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ESTIMATION OF THE OFFSPRING MEAN IN A GENERAL SINGLE-TYPE SIZE-DEPENDENT BRANCHING PROCESS

Christine Jacob, Nadia Lalam

We consider a general single-type size-dependent branching process $\{N_n\}_n$ satisfying $N_n = \sum_{i=1}^{\xi(N_{n-1})} Y_{n,i}(N_{n-1})$, for all n, where the offsprings $\{Y_{n,i}\}_i$ are identically distributed with mean $m_n(N)$ when the size of the population is N. We assume that $m_n(N)$ may be written: $m_n(N) = m_n^{(1)}(N)+m_n^{(2)}(N)$, where $m_n^{(1)}(.)$ depends on an unknown parameter θ_0 of finite dimension and $m_n^{(2)}(.)$ may be unknown and is assumed to be negligible relatively to $m_n^{(1)}(.)$, as $n \to \infty$. We estimate θ_0 on the non-extinction set $\{\overline{\lim} N_n \neq 0\}$ from observations in the time interval $[h+1, n]$ by using the conditional least squares method. The estimation is done in the approximate model $m_n^{(1)}(.) + \widehat{m_n^{(2)}}(.)$, where $\widehat{m_n^{(2)}}(.)$ is given. We study the strong consistency of the estimator with either h or $n-h$ remaining constant as $n \to \infty$. We prove the strong consistency of the estimator for any value $\{\nu_n\}_n$ of the nuisance parameter, under some very general conditions on the behavior of the process. The rate of convergence is calculated for a particular subclass of this model.

1. Introduction

In this paper, we deal with the estimation of the offspring mean of a general single-type size-dependent and Markovian branching process in discrete time.

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The size N_{n+1} of the population at generation $n+1$ is defined by

$$N_n = \sum_{i=1}^{\xi(N_{n-1})} Y_{n,i}(N_{n-1}), \tag{1}$$

where $\xi(N) \in \mathbf{N} \setminus \{0\}$ and the offspring variables $\{Y_{n,i}(N)\}_i$ are assumed identically distributed. Their law, given the past of the process may depend on the time and on the size N of the population. Denote $m_n(N) = E(Y_{n,1}(N))$, $\sigma_n^2(N) = Var(Y_{n,1}(N))$ and $\sigma_{n,i,j}(N) = Cov(Y_{n,i}(N), Y_{n,j}(N))$. We assume

$$\sigma_n^2(N) \leq \sigma_N^2 N_+^{\beta_N}, \sup_N \sigma_N^2 < \infty \tag{2}$$

$$\sigma_{n,i,j}(N) \leq \sigma_N^2 \xi^{-1}(N) N_+^{\beta_N} \tag{3}$$

where $N_+ = \max\{N, 1\}$ and β_N is a known real deterministic sequence. For simplifying the notations, we will write $m(.)$, $\sigma^2(.)$, $\sigma_{i,j}(.)$ instead of $m_n(.)$, $\sigma_n^2(.)$, $\sigma_{n,i,j}(.)$. The initial size N_0 is given. At the opposite of usual branching models, we do not assume here the independency of the $\{Y_{n,i}(N)\}_i$. The relaxing of this assumption allows to describe under this general model a very large number of models with independent individuals, including the Bienaym-Galton-Watson process, the usual size-dependent process with independent offsprings, processes with individual or familial migrations, or processes with an increasing number of ancestors:

1. The BGW (Bienaymé-Galton-Watson) process which was introduced by Bienaymé [2] and Galton and Watson [12] for modelling the evolution of family names and explaining mathematically their extinction:

$$\begin{aligned} N_n &= \sum_{i=1}^{N_{n-1}} Z_{n,i}; \{Z_{n,i}\}_i \text{ i.i.d. } (m_Z, \sigma_Z^2) \\ &= \sum_{i=1}^{N_{n-1,+}} Z_{n,i} 1_{\{N_{n-1}>0\}}, \end{aligned}$$

Therefore $Y_{n,i}(N_{n-1}) = Z_{n,i} 1_{\{N_{n-1}>0\}}$, $\xi(N) = N_+ \overset{definition}{=} \max\{N, 1\}$,

$$m(N) = m_Z 1_{\{N>0\}}; \ \sigma^2(N) = \sigma_Z^2 1_{\{N>0\}}; \ \sigma_{i,j}(N) = 0.$$

2. The controlled BGW branching process which allows familial migrations and the behavior of which was studied by Sevast'yanov and Zubkov [48], Yanev

[58], Yanev G. and Yanev [61], Gonzlez, Molina and del Puerto [14], ...:

$$N_n = \sum_{i=1}^{\phi(N_{n-1})} Z_{n,i} = \sum_{i=1}^{\phi_+(N_{n-1})} Z_{n,i} 1_{\{\phi(N_{n-1})>0\}},$$

where the $\{Z_{n,i}\}_i$ are i.i.d.(m_Z, σ_Z^2). Consequently $Y_{n,i}(N) = Z_{n,i}1_{\{\phi(N)>0\}}$, $\xi(N) = \phi_+(N) = \max\{\phi(N), 1\}$ and

$$m(N) = m_Z 1_{\{\phi(N)>0\}},\ \sigma^2(N) = \sigma_Z^2 1_{\{\phi(N)>0\}},\ \sigma_{i,j}(N) = 0.$$

The number of familial migrations when the size of the population is N is equal to $N - \phi(N)$.

3. The size-dependent model with independent offsprings:

$$N_n = \sum_{i=1}^{N_{n-1}} Z_{n,i}(N_{n-1})$$

where the $\{Z_{n,i}(N)\}_i$ are i.i.d. $(m_Z(N), \sigma_Z^2(N))$ with

$$\begin{aligned} m_Z(N) &= m + O(N^{-\alpha});\ m \geq 1,\ \alpha > 0 \\ \sigma_Z^2(N) &\leq \sigma^2 N^{\beta};\ 0 \leq \beta < 1. \end{aligned}$$

Then

$$N_n = \sum_{i=1}^{N_{n-1},+} Z_{n,i}(N_{n-1}) 1_{\{N_{n-1}>0\}}$$

implying $Y_{n,i}(N) = Z_{n,i}(N)1_{\{N>0\}}$, $\xi(N) = N_+$ and

$$m(N) = m_Z(N)1_{\{N>0\}},\ \sigma^2(N) = \sigma_Z^2(N)1_{\{N>0\}},\ \sigma_{i,j}(N) = 0.$$

The size-dependent model which was introduced by Levina *et al.* [35] was studied from its asymptotic behavior point of view by Fujimagari [11], Hopfner [18], Klebaner ([26], [27]), Kuester [28], Kersting ([24], [25]), Jagers ([20], [21]), Labkovskij [29], Pierre Loti Viaud ([43], [44]), and in the particular case of the Q-PCR (Quantitative Polymerase Chain Reaction), by Jagers and Klebaner [22]. Lalam, Jacob and Jagers [33] gave a CLSE (Conditional Least Squares Estimator) of the offspring mean.

We will refer to this model as the "Klebaner-Kersting" model.

4. The size-dependent model with thresholds, which is just a generalization of the previous model. We assume that there exists a finite partition of $\mathbf{N}$,

$\mathbf{N} = \oplus_{i=1}^{I}\mathbf{N}_i$, such that $m(N) = \sum_{i=1}^{I} m_i(N)1_{\{N\in\mathbf{N}_i\}}$. An example is given by Lalam *et al.* [19], [33] in the frame of the Q-PCR, where they model the amplification process first by a BGW until a threshold and then by a near-critical size-dependent process, and they give a CLSE of the offspring mean.

5. The BGW process with migrations

$$N_n = \sum_{i=1}^{N_{n-1}} Z_{n,i} + \delta_n^E(N_{n-1})E_n + \delta_n^I(N_{n-1})I_n,$$

where the $\{Z_{n,i}\}_{n,i}\}_i$ are i.i.d (m_Z, σ_Z^2), the $\{E_n\}_n$ are i.i.d. (λ_E, b_E^2) the $\{I_n\}_n$ are i.i.d. (λ_I, b_I^2), the $\{\delta_n^E(N)\}_n$ are independent Bernoulli variables with $E(\delta_n^E(N))=$ $p(N)$ with $p(0) = 0$, the $\{\delta_n^I(N)\}_n$ are independent Bernoulli variables with $E(\delta_n^I(N)) = q(N), p(N) + q(N) \leq 1$ and the different kinds of variables $\{Z_{n,i}\}_{n,i}$, $\{\delta_n^E(N)E_n\}_{n,i}$, $\{\delta_n^I(N)I_n\}_{n,i}$ are mutually independent. This implies

$$\begin{aligned}
N_n &= \sum_{i=1}^{N_{n-1,+}} Y_{n,i}(N_{n-1}) \\
Y_{n,i}(N) &= Z_{n,i}1_{\{N>0\}} + \delta_n^E(N)\frac{E_n}{N_+} + \delta_n^I(N)\frac{I_n}{N_+} \\
m(N) &= m_Z 1_{\{N>0\}} + \frac{p(N)\lambda_E + q(N)\lambda_I}{N_+} \\
\sigma^2(N) &= \sigma_Z^2 1_{\{N>0\}} + \frac{p(N)\widetilde{b}_E^2(N) + q(N)\widetilde{b}_I^2(N)}{N_+^2} \\
\sigma_{i,j}(N) &= \frac{p(N)\widetilde{b}_E^2(N) + q(N)\widetilde{b}_I^2(N)}{N_+^2}, \ i \neq j \\
\widetilde{b}_E^2(N) &= b_E^2 + \lambda_E^2(1 - p(N)) \\
\widetilde{b}_I^2(N) &= b_I^2 + \lambda_I^2(1 - q(N)) \\
\xi(N) &= N_+ = \sup\{N, 1\}.
\end{aligned}$$

In the case $p(.) = 0$, $q(.) = 1$, the process is a BGW process with immigration in every state. Its behavior was studied by Quine and Seneta [45], Pakes ([41], [42]) Heyde and Seneta [16], Heyde and Leslie [17], Yanev ([56],[57], [59]), Venkataraman ([50],[51]), Wei and Winnicki ([52],[53]), ... The process with immigration stopped at 0 was also studied by several authors, mainly Seneta (see for example [47]). The BGW processes with random migrations defined by $p(.) = p1_{\{N>0\}}$, $q(.) = q$, $p + q \leq 1$ was studied by Nagaev [40], Yanev G. and Yanev ([62], [63], [64]). In the frame of a Bellman-Harris process, Mitov and Yanev ([36], [37], [38]) studied the BGW processes with immigration in the state 0 defined by $p(.) = 0$, $q(N) = 1_{\{N=0\}}$ and which is a particular case of the previous BGW with random migrations.

6. The branching process with an increasing random number of ancestors $N_{0,n} \stackrel{n\to\infty}{\to} \infty$. This kind of process may be useful for studying asymptotics properties of estimators in a branching process which dies out a.s. when N_0 is fixed (subcritical or critical processes). This model has been introduced by Chauvin ([3], [4]) for studying the branching process with immigration as a branching process without immigration. Estimators of the offspring mean have been studied by Dion and Yanev ([6], [7],[8], [9], [10]). At time n, the number of ancestors is $N_{0,n}$. Writing $N_{n,(N_{0,n})}$ for N_n, and $N_{n-1,(N_{0,n})}$ for the size of the population at time $n-1$, the model is defined by

$$N_n = N_{n,(N_{0,n})} = \sum_{i=1}^{N_{n-1,(N_{0,n})}} Z_{n,i} 1_{\{N_{n-1,(N_{0,n})}>0\}},$$

where the $\{Z_{n,i}\}_i$ are i.i.d. (m_Z, σ_Z^2). We assume here that the variations $\{\Delta_n\}_n$ defined by $\Delta_n = N_{0,n} - N_{0,n-1}$ are independently distributed $(m_{\Delta_n}, \sigma^2_{\Delta_n})$. Then if we decompose the population of size $N_{n-1,(N_{0,n})}$ in two subpopulations, one of size $N_{n-1,(N_{0,n-1})}$ and the other of size $N_{n-1,(N_{0,n}-N_{0,n-1})}$, $\{N_n\}_n$ has the same distribution as the following process

$$N_n = \sum_{i=1}^{N_{n-1,(N_{0,n-1})}} Z_{n,i} 1_{\{N_{n-1,(N_{0,n-1})}>0\}} + N_{n,(N_{0,n}-N_{0,n-1})}.$$

Consequently since we set $N_n = N_{n,(N_{0,n})}$, for all n,

$$N_n = \sum_{i=1}^{N_{n-1,+}} [Z_{n,i} 1_{\{N_{n-1}>0\}} + \frac{N_{n,(\Delta_n)}}{N_{n-1,+}}]$$

implying

$$\begin{aligned} Y_{n,i}(N) &= Z_{n,i} 1_{\{N>0\}} + \frac{N_{n,(\Delta_n)}}{N_+} \\ m(N) &= m_Z 1_{\{N>0\}} + \frac{m_Z^n m_{\Delta_n}}{N_+}, m_Z \le 1 \\ \sigma^2(N) &= \sigma_Z^2 1_{\{N>0\}} + \frac{O(m_Z^{2n})}{N_+^2} \end{aligned}$$

$$\sigma_{i,j}(N) = \frac{O(m_Z^{2n})}{N_+^2}$$
$$\xi(N) = N_+.$$

7. Since the law of the offspring may depend on n, models with a branching structure only on some time intervals could also belong to this general class. Regenerative processes are such examples (see [39] for example).

We assume in the frame of the general model that there exists a finite partition of $\mathbf{N}$, $\mathbf{N} = \oplus_{i=1}^I \mathbf{N}_i$, such that for $N \in \mathbf{N}_i$, $m(N) = m_i(N)$, $\sigma^2(N) = \sigma_i^2(N)$, $\sigma^2_{i',j'}(N) = \sigma^2_{i;i',j'}(N)$, with

$$\begin{aligned} m_i(N) &= m_i^{(1)}(N) + m_i^{(2)}(N) \\ \sigma_i^2(N) &\leq \sigma_i^2 N_+^{\beta_i} \\ \sigma_{i;i',j'}(N) &\leq \sigma_i^2 \xi^{-1}(N) N_+^{\beta_i} \end{aligned}$$

where $m_i^{(1)}(.)$ depends on an unknown finite dimensional parameter $\theta_i \in \overset{\circ}{\Theta}_i$, $\Theta_i \subset \mathbf{R}^{d_i}$, that we will estimate and $m_i^{(2)}(.)$ is a nuisance part and therefore must be assumed negligible with respect to $m_i^{(1)}(.)$, as $n \to \infty$; $m_i^{(2)}(.)$ may depend on $\theta = \{\theta_i\}_i$ and an unknown nuisance parameter ν_i that may be of infinite dimension.

It is the case for all the examples above in which $m_i(.)$ belong to the following class

$$m_i(N) = m_i + \frac{f(\mu_i, n)}{N_+^{\alpha_i}} + O_i(N_+^{-\overline{\alpha_i}}), \tag{4}$$
$$0 < \alpha_i < \overline{\alpha_i},\ i = 1, 2,\ f(\mu_i, n) = O(1)$$

where $\mathbf{N}_1 = \{0\}$, $\mathbf{N}_2 = \mathbf{N} \setminus \{0\}$, and $\beta_N = \beta_i$, for $N \in \mathbf{N}_i$.

Our aim is the estimation of θ_0 from a single trajectory of the process, in the approximate model $m_{\theta_0,\nu}(.)$, where $\nu = \{\nu_i\}_i$ has any given value. The derivability of $m(.)$ with respect to θ is not necessary for proving the consistency of the estimator. We must assume that θ is identifiable from the single trajectory, which is not *a priori* obvious especially when the chosen trajectory satisfies $\lim_n N_n = \infty$ and $m(.)$ contains a transient part which disappears as $N \to \infty$. For example, in the frame of (4), only m_i or μ_i are separately identifiable on $\lim_n N_n = \infty$. Therefore either $\theta_i = m_i$ or $\theta_i = \mu_i$. In the first case the negligible part is $m_i^{(2)}(N) = f(\mu_i, n) N_+^{-\alpha_i} + O_i(N_+^{-\overline{\alpha_i}})$, and in the second case, since m_i is not negligible relatively to $N_{n+}^{-\alpha_i}$, *i.e.* $m_i N_{n+}^{\alpha_i}$ is not negligible, then either we assume that m_i is known, or we may replace m_i by its estimator, if $(\widehat{m}_{i;h,n} - m_i) N_{n+}^{\alpha_i}$ is negligible as $n \to \infty$.

We use the conditional least square method. This method, widely used in autoregressive processes, has many interesting properties, among them, it does not need the knowledge of the exact law of $\{Y_{n,i}(N)\}_{i,n}$ at the opposite of methods based on the likelihood, it is easily generalizable to noisy obervations and it is simply written, even when the observations are not all taken into account. In that aim, we write the model according to a stochastic nonlinear regression model:

$$
\begin{aligned}
N_{n+1} &= m(N_n)\xi(N_n) + \eta_{n+1}(N_n) \\
\eta_{n+1}(N_n) &= \sum_{i=1}^{\xi(N_n)} [Y_{n+1,i}(N_n) - m(N_n)],
\end{aligned} \tag{5}
$$

where $\{\eta_{n+1}(N_n)\}_n$ that satisfies $E(\eta_{n+1}^2(N)) \leq \xi(N)\sigma^2 N^{\beta_N}$ is a martingale difference sequence. Model (5) belongs to the class of nonlinear (but asymptotically linear) and explosive (when $\xi(N_n) N_n^{\beta_{N_n}} \to \infty$, as $n \to \infty$) autoregressive process of order one (AR(1)). In the frame of linear autoregressive processes, many results are already available concerning the estimation problem. White [54] studied the LSE (least squares estimator) $\{\widehat{m}_n\}_n$ of the autoregressive parameter in an explosive linear $AR(1)$ process with normally i.i.d. errors under the assumption $m(N) = m$, for all N. Lai and Wei [30] showed the strong consistency of the LSE for a linear $AR(p)$ process where the noise $\{\eta_n\}_n$ is a martingale difference sequence satisfying $E(\eta_n^2 | F_{n-1}) = \sigma^2$ and $\sup_n E(\eta_n^{2+\delta} | F_{n-1}) < \infty$ for some $\delta > 0$. Basawa et al. [1] studied the asymptotic behavior of the standard bootstrap for $\{\widehat{m}_n\}_n$ for an explosive linear $AR(1)$ model with finite error variance. Datta [5] established consistency of the LSE in an explosive linear $AR(p)$ model for independent errors $\{\eta_n\}_n$ satisfying the uniform integrability of $\log^+(|\eta_n|)$. Estimation in stochastic nonlinear regression models for which the noise is a martingale difference has been studied when the noise has a finite moment of order two especially by Lai [31] and Skouras [49]. But their methods require assumptions that are difficult to check in practice (see [32]) so that we do not rely on those methods to establish the strong consistency.

Our estimator $\widehat{\theta}_{h,n,\nu_n}$ will minimize over θ the conditional least squares:

$$
\widehat{\theta}_{h,n,\nu_n} = \arg\min_{\theta} \widetilde{S}_{h,n,\nu_n}(\theta) \tag{6}
$$

$$
\widetilde{S}_{h,n,\nu_n}(\theta) = \sum_{k \in \mathcal{O}_{h,n}} (N_k - m_{\theta,\nu_n}(N_{k-1})\xi(N_{k-1}))^2 \xi^{-1}(N_{k-1}) N_{k-1,+}^{-\beta_{N_{k-1}}}, \tag{7}
$$

where $m_{\theta,\nu_n}(.)$ is the offspring mean in which the nuisance parameter ν is set to any value ν_n, and $\mathcal{O}_{h,n} = [h+1,n] \cap \mathbf{K}^{obs}$, where $\mathbf{K}^{obs}$ is the set of observation times that are taken into account for the estimation of θ. It can refer either to non censored observation times or to non censored observations, or to observation times on which the process has a nonnull branching structure, or to the associated parameters that are estimated. In the last case $\mathbf{K}^{obs} = \{k : N_{k-1} \in \cup_{i\in\mathcal{I}}\mathbf{N}_i\}$, when $\theta = \{\theta_i\}_{i\in\mathcal{I}}$, $\mathcal{I}$ being any subset of $\{1,\dots,I\}$. Since the $\{\mathbf{N}_i\}_i$ are disjoint sets, it may be convenient to estimate the $\{\theta_i\}_i$ separately when they are not identifiable together. In that case $\mathbf{K}^{obs} = \{k : N_{k-1} \in \mathbf{N}_i\}$.

It is well-known that the CLSE built from (6) and (7) is optimal if N^{β_N} is the exact order of $\sigma^2(N)$ ([13],[34]). However this general method allows to build easily strong consistent estimators.

In the frame of submodel (4) with $f(\mu_i, n) = \mu_i$, writing $\mathcal{O}_{i;h,n}$ for $\mathcal{O}_{h,n} \cap \{k : N_{k-1} \in \mathbf{N}_i\}$, if we set $O(N^{-\overline{\alpha}}) = 0$, the CLSE of m_i and μ_i have the explicit form

$$(8)\ \widehat{m}_{i;h,n} = \frac{\sum_{k\in\mathcal{O}_{i;h,n}} N_k N_{k-1,+}^{-\beta_i}}{\sum_{k\in\mathcal{O}_{i;h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_i}} - \widehat{\mu}_{i;h,n} \frac{\sum_{k\in\mathcal{O}_{i;h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_i-\alpha_i}}{\sum_{k\in\mathcal{O}_{i;h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_i}}$$

$$(9)\ \widehat{\mu}_{i;h,n} = \frac{\sum_{k\in\mathcal{O}_{i;h,n}} N_k N_{k-1,+}^{-\beta_i-\alpha_i}}{\sum_{k\in\mathcal{O}_{i;h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_i-2\alpha_i}} - \widehat{m}_{i;h,n} \frac{\sum_{k\in\mathcal{O}_{i;h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_i-\alpha_i}}{\sum_{k\in\mathcal{O}_{i;h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_i-2\alpha_i}}.$$

The estimator (8) generalizes the well-known Harris estimator and the moment estimator of the BGW frame.

The asymptotic properties of $\{\widehat{\theta}_{h,n,\nu_n}\}_n$ will be studied on the set of nonextinction $\mathcal{E}_\infty = \{\overline{\lim} N_n \neq 0\}$ by increasing n to ∞, with either h or $n-h$ fixed, and we will show that they do not depend on $\{\nu_n\}_n$. In particular, when ν is of finite dimension, ν_n may take the value $\widehat{\nu}_{h,n}$ defined by $(\widehat{\theta}_{h,n}, \widehat{\nu}_{h,n}) = \text{argmin}_{\theta,\nu} \widetilde{S}_{h,n,\nu}(\theta)$; and when ν is infinite dimensional, then $m^{(2)}(.)$ will be set to 0. In section 2, we define and study the identifiability of parameters and prove the strong consistency of $\{\widehat{\theta}_{h,n,\nu_n}\}_n$ on $\mathcal{E}_\infty$. The proof relies on a sufficient condition concerning minimum contrast method (Wu [55]), on the branching structure of the process and the martingale properties of the noise $\{\eta_n\}_n$ and do not need the differentiability of $m(.)$ in θ. Assuming that θ_0 is asymptotically identifiable from the observations at the rate $N_.^{\psi_{N_.}}$, the proof need some general conditions on the a.s. asymptotic behavior of $\{N_n\}_n$ on $\mathcal{E}_\infty$: $\{\sum_{k\in\mathcal{O}_{h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_{N_{k-1}}-2\psi_{N_{k-1}}}\}$ is an increasing sequence with $\lim_{n\to\infty} \sum_{k\in\mathcal{O}_{h,n}} \xi(N_{k-1}) N_{k-1,+}^{-\beta_{N_{k-1}}-2\psi_{N_{k-1}}} \overset{a.s}{=} \infty$, and for a fixed h_0,

$\sum_{k\in\mathcal{O}_{h_0,\infty}} \xi(N_{k-1})N_{k-1,+}^{-\beta_{N_{k-1}}-2\psi_{N_{k-1}}}[\sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\beta_{N_{l-1}}-2\psi_{N_{l-1}}}]^{-2} < \infty$, a.s. In the particular case $\xi(N) = N_+$ with $\beta_N + 2\psi_N = 1$ and h fixed, these properties are always satisfied. We see in section 2 that these properties are satisfied in the general case under some weak conditions.

2. Identifiability and strong consistency

Let $m(N)$ depend on (θ_0, ν_0), where $\theta_0 \in \overset{\circ}{\Theta}$, Θ being a bounded set of $\mathbf{R}^d$, $d \in \mathbf{N}$. Denote $m_{\theta_0,\nu_0}(N)$ for $m(N)$.
Let $B_\delta^c = \{\theta = (\theta_1, \dots, \theta_d) \in \Theta : \sum_{k=1}^d |\theta_k - \theta_{0,k}| \geq \delta\}$, where $\delta > 0$ and let $\|.\|_n$ be a semi-norm on the space of functions $\{f(k)\}_{k\leq n}$.

Throughout this article, we give a value to the unknown nuisance parameter ν: when it is finite dimensional, it can be any estimator; and when it is infinite dimensional, we set $m_{\theta,\nu}^{(2)}(.) = 0$. For simplifying the notations, we write $m_{\theta,\nu}(N_n)$ and $\widetilde{S}_{h,n,\nu}(\theta)$ even if ν depends on n and we write ψ_{k-1}, β_{k-1} instead of $\psi_{N_{k-1}}$, $\beta_{N_{k-1}}$.

Definition 1. *(i) θ_0 is identifiable in $\{m_{\theta_0,\nu}(N_k)\}_{k\leq n}$ for the semi-norm $\|.\|_n$ if there exists a function $v(N_.)$ such that, for all $\delta > 0$, $\inf_{\theta\in B_\delta^c} \|[m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.)]v(N_.)\|_n \neq 0$ a.s..*
(ii) θ_0 is asymptotically identifiable in $m_{\theta_0,\nu}(N_.)$ for $\{\|.\|_n\}_n$ if there exists $v(N_.)$ such that, for all $\delta > 0$, $\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} \|[m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.)]v(N_.)\|_n \neq 0$ a.s.; if moreover this quantity is finite, $v(N_.)$ is called the rate of identifiability of θ_0.

Definition 2. *$g_\nu(.)$ is asymptotically negligible if $\overline{\lim}_{n\to\infty}\|g_\nu(.)\|_n \overset{a.s.}{=} 0$.*

Definition 3. *Let $\{\widehat{\theta}_n\}_n$ an estimator of θ_0. This estimator is called a.s. $m_{\theta_0,\nu}(N_.)$-consistent for $\{\|.\|_n\}_n$ if there exists $v(N_.)$ such that*
$\lim_{n\to\infty} \|[m_{\theta_0,\nu}(N_.) - m_{\widehat{\theta}_n,\nu}(N_.)]v(N_.)\|_n \overset{a.s.}{=} 0$.

Lemma 1. *Let $\mathcal{M}_{v,\nu} = \{\{\widehat{\theta}_n\}_n : \{\widehat{\theta}_n\}_n$ is a.s. $m_{\theta_0,\nu}(N_.)$-consistent at the rate $v(N_.)\}$. Then, θ_0 is asymptotically identifiable in $m_{\theta_0,\nu}(N_.)$ at the rate $v(N_.)$ for $\{\|.\|_n\}_n$ implies that, for all $\{\widehat{\theta}_n\}_n \in \mathcal{M}_{v,\nu}$, $\{\widehat{\theta}_n\}_n$ is consistent.*
The reciprocal implication holds true if we assume in addition condition C: for all closed set $F \subset \Theta$, $\inf_{\theta\in F} \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n$ is attained at some $\widetilde{\theta}$.

Proof. Assume that there exists a sequence $\{\widehat{\theta}_n\}_n \in \mathcal{M}_{v,\nu}$ that is not a.s. consistent. Then, for all $\delta > 0$, there exists n_δ and a subsequence $\{\widehat{\theta}_{n_i}\}_{n_i>n_\delta}$ such

that $\widehat{\theta}_{n_i} \in B_\delta^c$. Since $\{\widehat{\theta}_n\}_n \in \mathcal{M}_{v,\nu}$, $\lim_{n\to\infty} \|(m_{\theta_0,\nu}(N_.) - m_{\widehat{\theta}_n,\nu}(N_.))v(N_.)\|_n \overset{a.s.}{=} 0$ entailing that $\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n \overset{a.s.}{=} 0$. Hence, θ_0 is not asymptotically identifiable at the rate $v(N_.)$. Conversely, assume that condition C holds and that θ_0 is not asymptotically identifiable in $m_{\theta_0,\nu}(.)$ at the rate $v(N_.)$. Then, there exists $\delta > 0$ such that $\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n \overset{a.s}{=} 0$. Let $\theta_n = \text{argmin}_{\theta\in B_\delta^c}\|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n$. The sequence $\{\theta_n\}_n$ belongs to $\mathcal{M}_{v,\nu}$ but is not consistent since it belongs to B_δ^c.
□

Since $\widetilde{S}_{h,n,\nu}(\theta)$ defined by (7) may be written as

$$\widetilde{S}_{h,n,\nu}(\theta) = \sum_{k\in\mathcal{O}_{h,n}} \Big(\frac{X_k}{\xi(N_{k-1})^{1/2} N_{k-1,+}^{\frac{-\beta_{k-1}-2\psi_{k-1}}{2}}} + \delta_{k,\theta,\nu}\Big)^2 \times \xi(N_{k-1}) N_{k-1,+}^{-2\psi_{k-1}-\beta_{k-1}}. \tag{10}$$

where $X_k = [N_k - m_{\theta_0,\nu_0}(N_{k-1})N_{k-1}]\xi(N_{k-1})^{-1/2}N_{k-1,+}^{-\beta_{k-1}/2}$ and
$\delta_{k,\theta,\nu} = (m_{\theta_0,\nu_0}(N_{k-1}) - m_{\theta,\nu}(N_{k-1}))N_{k-1,+}^{\psi_{k-1}}$, the estimator $\widehat{\theta}_{h,n,\nu}$ also satisfies $\widehat{\theta}_{h,n,\nu} = \text{argmin}_{\theta\in\Theta} S_{h,n,\nu}(\theta)$, where
$S_{h,n,\nu}(\theta) = \widetilde{S}_{h,n,\nu}(\theta)[\sum_{k\in\mathcal{O}_{h,n}} \xi(N_{k-1})N_{k-1,+}^{-2\psi_{k-1}-\beta_{k-1}}]^{-1}$.

This leads to the natural following semi-norm $\{\|.\|_n\}_n$:

$$\|u(N_.)\|_n^2 = \frac{\sum_{k\in\mathcal{O}_{h,n}} u^2(N_{k-1})\xi(N_{k-1})N_{k-1,+}^{-2\psi_{k-1}-\beta_{k-1}}}{\sum_{k\in\mathcal{O}_{h,n}} \xi(N_{k-1})N_{k-1,+}^{-2\psi_{k-1}-\beta_{k-1}}}. \tag{11}$$

We then show in the following proposition that if θ_0 is asymptotically identifiable in $m_{\theta_0,\nu}(N_.)$ for the sequence $\{\|.\|_n\}_n$ at the rate $N_.^{\psi_.}$ and if $m_{\theta,\nu}^{(2)}(N_.)N_.^{\psi_.}$ is asymptotically negligible, then the strong consistency of $\{\widehat{\theta}_{h,n,\nu}\}_n$ is ensured under some general conditions on the behavior of the process. Let $\|u(N_.)\|_{n,\infty} = \sup_{k\in\mathcal{O}_{h,n}} |u(N_{k-1})|$. Denote $\varepsilon_k = \beta_k + 2\psi_k$, for all k, and $D_n = \sum_{k\in\mathcal{O}_{h,n}} \xi(N_{k-1})N_{k-1,+}^{-2\psi_{k-1}-\beta_{k-1}}$.

Proposition 1. *Let $\|.\|_n$ be defined by (11). Assume the following conditions:*
1) θ_0 is asymptotically identifiable in $m_{\theta_0,\nu}(N_.)$ at the exact rate $v(N_.) = N_.^{\psi_.}$ for the semi-norm $\{\|.\|_n\}_n$, ie
B1: $\forall\delta > 0$, $\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} \|(m_{\theta_0,\nu_.}(N_.) - m_{\theta,\nu_.}(N_.))v(N_.)\|_n \overset{a.s.}{\neq} 0$;
B2: $\overline{\lim}_{n\to\infty} \sup_{\theta\in B_\delta^c} \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_{n,\infty} \overset{a.s.}{<} \infty$ a.s.;

2) $(m_{\theta_0,\nu_0}(N_.) - m_{\theta_0,\nu}(N_.))v(N_.)$ *is asymptotically negligible, ie*
B3: $\overline{\lim}_{n\to\infty}\|(m_{\theta_0,\nu_0}(N_.) - m_{\theta_0,\nu}(N_.))v(N_.)\|_n \overset{a.s.}{=} 0$*;*
3) B4: (i) $\lim_{n\to\infty} D_n \overset{a.s.}{=} \infty$*;* $\{D_n\}_n$ *is an increasing sequence;*
(ii) There exists h_0 *such that* $\sum_{k\in\mathcal{O}_{h_0,\infty}} \xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}} D_k^{-2} \overset{a.s.}{<} \infty$*;*
4) B5: for all $\delta > 0$*, for all* N*,* $\sup_{\theta\in B_\delta^c}(m_{\theta_0,\nu}(N) - m_{\theta,\nu}(N))$ *is attained for some* θ_N^{sup} *(respectively* $\inf_{\theta\in B_\delta^c}(m_{\theta_0,\nu}(N) - m_{\theta,\nu}(N))$ *is attained for some* θ_N^{inf}*).*
Then, $\{\widehat{\theta}_{h,n,\nu}\}_n$ *is strongly consistent on* $\mathcal{E}_\infty$*.*

Remarks. 1. Let $\xi(N) = N$, for all N. Assume that there exists a random variable W with $\mathcal{E}_\infty = \{W > 0\}$ and a deterministic sequence $\{a_n\}_n$ such that $\lim_n N_n a_n^{-1} \overset{a.s.}{=} W$. Then it is sufficient to check $B4$ in which N_k is replaced by a_k, for all k.

2. Let h fixed with $(a) : \xi(N)N_+^{-\varepsilon_N} \geq 1$, for all N and either $(b1)$ or $(b2)$:

$$(b1) : \sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}} \overset{a.s.}{=} O(card(\mathcal{O}_{h,k}))$$

$$(b2) : \exists u > 0 : \sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}} \overset{a.s.}{\geq} O(card(\mathcal{O}_{h,k}))^{1+u}.$$

Then $B4$ (i) is satisfied under (a). As concerning $B4$ (ii), denote
$A = \sum_{k\in\mathcal{O}_{h,\infty}} \xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}(\sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}})^{-2}$. Then

$$\begin{aligned} A &= \sum_{k\mathcal{O}_{h,\infty}} \frac{1}{(\sum_{l\in\mathcal{O}_{h,k}} N_{l-1,+}^{-\varepsilon_{l-1}})^{1+x_k}} \frac{\xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}}{(\sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}})^{1-x_k}} \\ &= \sum_{k\cap\mathcal{O}_{h,\infty}} \frac{1}{(\sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}})^{1+x_k}}, \end{aligned}$$

where x_k is defined by

$$x_k = 1 - \frac{\ln \xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}}{\ln \sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}}}.$$

Under $(b1)$, since $card(\mathcal{O}_{h,k}) \leq \sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}} \overset{a.s.}{=} O(card(\mathcal{O}_{h,k}))$, there necessarily exists M such that $\xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}} \leq M$, for all $l \in \mathcal{O}_{h,\infty}$. This implies, that $\lim_{k\to\infty} x_k = 1$. Consequently there exists $u > 0$ and $0 < c < \infty$ such that

$$(12) \sum_{k\in\mathcal{O}_{h,\infty}} \frac{1}{(\sum_{l\in\mathcal{O}_{h,k}} \xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}})^{1+x_k}} \leq c + \sum_{k\in\mathcal{O}_{h,\infty}} \frac{1}{(card(\mathcal{O}_{h,k}))^{1+u}} < \infty.$$

Now, assuming $(b2)$

$$(13)\quad \sum_{k\in\mathcal{O}_{h,\infty}} \frac{1}{(\sum_{l\in\mathcal{O}_{h,k}}\xi(N_{l-1})N_{l-1,+}^{-\varepsilon_{l-1}})^{1+x_k}} \le c \sum_{k\in\mathcal{O}_{h,\infty}} \frac{1}{(card(\mathcal{O}_{h,k}))^{1+u}} < \infty.$$

Then (12) or (13) imply $B4$ (ii).

Proof. We apply Wu's lemma [55] on the normalized contrast $S_{h,n,\nu}(\theta)$: if for all $\delta > 0$, $\underline{\lim}_{n\to\infty}(\inf_{\theta\in B_\delta^c} S_{h,n,\nu}(\theta) - S_{h,n,\nu}(\theta_0)) > 0$ a.s., then $\{\widehat{\theta}_{h,n,\nu}\}_n$ is strongly consistent (proof: assume that $\widehat{\theta}_{h,n,\nu}$ is not a.s. consistent. Then there exist δ and an infinite subsequence $\{\widehat{\theta}_{h,n_i,\nu}\}_{n_i}$ such that $\widehat{\theta}_{h,n_i,\nu} \in B_\delta^c$ implying that $S_{h,n_i,\nu}(\widehat{\theta}_{h,n_i,\nu}) > S_{h,n_i,\nu}(\theta_0)$, which is in contradiction with the definition of $\widehat{\theta}_{h,n_i,\nu}$).

Let $\delta > 0$. According to (10),

$$S_{h,n,\nu}(\theta) - S_{h,n,\nu}(\theta_0) = S_{1n}(\theta) + 2S_{2n}(\theta) + 2S_{3n}(\theta),$$

where

$$S_{1n}(\theta) = \sum_{k\in\mathcal{O}_{h,n}} (m_{\theta_0,\nu}(N_{k-1}) - m_{\theta,\nu}(N_{k-1}))^2 N_{k-1,+}^{2\psi_{k-1}}\xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}D_n^{-1},$$

$$S_{2n}(\theta) = \sum_{k\in\mathcal{O}_{h,n}} \delta_{k,\theta_0,\nu}(m_{\theta_0,\nu}(N_{k-1}) - m_{\theta,\nu}(N_{k-1}))N_{k-1,+}^{\psi_{k-1}}\xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}D_n^{-1},$$

$$S_{3n}(\theta) = \sum_{k\in\mathcal{O}_{h,n}} \frac{X_k}{\xi(N_{k-1})^{\frac{1}{2}}N_{k-1,+}^{\frac{-\varepsilon_{k-1}}{2}}}(m_{\theta_0,\nu}(N_{k-1}) - m_{\theta,\nu}(N_{k-1})) \times$$
$$\times N_{k-1,+}^{\psi_{k-1}}\xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}D_n^{-1}.$$

We study successively each term $S_{in}(\theta)$, $i \in \{1,2,3\}$.

- Since $S_{1n}(\theta) \ge \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n^2$, then

$$\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} S_{1n}(\theta) \ge \underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n^2.$$

According to $B1$, the right-hand side is strictly positive yielding

$$(14)\quad \underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} S_{1n}(\theta) > 0 \text{ a.s.}$$

- According to Hölder inequality,

$$|S_{2n}(\theta)| \le \|(m_{\theta_0,\nu_0}(N_.) - m_{\theta_0,\nu}(N_.))v(N_.)\|_n \times$$
$$\times\|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n.$$

Using the relationship $|\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} S_{2n}(\theta)| \le \overline{\lim}_{n\to\infty} \sup_{\theta\in B_\delta^c} |S_{2n}(\theta)|$,

$$\begin{aligned}|\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} S_{2n}(\theta)| \quad &\le \quad \overline{\lim}_{n\to\infty} \|(m_{\theta_0,\nu_0}(N_.) - m_{\theta_0,\nu}(N_.))v(N_.)\|_n \times \\ &\times \overline{\lim}_{n\to\infty} \sup_{\theta\in B_\delta^c} \|(m_{\theta_0,\nu}(N_.) - m_{\theta,\nu}(N_.))v(N_.)\|_n.\end{aligned}$$

The right-hand side is equal to 0 thanks to $B2$ and $B3$, implying

$$\underline{\lim}_{n\to\infty} \inf_{\theta\in B_\delta^c} S_{2n}(\theta) \overset{a.s.}{=} 0. \tag{15}$$

- Let

$$\begin{aligned}Z_k(\theta,\nu) \quad &= \quad \frac{X_k}{\xi(N_{k-1})^{\frac{1}{2}} N_{k-1,+}^{\frac{-\varepsilon_{k-1}}{2}}}(m_{\theta_0,\nu}(N_{k-1}) - m_{\theta,\nu}(N_{k-1})) \times \\ &\times N_{k-1,+}^{\psi_{k-1}} \xi(N_{k-1}) N_{k-1,+}^{-\varepsilon_{k-1}}.\end{aligned}$$

According to condition $B5$, denoting $(\widetilde{\theta}_k,\nu) = \text{argmin}_{\theta\in B_\delta^c} Z_k(\theta,\nu)$ and $(\overline{\theta}_k,\nu) = \text{argmax}_{\theta\in B_\delta^c} Z_k(\theta,\nu)$,

$$\frac{\sum_{k\in\mathcal{O}_{h,n}} Z_k(\widetilde{\theta}_k,\nu)}{D_n} \le \inf_{\theta\in B_\delta^c} S_{3n}(\theta) \le \frac{\sum_{k\in\mathcal{O}_{h,n}} Z_k(\overline{\theta}_k,\nu)}{D_n}.$$

Assume first h constant with $h_0 = h$. Let us show that $\{\sum_{k\in\mathcal{O}_{h,n}} Z_k(\overline{\theta}_k,\nu)\}_n$ (respectively $\{\sum_{k\in\mathcal{O}_{h,n}} Z_k(\widetilde{\theta}_k,\nu)\}_n$) is a martingale with respect to the filtration $\mathbf{F} = \{F_n\}_n$, , F_n being the σ-algebra generated by $\{N_0,\ldots,N_n\}$. We will then use a strong law of large numbers for martingales (theorem 2.18 of Hall and Heyde [15]) to show that $\lim_n[\sum_{k\in\mathcal{O}_{h,n}} Z_k(\overline{\theta}_k,\nu)]D_n^{-1} \overset{a.s.}{=} 0$ (respectively $\lim_n[\sum_{k\in\mathcal{O}_{h,n}} Z_k(\widetilde{\theta}_k,\nu)]D_n^{-1} \overset{a.s.}{=} 0$).
Let $L_n = \sum_{k\in\mathcal{O}_{h,n}} Z_k(\overline{\theta}_k,\nu)$. Since $E(|Z_k(\overline{\theta}_k,\nu)\||F_{k-1}) < \infty$, for all k, we have

$$\begin{aligned}&E(Z_n(\overline{\theta}_n,\nu)|F_{n-1}) = E[E(Z_n(\overline{\theta}_n,\nu)|F_{n-1},\overline{\theta}_n,\nu)|F_{n-1}] = \xi(N_{n-1})^{1/2}\times \\ &\quad \times N_{n-1,+}^{\frac{-\varepsilon_{k-1}}{2}} E[(m_{\theta_0,\nu}(N_{n-1}) - m_{\overline{\theta}_n,\nu}(N_{n-1}))E(X_n|F_{n-1},\overline{\theta}_n,\nu)|F_{n-1}].\end{aligned}$$

The right-hand side equals zero since on one hand X_n is independent of $\overline{\theta}_n$ and ν, and $E(X_n|F_{n-1}) = 0$ on the other hand. Consequently L_n is a martingale.

Let $s_k = E(Z_k^2(\overline{\theta}_k,\nu)|F_{k-1})D_k^{-2}$. The sequence $\{D_k\}_k$ increases to ∞ under $B4$(i). Moreover

$$\sum_{k\in\mathcal{O}_{h,\infty}} s_k \leq \sigma^2 \sup_{\theta\in B_\delta^c}\sup_{h+1\leq k}(m_{\theta_0,\nu}(N_{k-1})-m_{\theta,\nu}(N_{k-1}))^2 N_{k-1,+}^{2\psi_{k-1}}.$$

$$\sum_{k\in\mathcal{O}_{h,\infty}} \frac{\xi(N_{k-1})N_{k-1,+}^{-\varepsilon_{k-1}}}{D_k^2}$$

that is finite according to $B2$ and $B4$ (ii).

Assume now $n-h$ constant and denote $c=n-h$. Since h depends on n, L_n is no longer a martingale. But $L_n = \sum_{k\in\mathcal{O}_{c,n}} Z_k(\overline{\theta}_k,\nu) - \sum_{k\in\mathcal{O}_{c,h}} Z_k(\overline{\theta}_k,\nu)$, where each of the two terms of this difference is a martingale. Then assuming $B4$ (i) and $B4$ (ii) with $h_0=c$, we have as previously,

$$\lim_{n\to\infty}[\sum_{k\in\mathcal{O}_{c,n}} Z_k(\overline{\theta}_k,\overline{\nu}_k)]D_n^{-1} \overset{a.s.}{=} 0,$$

and using $D_n \geq D_h$, $\lim_{n\to\infty}[\sum_{k\in\mathcal{O}_{c,h}} Z_k(\overline{\theta}_k,\overline{\nu}_k)]D_h^{-1} \overset{a.s.}{=} 0$. Hence we get

$$\lim_{n\to\infty}\inf_{\theta\in B_\delta^c} S_{3n}(\theta) \overset{a.s.}{=} 0. \tag{16}$$

Relationships (14), (15) and (16) entail that Wu's lemma is satisfied. □

Remark. In the frame of (4) with $f(\mu_i,n)=\mu_i$, assume $O_i(N_+^{-\overline{\alpha}_i})=0$, for all N, and $\{N_n\to\infty\}\neq\emptyset$. Since (m_0,μ_0) is not asymptotically identifiable on $\{\overline{\lim}_n N_n=\infty\}$, we have to consider separately on this set the consistency of (8) and (9): $\widehat{m}_{i;h,n}$ will be strongly consistent under conditions $B4$, $B5$ of the proposition, if $(\widehat{\mu}_{i;h,n}-\mu_i)N_+^{-\alpha_i}$ is asymptotically negligible, which is easily checked, and $\widehat{\mu}_{i;h,n}$ will be strongly consistent if $(\widehat{m}_{i;h,n}-m_i)N_+^{\alpha_i}$ is asymptotically negligible, which requires a rate of convergence for the estimator of m_i sufficiently large. Now if $\{\overline{\lim}_n N_n\to\infty\}=\emptyset$, then (m_0,μ_0) is asymptotically identifiable on $\mathcal{E}_\infty$ implying the strong consistency of $(\widehat{m}_{i;h,n},\widehat{\mu}_{i;h,n})$ under $B4$, $B5$.

3. Rate of convergence

In order to study the asymptotic distribution of $\widehat{\theta}_{h,n,\nu}-\theta_0$ properly normalized, we consider the more accurate model

$$m_i(N) = m_i + \frac{\mu_i}{N_+^{\alpha_i}} + r_i(N),$$

$$\begin{array}{rcl} & & r_i(N) = O(N_+^{-\overline{\alpha_i}}), 0 < \alpha_i < \overline{\alpha_i}, \sup_\theta r_i(N) N^{\overline{\alpha}_i} \le M \\ \sigma^2(N) & = & \sigma^2 N^{\beta_i} - r_{i+}(N), r_+(N) = O(N^{\overline{\beta}_i}), \overline{\beta}_i < \beta_i, r_{i+}(.) \ge 0 \\ \sigma_{i,j}(N) & = & \sigma^2 N_+^{-1+\beta_i} \\ \exists \{a_{i,n}\}_n & : & \lim_{n: N_n \in \mathbf{N}_i} N_n a_{i,n}^{-1} \overset{a.s.}{=} W_i, \end{array}$$

where $\{a_{i,n}\}_n$ is a deterministic sequence. We will assume here that θ_i is equal either to m_i or to μ_i.

Since $\widetilde{S}_{h,n,\nu}(\theta) = \sum_i \widetilde{S}_{i;h,n,\nu}(\theta_i)$, we will study the rate of convergence of each $\widetilde{\theta}_{i;h,n,\nu_i}$, using the Taylor's series expansion at first order applied to $\widetilde{S}'_{i;h,n,\nu}(\theta_i)$, where the $'$ denotes the derivative with respect to θ_i:

$$(\widehat{\theta}_{i;h,n,\nu} - \theta_{i;0}) = -\frac{\widetilde{S}'_{i;h,n,\nu}(\theta_{i;0})}{\widetilde{S}''_{i;h,n,\nu}(\widetilde{\theta}_{i;n})}, \tag{17}$$

where $\widetilde{\theta}_{i;n}$ is a value between $\theta_{i;0}$ and $\widehat{\theta}_{i;h,n,\nu}$.

Then we will show that $\widetilde{S}'_{i;h,n,\nu}(\theta_{i;0})$ properly normalized converges in distribution, whereas $\widetilde{S}''_{i;h,n,\nu}(\widetilde{\theta}_{i;n})$ properly normalized converges a.s.. For simplifying the notations, since the proof is done separately on each $i \in \mathcal{I}$, we will forget the subscript "i" in the following and we will assume $card\mathcal{O}_{h,n} = [h+1, n]$.

Define $B_n = \sqrt{\sum_{k=h+1}^n a_{k-1}^{1-\beta-2\psi}}$ and let the assumptions
$B6$: when h is fixed, $\lim_{n\to\infty} B_n = \infty$;
$B7$:there exists a random variable Γ such that $\lim_{n\to\infty} B_n^{-1}\Sigma^1\theta_{k,i} \overset{P}{=} \Gamma$, where $\theta_{k,i} = E(\xi_{k,i}|F_{k-1}) = \overline{r}_{\theta_0,\nu}(N_{k-1}) m'_{\theta_0,\nu}(N_{k-1}) N_{k-1}^{-\beta} 1_{\{k\ge h+1\}}$, $\overline{r}_{\theta,\nu}(.) = r_{\theta_0,\nu_0}(.) - r_{\theta,\nu}(.)$.
$B8$ (Lindeberg's condition):

$$\forall x, \lim_{H\to\infty} \lim_{n\to\infty} \sup_{H+1\le k\le n} E(R_k^2 1_{\{R_k^2 \ge B_{k,n}^2 x^2\}} | F_{k-1}) \overset{a.s.}{=} 0,$$

where $R_k = [Y_{k,1} - m_{\theta_0,\nu_0}(N_{k-1})] m'_{\theta_0,\nu}(N_{k-1}) W_{k-1}^{-\beta} a_{k-1}^{\frac{1}{2}(-\beta+2\psi)}$ and $B_{k,n} = B_n a_{k-1}^{\frac{1}{2}[\beta+2\psi]}$.
$B9$: (i) $\theta \mapsto r_{\theta,\nu}(.)$ is twice continuously differentiable in a neighborhood of θ_0 and there exist $M' < \infty$, $M'' < \infty$ such that $\sup_\theta |r'_{\theta,\nu}(N)| N^{\overline{\psi}} \le M'$ and $\sup_\theta |r''_{\theta,\nu}(N)| N^{\overline{\psi}} \le M''$, for all N, where $\overline{\psi} = \alpha$, when $\theta = m$ and $\overline{\psi} = \overline{\alpha}$, when $\theta = \mu$;

(ii) for all N, $\sup_\theta r''_{\theta,\nu}(N)$ is attained for some θ_N^{sup} (respectively $\inf_\theta r''_\theta(N)$ is attained for some θ_N^{inf}).

Then according to (17), $\Phi_{h,n}^{-1}(\widehat{\theta}_{h,n,\nu} - \theta_0) = P_{h,n,\nu}[Q_{h,n,\nu}]^{-1}$, where

$$P_{h,n,\nu} = \widetilde{S}'_{h,n,\nu}(\theta_{i;0})B_n^{-1} = [\sum_{k=h+1}^{n} \sum_{i=1}^{N_{k-1}} [Y_{k,i} - m_{\theta_0,\nu}(N_{k-1})] \times$$
$$\times m'_{\theta_0,\nu}(N_{k-1})N_{k-1}^{-\beta}]B_n^{-1}$$

$\Phi_{h,n} = B_n D_n^{-1}$, and

$$(18) \quad Q_{h,n,\nu} = \frac{\widetilde{S}''_{h,n,\nu}(\widetilde{\theta}_{i;n})}{D_n} = \frac{\sum_{k=h+1}^{n}[\sum_{i=1}^{3} Ui_k(\theta_n)]W_{k-1}^{1-2\psi-\beta}a_{k-1}^{1-2\psi-\beta}}{D_n},$$

with $U1_k(\theta_n) = m'^2_{\theta_n,\nu}(N_{k-1})N_{k-1}^{2\psi}$,
$U2_k(\theta_n) = -(m_{\theta_0,\nu_0}(N_{k-1}) - m_{\theta_n,\nu}(N_{k-1}))N_{k-1}^{\psi}m''_{\theta_n,\nu}(N_{k-1})N_{k-1}^{\psi}$, and
$U3_k(\theta_n) = -X_k N_{k-1,+}^{\frac{\beta+2\psi-1}{2}} m''_{\theta_n,\nu}(N_{k-1})N_{k-1,+}^{\psi}$.

Proposition 2. *Assume B1 to B9. Then*

$$(19) \quad \lim_{n\to\infty} P_{h,n,\nu} \stackrel{d}{=} P,\ E[\exp(itP)] = E[\exp(-it\Gamma - \frac{t^2}{2}\sigma^2 W^{1-\beta-2\psi})],$$

$$(20) \qquad \lim_{n\to\infty} Q_{h,n,\nu} \stackrel{a.s.}{=} W^{1-\beta-2\psi}.$$

Remark. If $\Gamma = 0$ and $W^{1-\beta-2\psi} \stackrel{a.s.}{=} 1$, then $\lim_{n\to\infty} P_{h,n,\nu} \stackrel{d}{=} N(0,\sigma^2)$ and $\lim_{n\to\infty} Q_{h,n,\nu} \stackrel{a.s.}{=} 1$. Slutsky's theorem entails $\lim_{n\to\infty} P_{h,n,\nu}[Q_{h,n,\nu}]^{-1} \stackrel{d}{=} N(0,\sigma^2)$.

Proof.

1. Asymptotic distribution of $P_{h,n,\nu}$.

We check assumptions 1.27 to 1.30 of theorem 1.3 of Rahimov [46]. Let $\xi_{k,i} = [Y_{k,i} - m_{\theta_0,\nu}(N_{k-1})]m'_{\theta_0,\nu}(N_{k-1})N_{k-1}^{-\beta}1_{\{k\geq h+1\}}$,
$\sigma_{k,i}^2 = E((\xi_{k,i} - \theta_{k,i})^2|F_{k-1}) = \sigma^2(N_{k-1})m'^2_{\theta_0,\nu}(N_{k-1})N_{k-1}^{2(-\beta)}1_{\{k\geq h+1\}}$ and $\Sigma^1 = \sum_{k=1}^{n}\sum_{i=1}^{N_{k-1}}$.

• condition 1.27: there exists a random variable T such that $\lim_{n\to\infty} P(B_n^{-2}\Sigma^1\sigma_{k,i}^2 > T) = 0$. Here,

$$\frac{1}{B_n^2}\Sigma^1\sigma_{k,i}^2 = \frac{1}{B_n^2}\sum_{k\in\mathcal{O}_{h,n}} \sigma^2(N_{k-1})N_{k-1}^{-\beta}W_{k-1}^{1-\beta-2\psi}m'^2_{\theta_0,\nu}(N_{k-1})N_{k-1}^{2\psi}a_{k-1}^{1-\beta-2\psi}.$$

Thanks to $B9$(i) and using Toeplitz lemma and $B6$, we obtain $\lim_{n\to\infty} B_n^{-2}\Sigma^1\sigma_{k,i}^2 \overset{a.s.}{=} \sigma^2 W^{1-\beta-2\psi}$.

• condition 1.28 : $\lim_{n\to\infty} \max_{h+1\le k\le n}\sigma_{k,i}^2 B_n^{-2} \overset{P}{=} 0$. We have

$$\max_{h+1\le k\le n} \frac{\sigma^2(N_{k-1})N_{k-1}^{-2\beta)}m'^2_{\theta_0,\nu}(N_{k-1})}{B_n^2} \le$$
$$\sigma^2 \max_{h+1\le k\le n} W_{k-1}^{-\beta-2\psi} m'^2_{\theta_0,\nu}(N_{k-1})N_{k-1}^{2\psi}.\frac{\max_{h+1\le k\le n} a_{k-1}^{-\beta-2\psi}}{B_n^2}.$$

We obtain $\lim_{n\to\infty} \max_{h+1\le k\le n} W_{k-1}^{-\beta-2\psi} m'^2_{\theta_0,\nu}(N_{k-1})N_{k-1}^{2\psi} < \infty$ and $\lim_{n\to\infty} \max_{h+1\le k\le n} a_{k-1}^{-\beta-2\psi} B_n^{-2} = 0$, which imply

$$\lim_{n\to\infty} \max_{h+1\le k\le n} \sigma^2(N_{k-1})N_{k-1}^{-2\beta} m'^2_{\theta_0,\nu}(N_{k-1})B_n^{-2} \overset{a.s.}{=} 0. \tag{21}$$

• condition 1.29: there exists a random variable Γ such that $\lim_{n\to\infty} B_n^{-1} \Sigma^1\theta_{k,i} \overset{P}{=} \Gamma$. This is the assumption $B7$.

• condition 1.30: there exists a function $x \mapsto K(x)$ such that, for all real x,

$$\lim_{n\to\infty} \Sigma^1 E([\frac{\xi_{k,i}-\theta_{k,i}}{B_n}]^2 1_{\{\frac{\xi_{k,i}-\theta_{k,i}}{B_n}<x\}}|F_{k-1}) \overset{P}{=} K(x).$$

We define R_k from $\xi_{k,i}$ such that R_k has finite variance:

$$R_k = (\xi_{k,1}-\theta_{k,1})a_{k-1}^{\frac{1}{2}[\beta+2\psi]}; \qquad \textit{let } B_{k,n} = B_n a_{k-1}^{\frac{1}{2}[\beta+2\psi]}.$$

(i) Assume first h constant.
Denote $A_n = \Sigma^1 B_n^{-2}E((\xi_{k,i}-\theta_{k,i})^2 1_{\{(\xi_{k,i}-\theta_{k,i})B_n^{-1}<x\}}|F_{k-1})$. Therefore $A_n = \Sigma^1 B_{k,n}^{-2}E(R_k^2 1_{\{R_k B_{k,n}^{-1}<x\}}|F_{k-1})$. Let us show that, for all $x>0$, $\{A_n\}_n$ converges a.s. to $\sigma^2 W^{1-\beta-2\psi}$ and that, for all $x<0$, it converges a.s. to 0.

• Let $x>0$, $H\in\mathbf{N}$, $\Delta_n = |A_n - \sigma^2 W_{k-1}E(R_k^2|F_{k-1}) + \sigma^2 W^{1-\beta-2\psi}|$.

$$\Delta_n \le (\sup_{h+1\le k\le H} W_{k-1}E(R_k^2|F_{k-1}) + \sigma^2 W^{1-2\psi-\beta})\frac{\sum_{k=h+1}^H a_{k-1}^{1-\beta-2\psi}}{B_n^2} +$$
$$\sup_{H+1\le k\le n} |W_{k-1}E(R_k^2 1_{\{R_k<B_{k,n}x\}}|F_{k-1}) - \sigma^2 W^{1-\beta-2\psi}|.$$

Using $E(R_k^2|F_{k-1}) = \sigma^2(N_{k-1})N_{k-1}^{-\beta}W_{k-1}^{-\beta-2\psi}m'^2_{\theta_0,\nu}(N_{k-1})N_{k-1}^{2\psi}$, we get

$$\sup_{h+1\le k\le H} W_{k-1}E(R_k^2|F_{k-1}) < \infty.$$

Using $\lim_n B_n = \infty$, the first term of the right-hand side of the inequality converges a.s. to 0 as $n \to \infty$. As concerning the second term,

$$\sup_{H+1\le k\le n} |W_{k-1}E(R_k^2 1_{\{R_k<B_{k,n}x\}}|F_{k-1}) - \sigma^2 W^{1-\beta-2\psi}| \le$$
$$\sup_{H+1\le k\le n} W_{k-1} \sup_{H+1\le k\le n} |E(R_k^2 1_{\{R_k<B_{k,n}x\}}|F_{k-1}) - \sigma^2 W^{-\beta-2\psi}| +$$
$$\sigma^2 W^{-\beta-2\psi} \sup_{H+1\le k\le n} |W_{k-1} - W|.$$

We have $\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n} W_{k-1} < \infty$ a.s. and $\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n}|W_{k-1} - W| \overset{a.s.}{=} 0$. Let us show that $\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n}|E(R_k^2 1_{\{R_k<B_{k,n}x\}}|F_{k-1}) - \sigma^2 W^{-\beta-2\psi}| \overset{a.s.}{=} 0$.

$$\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n} |E(R_k^2 1_{\{R_k<B_{k,n}x\}}|F_{k-1}) - \sigma^2 W^{-\beta-2\psi}| \le$$
$$\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n} |E(R_k^2|F_{k-1}) - \sigma^2 W^{-\beta-2\psi}| +$$
$$\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n} E(R_k^2 1_{\{R_k\ge B_{k,n}x\}}|F_{k-1}).$$

The first term of the right-hand side converges a.s. to 0 by letting first $n \to \infty$ and then $H \to \infty$ and according to $B8$, $\lim_{H\to\infty}\lim_{n\to\infty}\sup_{H+1\le k\le n} E(R_k^2 1_{\{R_k\ge B_{k,n}x\}}|F_{k-1}) \overset{a.s.}{=} 0$.

- Let $x < 0$. Assumption $B8$ entails that $\lim_{n\to\infty} A_n \overset{a.s.}{=} 0$.

(ii) Consider now $n - h$ constant.

The proof is the same as in the case h fixed. Hence, under $B1$ to $B8$, Rahimov's theorem [46] ensures that (19) holds.

2. Study now the asymptotic behavior of $\{Q_{h,n,\nu}\}_n$. We first show

$$(22)\quad \lim_{n\to\infty} \frac{\sum_{k=h+1}^n [U1_k(\theta_n) + U2_k(\theta_n)]W_{k-1}^{1-2\psi-\beta}a_{k-1}^{1-2\psi-\beta}}{D_n} \overset{a.s.}{=} W^{1-2\psi-\beta},$$

and then we show

$$(23)\quad \lim_{n\to\infty} \frac{\sum_{k=h+1}^n U3_k(\theta_n)W_{k-1}^{1-2\psi-\beta}a_{k-1}^{1-2\psi-\beta}}{D_n} \overset{a.s.}{=} 0.$$

Let us prove (22).

$$\left|\frac{\sum_{k=h+1}^{n}[U1_k(\theta_n)+U2_k(\theta_n)]W_{k-1}^{1-2\psi-\beta}a_{k-1}^{1-2\psi-\beta}}{D_n}-W^{1-2\psi-\beta}\right|\leq$$

$$\leq\frac{\sum_{k=h+1}^{n}\sup_{\theta,\nu}[|U1_k(\theta_n)|+U2_k(\theta_n)]|W_{k-1}^{1-2\psi-\beta}-W^{1-2\psi-\beta}|a_{k-1}^{1-2\psi-\beta}}{D_n}+$$

$$+\frac{\sum_{k=h+1}^{n}\sup_{\theta,\nu}[|U1_k(\theta_n)-1|+|U2_k(\theta_n)|]W^{1-2\psi-\beta}a_{k-1}}{D_n}.$$

• Using $m'_{\theta,\nu}(N)=N^{-\psi}+r'_{\theta,\nu}(N)$ and $B9$(i),

$$\sup_{\theta,\nu}|U1_k(\theta_n)-1|\leq 2M'N_{k-1}^{\psi-\overline{\psi}}+M'^2N_{k-1}^{2(\psi-\overline{\psi})}. \tag{24}$$

• $m_{\theta_0,\nu_0}(N)-m_{\theta_n,\nu}(N)=(\theta_0-\theta_n)N^{-\psi}+\overline{r}_{\theta_n,\nu}(N)$, where $\overline{r}_{\theta,\nu}(.)=r_{\theta_0,\nu_0}(.)-r_{\theta,\nu}(.)$. Then there exists a constant C such that

$$\sup_{\theta,\nu}|U2_k(\theta_n)|\leq M''(C+2MN_{k-1}^{\psi-\overline{\psi}})N_{k-1}^{\psi-\overline{\psi}}. \tag{25}$$

Then (24), (25) and $\psi<\overline{\psi}$ entail (22) according to Toeplitz lemma and $\lim_{n\to\infty}D_n=\infty$ when h is fixed. Using the same reasoning as the one used in the proof of proposition 1 while applying theorem 2.18 of Hall and Heyde, we obtain (23). □

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Christine JACOB
INRA, Laboratoire de Biométrie
78352 Jouy-en-Josas, Cedex, FRANCE

Nadia Lalam
INRA, Laboratoire de Biométrie
78352 Jouy-en-Josas, Cedex, FRANCE