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Pliska Stud. Math. Bulgar. **17** (2005), 85–96

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A NOTE ON THE EXTINCTION PROBLEM FOR CONTROLLED MULTITYPE BRANCHING PROCESSES

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In this paper we consider a discrete time controlled multitype branching process with random control in discrete time. We provide sufficient conditions for the almost sure extinction of the process as well as for its indefinite growth with a positive probability. Moreover an illustrative example is shown and some simulations are given.

1. Introduction

Controlled branching processes in discrete time are stochastic models to describe the evolution of populations where the number of potential progenitors is controlled by means of a fixed or random mechanism. The univariate case has been widely investigated, with both deterministic (see [1], [5, 2004b], [12], [14], [20]) and random control (see [7, 2003, 2004d, 2004e], [19]). On the other hand, with respect to the multidimensional version of these processes, [14] introduced the deterministic case but without a further development of its theory. More recently [6] have provided a multidimensional controlled process with random control where the reproduction depends on population size.

[1]Research supported by the Ministerio de Ciencia y Tecnología and the FEDER through the Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica, grant BFM2003-06074.

2000 *Mathematics Subject Classification:*

Key words: Controlled multitype branching processes, random control, homogeneous multitype Markov chains.

In this paper, we deal with the extinction problem for a controlled multitype branching process with random control. The results are deduced from the study of the unlimited growth for a class of homogeneous multitype Markov chains in discrete time considered by [6]. The paper consists of five sections. In Section 2 the probability model is defined. Its basic properties are investigated in Section 3. In Section 4 we provide sufficient conditions for the almost sure extinction of the process as well as for its survival with a positive probability. Finally Section 5 is devoted to show an illustrative example and some simulations.

2. The probability model

In this section we define a controlled multitype model where the control is realized by means of a random mechanism. Moreover, we consider a possible dependence at the reproduction time between individuals of the same generation. This dependence represents an important novelty with respect to classical branching models, whose implicit assumption of independence can only be considered as a mere theoretical simplification of the more complex types of reproductive behaviours in the nature.

Mathematically, we denominate Controlled Multitype Branching Process with Random Control (CMP) to a sequence of m-dimensional random vectors $\{Z(n)\}_{n\geq 0}$, defined in recursive way by:

$$Z(0) = z \in \mathbb{N}_0^m,\ \ Z(n+1) = (Z_1(n+1), \dots, Z_m(n+1)) = \sum_{i=1}^{m} \sum_{j=1}^{\phi_i^n(Z(n))} X^{i,n,j},\ \ n \geq 0,$$

being $\{X^{i,n,j} : i = 1, \dots, m; n = 0, 1, \dots; j = 1, 2, \dots\}$ and $\{\phi^n(z) : n = 0, 1, \dots; z \in \mathbb{N}_0^m\}$ two independent sequences of m-dimensional, non negative, integer valued random vectors defined on the same probability triple $(\Omega, \mathcal{A}, P)$, such that:

(i) The stochastic processes $\{\phi^n(z) : z \in \mathbb{N}_0^m\}$, $n = 0, 1, \dots$ are independent and for each $z \in \mathbb{N}_0^m$, the vectors $\{\phi^n(z) : n = 0, 1, \dots\}$ are identically distributed.

(ii) The stochastic processes $\{X^{i,n,j} : i = 1, \dots, m; j = 1, 2, \dots\}$, $n = 0, 1, \dots$ are independent and identically distributed. Moreover, for each $i = 1, \dots, m$ the vectors $\{X^{i,n,j} : n = 0, 1, \dots; j = 1, 2, \dots\}$ are identically distributed.

As usually the empty sum is considered as zero. Intuitively, the m-dimensional random vector $Z(n)$ represents the total number of individuals of each type at the nth generation, so that the process starts with $Z(0) = z$ ancestors. Provided that z^* individuals coexist at the nth generation, i.e. $Z(n) = z^*$, the number of potential progenitors of each type at this generation is given by $\phi^n(z^*)$, whose distribution is the same for every n. Once we know the concrete value of such random vector, for instance $\phi^n(z^*) = \widetilde{z}$, through the random mechanism $(X^{1,n,1}, \ldots, X^{m,n,\widetilde{z}_m})$, is possible to calculate the number of individuals at the generation $n+1$, i.e. $Z(n+1)$. $X^{i,n,j}$ denotes the vector whose coordinates are the number of individuals of each type originated by the jth progenitor of i-type. Note that the offspring distribution $(X^{1,n,1}, \ldots, X^{m,n,\widetilde{z}_m})$ is determined jointly so that the vectors $X^{i,n,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, \widetilde{z}_i$ are not necessarily independent. This fact implies a sort of interaction between individuals at the reproduction time.

Controlled branching processes proposed by [14] and [19] are particular cases of the CMP. Moreover, the CMP is a particular case of a more general controlled model proposed in [6], where the reproduction depends on population size. Notice that the random control allows to consider multitype branching processes with migration related to those studied by [16, 1996, 2004].

3. Basic properties

From its definition, it is not hard to deduce that a CMP is a homogeneous multitype Markov chain in discrete time, whose states are vectors with non-negative integer coordinates.

In the following result, some markovian properties of its states are provided.

Proposition 1. *Let $\{Z(n)\}_{n\geq 0}$ be a CMP. The null vector $\mathbf{0}$ is absorbing if and only if*

$$P\left[\phi^0(\mathbf{0}) = \mathbf{0}\right] = 1. \tag{1}$$

If, in addition, z is a non null state such that

$$P\left[\bigcap_{i=1}^{m} \left(\{\phi_i^0(z) = 0\} \cup \{\phi_i^0(z) > 0, X^{i,0,j} = \mathbf{0},\ j = 1, \ldots, \phi_i^0(z)\}\right)\right] > 0, \tag{2}$$

then this state is transient.

Proof. Since the empty sum is considered as $\mathbf{0}$, then the condition (1) is equivalent to the null state to be absorbing. Moreover, let $z \in \mathbb{N}_0^m$ be a non null state such that (2) holds. Then

$$P[Z(n) = z \ \text{ for some } n > 0 | Z(0) = z] \leq 1 - P[Z(1) = \mathbf{0} | Z(0) = z] < 1,$$

and consequently the state z is transient. □

Applying Markov chain theory, it is easy to obtain that if every non null state is transient then only two limiting behaviours are possible for the CMP: either all its coordinates go to zero (i.e. the process becomes extinct) or it has an unlimited growth (i.e. the total number of individuals grows indefinitely). This duality, typical in many homogeneous branching processes, is stated mathematically in the formula:

$$P[Z(n) \to \mathbf{0}] + P[\|Z(n)\| \to \infty] = 1, \tag{3}$$

being $\|\cdot\|$ an arbitrary norm on $\mathbb{R}^m$.

Next, we obtain the expression for the vector of conditional means. Taking into account the independence between control and reproduction, it is verified for every $z \in \mathbb{N}_0^m$:

$$E[Z(n+1)|Z(n)=z] = E[\phi^0(z)]R,$$

being $R := (r_{ij})_{1\le i,j\le m}$ the matrix of means of the offspring distribution, i.e. $r_{ij} := E[X_j^{i,0,1}]$.

¿From now on, we suppose that for each type $i = 1, \ldots, m$ there exists $\lambda_i \geq 0$ and $h_i(z)$ satisfying that

$$E[\phi_i^0(z)] = z_i\lambda_i + h_i(z) \quad \text{and} \quad h_i(z) = o(\|z\|). \tag{4}$$

This condition means that the average number of potential progenitors of a type is proportional to the number of individuals of this type plus/minus certain quantity of progenitors which is negligible with respect to the total amount of the population. Notice that under (4), immigration/emigration of progenitors of each type is allowed. Immigration is possible even if there are not individuals of a type. This could not happen if $h_i(z) = z_i o(1)$, though, in this situation we could determine λ_i explicitly as

$$\lambda_i = \lim_{\|z\|\to\infty : z_i \neq 0} \frac{E[\phi_i^0(z)]}{z_i}.$$

We also suppose that the matrix $M := (\lambda_i r_{ij})_{1\le i,j\le m}$ is irreducible in the sense of [13]. Notice that M is irreducible iff λ_i is non null for all $i = 1, \ldots, m$ and the matrix R is irreducible. In this case, there exist ρ, the Perron-Frobenius eigenvalue of M, and $\mu \in \mathbb{R}_+^m$, a right eigenvector of M associated to ρ, which will play an important role in the study extinction problem.

4. The extinction problem

In this section we provide sufficient conditions for the almost sure extinction of the process, i.e. $P[Z(n) \to \mathbf{0}] = 1$, as well as for its indefinite growth with a positive probability, i.e. $P[\|Z(n)\| \to \infty] > 0$. We obtain these conditions from the study considered by [6] on the unlimited growth for homogeneous multitype Markov chains in discrete time, whose states are vectors with non-negative integer coordinates, because a CMP belongs to such a class.

In order to apply such results, let us find suitable bounds for

$$\xi_z^\gamma := E\left[\left|\sum_{i=1}^{m}\left(\sum_{j=1}^{\phi_i^n(z)} X^{i,n,j}\mu - E[\phi_i^n(z)]E[X^{i,n,1}]\mu\right)\right|^\gamma\right],$$

with $\gamma \geq 1$, $i = 1, \ldots, m$ and $z \in \mathbb{N}_0^m$. Taking into account that

$$\sum_{j=1}^{\phi_i^n(z)} X^{i,n,j}\mu - E[\phi_i^n(z)]E[X^{i,n,1}]\mu = \sum_{j=1}^{\phi_i^n(z)} (X^{i,n,j} - E[X^{i,n,1}])\mu + (\phi_i^n(z) - E[\phi_i^n(z)])E[X^{i,n,1}]\mu,$$

from the C_r-inequality (see e.g. [11]), we get

$$\xi_z^\gamma \leq A_1\sum_{i=1}^{m} E\left[\left|\sum_{j=1}^{\phi_i^n(z)} (X^{i,n,j} - E[X^{i,n,1}])\mu\right|^\gamma\right] + A_2\sum_{i=1}^{m} E[|\phi_i^n(z) - E[\phi_i^n(z)]|^\gamma](E[X^{i,n,1}]\mu)^\gamma, \tag{5}$$

for certain constants $A_1, A_2 > 0$. Moreover, in the general case, i.e. when the random vectors $X^{i,n,j}$, $j = 1, 2, \ldots$, $i = 1, \ldots, m$ are not necessarily independent for each fixed $n \geq 0$, it can be shown

$$E\left[\left|\sum_{j=1}^{\phi_i^n(z)} (X^{i,n,j} - E[X^{i,n,1}])\mu\right|^\gamma\right] \leq E[\phi_i^n(z)^\gamma]E\left[|(X^{i,n,1} - E[X^{i,n,1}])\mu|^\gamma\right]. \tag{6}$$

On the other hand, if such vectors are independent, using von Bahr-Esseen and Marcinkiewicz-Zygmund inequalities (see [15] and [2], respectively), one finds that (6) can be bounded either by

$$E[\phi_i^n(z)]E\left[|(X^{i,n,1} - E[X^{i,n,1}])\mu|^\gamma\right] \quad \text{if } 1 \leq \gamma < 2, \tag{7}$$

or by

$$E[\phi_i^n(z)^{\gamma/2}]E\left[|(X^{i,n,1} - E[X^{i,n,1}])\mu|^\gamma\right] \quad \text{if } \gamma \geq 2\,. \tag{8}$$

Let us introduce the notation $g(z) := (\rho-1)(z\mu) + \sum_{i=1}^m \sum_{j=1}^m h_i(z) r_{ij}\mu_j$ and $\sigma^2(z) := Var[Z(n+1)\mu|Z(n) = z]$, for each $z \in \mathbb{N}_0^m$.

Applying Theorem 3 and Theorem 4 of [6], we establish the following sufficient conditions for the process to become extinct almost surely.

Theorem 1. *Let $\{Z(n)\}_{n\geq 0}$ be a CMP such that the equality (3) holds. Then $P[Z(n) \to \mathbf{0}] = 1$ if at least one of the following conditions is satisfied:*

i) $\rho < 1$.

ii) $\rho = 1$, *and $h_i(z) \leq 0$ for each $i = 1, \dots, m$ and $\|z\|$ large enough.*

iii) $\rho = 1$,

$$\limsup_{\|z\|\to\infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} < 1 \tag{9}$$

and for some $0 < \delta \leq 1$ and for every $i = 1, \dots, m$,

$$E[\phi_i^0(z)^{2+\delta}] = o((z\mu)^{1+\delta}\sigma^2(z)) \tag{10}$$

and

$$E[|\phi_i^0(z) - E[\phi_i^0(z)]|^{2+\delta}] = o((z\mu)^{1+\delta}\sigma^2(z)). \tag{11}$$

Proof. If *i)* and *ii)* are satisfied, then we deduce from Theorem 3 of [6] that $P[\|Z(n)\| \to \infty] = 0$ and therefore, since the equality (3) holds, the result.

If *iii)* holds, taking into account the inequalities (5) and (6), we obtain $\xi_z^{2+\delta} = o((z\mu)^{1+\delta}\sigma^2(z))$. Finally, applying Theorem 4 of [6] we conclude the proof. □

Remark. If the components of the vectors determining the reproduction are assumed to be independent and taking into consideration (8), condition (10) can be replaced by

$$E[\phi_i^0(z)^{1+\delta/2}] = o((z\mu)^{1+\delta}\sigma^2(z))$$

obtaining a weaker condition. Anyway, since (9) holds, sufficient conditions to guarantee (10) and (11) are

$$E[\phi_i^0(z)^{2+\delta}] = o((z\mu)^{2+\delta}g(z)) \quad \text{and} \quad E[|\phi_i^0(z) - E[\phi_i^0(z)]|^{2+\delta}] = o((z\mu)^{2+\delta}g(z)),$$

respectively.

Remark. Notice that although (3) doesn't hold, under conditions of previous theorem it is satisfied that $P[\|Z(n)\| \to \infty] = 0$ (see [6]).

From Theorem 1 and Theorem 2 of [6], we deduce the following sufficient conditions for an unlimited growth of a CMP.

Theorem 2. *Let* $\{Z(n)\}_{n\geq 0}$ *be a* CMP *such that for every non null* $z \in \mathbb{N}_0^m$

$$P[\phi_i^0(z) > z_i, X^{i,0,j} \neq \mathbf{0}, i \in I_{(z)}, j = 1, \dots, \phi_i^0(z)] > 0, \tag{12}$$

with $I_{(z)} = \{i \in \{1, \dots, m\} : z_i \neq 0\}$. *Then* $P[\|Z(n)\| \to \infty] > 0$ *if at least one of the following conditions is satisfied:*

i) $\rho > 1$, *for some* $\delta > 0$ *and for every* $i = 1, \dots, m$,

$$E[\phi_i^0(z)^{1+\delta}] = O(\|z\|^\delta) \tag{13}$$

and

$$E[|\phi_i^0(z) - E[\phi_i^0(z)]|^{1+\delta}] = O(\|z\|^\delta).$$

ii) $\rho = 1$,

$$\liminf_{\|z\|\to\infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} > 1 \tag{14}$$

and for some $0 < \delta \leq 1$, $\alpha > 0$ *and for every* $i = 1, \dots, m$,

$$E[\phi_i^0(z)^{2+\delta}] = o\left((z\mu)^{1+\delta} g(z)/(\log(z\mu))^{1+\alpha}\right) \tag{15}$$

and

$$E[|\phi_i^0(z) - E[\phi_i^0(z)]|^{2+\delta}] = o\left((z\mu)^{1+\delta} g(z)/(\log(z\mu))^{1+\alpha}\right). \tag{16}$$

Proof. If *i)* holds, taking into account the inequalities (5) and (6), we obtain $\xi_z^{1+\delta} = O(\|z\|^\delta)$. Moreover, if *ii)* is satisfied then $\xi_z^{2+\delta} = o\left((z\mu)^{1+\delta} g(z)/(\log(z\mu))^{1+\alpha}\right)$. Applying Theorem 1 and Theorem 2 of [6], respectively, we deduce the result. □

Remark. The condition (12) guarantees a positive probability for the total number of individuals in some generation to be large enough.

Remark. Since (14) holds, sufficient conditions to guarantee (15) and (16) are

$$E[\phi_i^0(z)^{2+\delta}] = o\left((z\mu)^{\delta}\sigma^2(z)/(\log(z\mu))^{1+\alpha}\right)$$

and

$$E[|\phi_i^0(z) - E[\phi_i^0(z)]|^{2+\delta}] = o\left((z\mu)^{\delta}\sigma^2(z)/(\log(z\mu))^{1+\alpha}\right),$$

respectively. Under the assumption of independence of the vectors determining the reproduction and taking into consideration (7) and (8), condition (13) can be replaced by

$$E[\phi_i^0(z)] = O(\|z\|^{\delta}), \quad \text{if } \delta \leq 1$$

or

$$E[\phi_i^0(z)^{(1+\delta)/2}] = O(\|z\|^{\delta}), \quad \text{if } \delta > 1,$$

and condition (15) by

$$E[\phi_i^0(z)^{1+\delta/2}] = o\left((z\mu)^{1+\delta}g(z)/(\log(z\mu))^{1+\alpha}\right).$$

From the previous study and analogously to the classification of the multitype Galton-Watson process, we propose the following classification for a CMP. We say that the process is:

i) *subcritical* if $\rho < 1$.

ii) *near-critical* if $\rho = 1$.

iii) *supercritical* if $\rho > 1$.

Unlike classical multitype branching process, when $\rho = 1$ and under some extra conditions, the process may become extinct almost surely (see Theorem 1) or may have a positive probability of growing indefinitely (see Theorem 2). Taking into account this dual behaviour, we have called this case *near-critical* instead of the classical *critical*.

5. Illustrative example

Finally, we illustrate with an example the possible behaviours of CMP along the generations. Let us consider a CMP such that the random variables $\{X_j^{i,n,k} : i,j = 1,\ldots,m; n = 0,1,\ldots; k = 1,2,\ldots\}$ are independent with mean and variance equal to 1, i.e., $r_{ij} = Var[X_j^{i,0,1}] = 1$. Moreover, we suppose that the random variables $\{\phi_i^n(z) : i = 1, \dots, m; n = 0, 1, \dots; z \in \mathbb{N}_0^m\}$ are also independent and have Poisson distributions with $E[\phi_i^0(z)] = az_i + b$, $a, b \geq 0$ for $z \neq \mathbf{0}$ and $\phi_i^0(\mathbf{0}) = 0$.

It is not hard to prove that the null state is absorbing. Since (4) holds, with $\lambda_i = a$ and

$$h_i(z) = \begin{cases} b, & \text{if } z \neq \mathbf{0} \\ 0, & \text{if } z = \mathbf{0}, \end{cases}$$

and $r_{ij} = 1$, then the matrix $M = (a)_{1 \leq i,j \leq m}$ is irreducible, the Perron-Frobenius eigenvalue $\rho = ma$ and $\mu = \mathbf{1}$, the vector whose components are all equal to 1. Moreover $g(z) = (ma - 1)z\mu + m^2 b$ and $\sigma^2(z) = a(m^2 + m)(z\mathbf{1}) + m^2(m+1)b$, and consequently if $ma = 1$, then

$$\lim_{\|z\| \to \infty} \frac{2(z\mathbf{1})g(z)}{\sigma^2(z)} = \lim_{\|z\| \to \infty} \frac{2m^2 b(z\mathbf{1})}{a(m^2 + m)(z\mathbf{1}) + m^2(m+1)b} = \frac{2m^2 b}{m+1}.$$

Also, from properties of Poisson distribution we get that (2) and (12) hold and

$$E[(\phi_i^0(z))^{3/2}] = O(z_i^{3/2}) \quad \text{and} \quad E[|\phi_i^0(z) - E[\phi_i^0(z)]|^3] = O(z_i^{3/2}), \ i = 1, \dots, m.$$

We consider four different situations:

S1. If $a < m^{-1}$, then $\rho < 1$ (*subcritical case*) and from Theorem 1 *i*), $P[Z(n) \to \mathbf{0}] = 1$.

S2. If $a > m^{-1}$, then $\rho > 1$ (*supercritical case*) and applying Theorem 2 *i*) with $\delta = 2$ we deduce $P[\|Z(n)\| \to \infty] > 0$.

S3. If $a = m^{-1}$ and $b < (2m^2)^{-1}(m+1)$, then $\rho = 1$ (*near-critical case*). From Theorem 1 *iii*) with $\delta = 1$, we deduce $P[Z(n) \to \mathbf{0}] = 1$.

S4. If $a = m^{-1}$ and $b > (2m^2)^{-1}(m+1)$, then $\rho = 1$ (*near-critical case*) and applying Theorem 2 *ii*) with $\delta = 1$, we obtain $P[\|Z(n)\| \to \infty] > 0$.

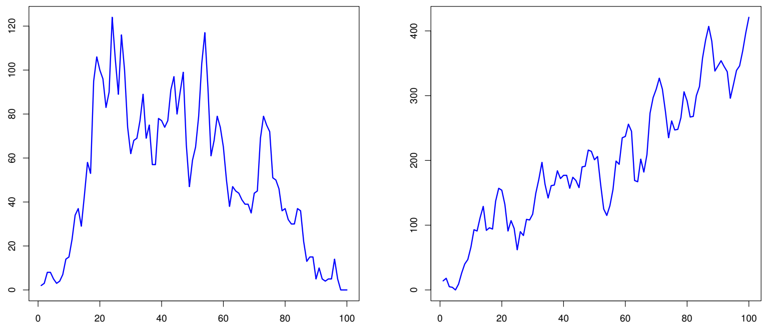
To illustrate such behaviours we have simulated some particular paths of this process until generation 100 with $m = 2$, starting with $Z(0) = (1, 2)$ and with reproduction law having independent marginal Poisson distributions. More specifically we have made three types of simulations, with two different values for ρ.

First we have taken $\rho = 1$ and have considered two cases with the following parameters:

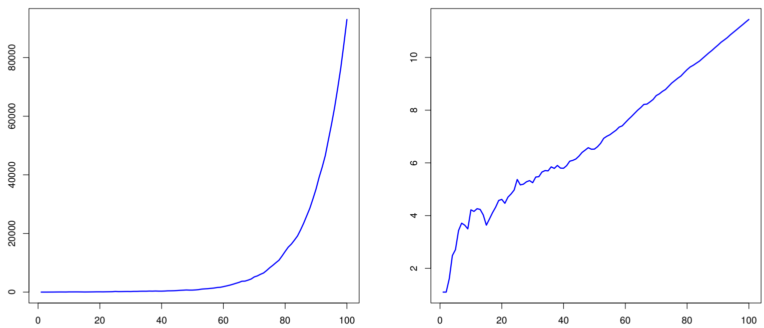
i) $a = 0.5$ and $b = 0$, belonging to S3.

ii) $a = 0.5$ and $b = 0.5$, belonging to S4.

Whereas all the paths corresponding to case i) become extinct, some of the paths of case ii) may survive and they show a possible linear growth. In the figure below we show the evolution of one of the paths for i) (left graphic) and one (perhaps!) non extinct path for ii) (right graphic).



Finally, we have supposed $\rho = 1.1$ with parameters $a = 0.55$ and $b = 0$, so it belongs to S2. We observe what seems to be, not only the survival of some paths, but also an exponential growth. The evolution of the simulated population (left graphic) and its logarithm (right graphic) are given in the following figure:



Remark. For the development of the simulated example, the statistical software **R** ("GNU S"), a language and environment for statistical computing and graphics (see [3]), has been used.

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