.1 – 1.3 can be investigated by techniques developed in [1, 2, 3]. Here we present a solution to the problems 1.1 – 1.2 for the simplest version.

Suppose that $A(t) \equiv 0$ and

$$\frac{dx}{dt} \in \tilde{Q}(t),$$

$$\frac{dx_0}{dt} = 0,$$

$$x(t_0) = x^0,\ x_0(t_0) = r_0.$$

Moreover, to simplify the formulas assume the covariance matrix V to be identity one. We shall formulate the results for the problems 1.1 – 1.2 with $\alpha = \beta = 0$. Assume also the following regularity condition to be fulfilled:

$$\|x^0 - er_0\| - \max_{\|l\|=1} \int_{t_0}^{\Theta} \rho(l|\tilde{Q}(t))dt = d > 0. \tag{1.7}$$

This inequality ensures $\|x(t) - r_0 e\| \geq d > 0$ that, in turn, means the impossibility for all risky expected returns simultaneously to be close to risk-free rate of interest.

The solvability conditions to the problem 1.1 – 1.3 are then defined by relations between the set-valued maps $\tilde{Q}(t)$ and $\mathcal{P}(t)$.

A sufficient solvability condition for the problem 1.2 a) can be written in the form:

$$d \cdot \rho(l|\mathcal{P}(t)) - \sigma^0 \rho(l|\tilde{Q}(t)) \geq \sigma^0 \cdot \max_{\|q\|=1} \rho(q|\tilde{Q}(t)) \tag{1.8}$$

$$\forall l \in \mathbb{R}^N, \quad t \in [t_0, \Theta]$$

For the cases b) – c) in this simplest version we need the following inequality:

$$d^2 \rho(l|\mathcal{P}(t)) - (\mu^0 - r_0)\rho(l|\tilde{Q}(t)) \geq (\mu^0 - r_0) \max_{\|q\|=1} \rho(q|\tilde{Q}(t)) \tag{1.9}$$

$$\forall l \in \mathbb{R}^N, \quad t \in [t_0, \Theta]$$

To specify a strategy that resolves the problems define the set-valued map (extremal aiming map):

$$U^e(t, z, w) = \begin{cases} \mathcal{P}(t), & \text{if } z = w \\ \partial_l \rho(z - w|\mathcal{P}(t)), & \text{if } z \neq w, \end{cases} \tag{1.10}$$

where $z, w \in \mathbb{R}^N$, $\partial_l \rho(l|\mathcal{P}(t))$ is the subdifferential of the function $\rho(l|\mathcal{P}(t))$ with respect to the argument l.

Theorem 1.1

i) Under the assumptions (1.7) – (1.8) *the solution to the problems* 1.1 – 1.2 a) *does exist and is given by the feasible control strategy*

$$U(t, \hat{x}, \hat{y}) = U^e(t, y, y^*), \tag{1.11}$$

where

$$y^* = \frac{(x - x_0 e)\sigma^0}{\|x - x_0 e\|} :$$

ii) Under the assumptions (1.7), (1.9) *the solution to the problems* 1.1 – 1.2 b), c) *is given by the feasible control strategy* (1.11) *with*

$$y^* = \frac{(x - x_0 e)}{\|x - x_0 e\|^2}(\mu^0 - r_0).$$

The proof of the theorem is based on the techniques developed in [1, 2, 3, 4]. The inequalities (1.8) – (1.9) guarantee that the point $(\mu[t], \sigma[t])$ which reflects the portfolio controlled by the strategy (1.11) follows the corresponding moving point from the efficient set.

2 Option pricing problems

In this section we consider the simplest example from the option pricing theory. We derive two well-known formulas, namely, that of Cox, Ross and Rubinstein [6] and the Black-Scholes formulas [7] for the standard European call option. The aim of this section is to demonstrate how these results can be obtained in the framework of guaranteed control approach rather than stochastic one. We expressly restrict ourselves by very simple cases, where both approaches lead to practically identical calculations to demonstrate the general idea. The applications to more complicated problems of the option pricing theory would contain a variety of technical details and are the subject matter for a separate paper.

Stochastic approach. Traditional scheme can be briefly described as follows. A probability space $\{\Omega, \mathcal{F}, P\}$ equipped with a filtration $\{\mathcal{F}_t\}$ ($\mathcal{F}_t \subseteq \mathcal{F}_\tau \subseteq \mathcal{F}$, when $t \leq \tau$) is given. The price S_t of an underlying risky asset is assumed to be a non-negative random variable or stochastic process adapted to $\{\mathcal{F}_t\}$. Without any dividends, taxes, consumption etc. the value of the call option at maturity T with exercise price K is given by the equality

$$(S_T - K)_+ = \max\{S_T - K, 0\} \tag{2.1}$$

We assume that there is the only risky asset and a risk-free financial instrument B_t (bond, bank account). The latter evolves according to the equation

$$B_{t+1} = (1 + r)B_t \tag{2.2}$$

in the discrete-time version or

$$dB_t = rB_t dt \tag{2.3}$$

in the continuous-time case.

Control strategy is also defined as a stochastic process $U = \{\beta_t, \gamma_t\}$ adapted to the given filtration where $|\beta_t|$ is a number of risk-free assets, while $|\gamma_t|$ is that of risky ones. As in the previous section the values β_t and γ_t are not necessarily non-negative.

The control strategies are assumed to have some properties. In particular, they should be self-financing. The latter means that the investor readjust his position without bringing or consuming any wealth. Starting from a capital V_0 the investor has

$$V_t = \beta_t B_t + \gamma_t S_t \tag{2.4}$$

at time t.

The price of the option is defined as the minimal value V_0 for which there exists a strategy U that guarantees with probability 1 the inequality

$$V_T \geq (S_T - K)_+. \tag{2.5}$$

Remark 1 *In the above very simplified description we have tried to present the general idea of option evaluation and have not concerned many important issues of this theory leaving aside the crucial role of martingals, problems related to American options, option pricing in the presence of transaction costs and valuing exotic options. The main results in these areas and related references are presented, in particular, in* [12, 13, 14].

Guaranteed approach. The above approach for evaluation of options is based on the idea that some property must be provided with probability 1.It seems to be natural to reformulate the problem in terms of the theory of guaranteed control for uncertain systems [1, 2, 3, 4], in the framework of which the desired result must be achieved surely, whatever the realizations of uncertain parameters, disturbances and trajectories would be. Doing this, we replace the martingal measures that correspond to the hedging strategies by positional, closed-loop synthesized strategies used in guaranteed control theory and differential games. We loss the stochastic interpretation, the possibility to use the advanced and sophisticated techniques of stochastic calculus, but we gain the possibility to apply not less advanced and sophisticated methods of control theory, multivalued analysis, viability theory etc. In the following we will assume that the dynamics of risk-free asset B_t is again described by equation (2.2) or (2.3). For evolution description of the risky one S_t we introduce $\mathbb{S}' = \mathbb{S}(t, S, \Delta t)$ – a compact valued map: $\mathbb{R}^3 \to \mathbb{R}^1$ continuous in its arguments. In particular, it can be the infinitesimal generator of the semigroup that corresponds to the differential inclusions of the type considered in the first section.

We further assume the following inequalities to be true for sufficiently small Δt:

$$S^-(t, S, \Delta t) < Se^{r\Delta t} < S^+(t, S, \Delta t), \tag{2.6}$$

where

$$S^-(t, S, \Delta t) = \min\{S | S \in \mathbb{S}(t, S, \Delta t)\}$$

$$S^+(t, S, \Delta t) = \max\{S | S \in \mathbb{S}(t, S, \Delta t)\}$$

Let P_m be a partition $0 = t_0 < t_1 < \ldots < t_m = T$. A trajectory of the price $S_t = S[t]$ is defined by the relations

$$S[t_0] = S_0,\ \ S[t_{i+1}] \in \mathbb{S}(t_i, S[t_i], t_{i+1} - t_i). \tag{2.7}$$

A feasible control strategy $U = \{\beta_t, \gamma_t\}$ we define as a rule that for every $P_m, S_0 \geq 0,\ k \in \overline{1, m}$, a trajectory path $\{S[t_j],\ j = 0, \ldots, k-1\}$ determines the values $\{\beta_{t_k}, \gamma_{t_k}\}$

satisfying the self-financing conditions:

$$(\beta_{t_k} - \beta_{t_{k-1}})B_{t_{k-1}} + (\gamma_{t_k} - \gamma_{t_{k-1}})S_{t_{k-1}} = 0$$

Then, starting from the initial endowment V_0 one can determine the wealth evolution

$$V_t = V_t(V_0, P_m, U, S[\cdot]).$$

The minimal number $V_0 = C_0(P_m)$ for which there exists a feasible strategy U that ensures the inequality

$$V_T \geq (S[T] - K)_+,$$

whatever trajectory defined by (2.7) is taken is said to be a price of the option that corresponds to discrete control.

The value

$$C^0 = \limsup_{\Delta(P_m)\to 0} C_0(P_m),$$

where $\Delta(P_m) = \max\{t_{i+1} - t_i | i = 0, \ldots, m-1\}$ that does not depend on (P_m) is called option's price.

The following assertion is true.

Theorem 2.1

i) Suppose that

$$S^+(t, S, \Delta t) = Se^{u\Delta t}, \ S^-(t, S, \Delta t) = Se^{d\Delta t},$$

$$t_{i+1} - t_i = \frac{T}{m}, \ d < r < u.$$

Then the value $C_0(P_m)$ *is determined by CRR formula* ($a = e^{d\Delta t} - 1, b = e^{u\Delta t} - 1, r' = e^{r\Delta t} - 1, \Delta t = \frac{T}{m}$)*:*

$$C(P_m) = S_0 \sum_{k=k_0}^{m} \binom{k}{m} \cdot p^k (1-p)^{m-k} - K(1+r')^{-m} \sum_{k=k_0}^{m} \binom{k}{m} p_*^k (1-p_*)^{m-k},$$

where $k_0 = 1 + \left[\ln \frac{K}{S_0(1+a)^m} / \ln \frac{1+b}{1+a}\right]$, $p = \frac{1+b}{1+r} p_*$, $p_* = \frac{r'-a}{b-a}$, $\binom{k}{m} = \frac{m!}{k!(m-k)!}$.

ii) In case, when $\mathbb{S}(t, S, \Delta t)$ *is defined by the relations*

$$S^+(t, S, \Delta t) = Se^{\sigma\sqrt{\Delta t}},$$

$$S^-(t, S, \Delta t) = Se^{-\sigma\sqrt{\Delta t}}$$

we come to Black-Scholes formula for C_0*:*

$$C^0 = \lim_{\Delta(P_m)\to 0} C_0(P_m) = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where $d_\pm = \dfrac{\ln(S_0/K) + T(r \pm \sigma^2/2)}{\sigma\sqrt{T}}$ *and* $\Phi(d)$ *stands for the normal distribution function.*

Conclusions

In this papers, we have considered two problems from financial mathematics in their simplest versions. The first one concerns with dynamic portfolio selection, while the second problem relates to the option pricing theory. It is shown that along with traditional methods of stochastic calculus, those of the guaranteed control theory can be applied. In particular, we have demonstrated the scheme of portfolio management via the extremal control strategy and an example of formalization of the option pricing problem based on the guaranteed approach.

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