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WEAK CONVERGENCE TO THE TANGENT PROCESS OF THE LINEAR MULTIFRACTIONAL STABLE MOTION

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The linear multifractional stable motion (LMSM), $Y = \{Y(t)\}_{t\in\mathbb{R}}$, is an α-stable $(0 < \alpha < 2)$ stochastic process which exhibits local self-similarity. It is constructed by using a stochastic integral representation of the linear fractional stable motion (LFSM) process $X_{H,\alpha}(t)$, where the self-similarity exponent H is replaced by a function $H(t) \in (0, 1)$ of time t.

Here, we focus on LMSM processes with continuous paths and study the convergence

$$\Big\{\frac{1}{d(\lambda)}\Big(Y(\lambda t + t_0) - Y(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{Z(t)\}_{t\in[-1,1]}, \quad \text{as } \lambda \downarrow 0,$$

where $\Rightarrow$ denotes the weak convergence of probability distributions on the space of continuous functions $\mathcal{C}[-1, 1]$ equipped with the uniform norm and where $d(\lambda) \downarrow 0$. We show that if the function $H(t)$ is sufficiently regular and if $1/\alpha < H(t_0) < 1$, then the above weak convergence holds with normalization $d(\lambda) = \lambda^{H(t_0)}$ and the limit (*tangent*) process $Z(t)$ is the LFSM $X_{H(t_0),\alpha}(t)$. We also show that one can have degenerate tangent processes $Z(t)$, when the function $H(t)$ is not sufficiently regular.

The LMSM process is closely related to the Gaussian multifractional Brownian motion (MBM) process. We establish similar weak convergence results for the MBM.

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1. Introduction

We shall focus on the local asymptotic properties of *linear multifractional stable motion* (LMSM) processes. The LMSM process, $Y = \{Y(t)\}_{t\in\mathbb{R}}$, is an α-stable stochastic process which exhibits local self-similarity, has infinite variance and skewed distributions. It was introduced in Stoev and Taqqu (2004*c*) by extending the definition of the multifractional Brownian motion (MBM) process proposed by Peltier and Lévy-Vehel (1995) (see also Benassi, Jaffard and Roux (1997), Cohen (1999), Ayache and Lévy-Vehel (2000), and Stoev and Taqqu (2004*a*)).

Recall that a process $X = \{X(t)\}_{t\in\mathbb{R}}$ is said to be *self-similar* with self-similarity parameter $H > 0$ (H-self-similar), if for all $c > 0$, $\{X(ct)\}_{t\in\mathbb{R}} =_d \{c^H X(t)\}_{t\in\mathbb{R}}$, where $=_d$ means equality of the finite-dimensional distributions. The process X has *stationary increments*, if for all $h \in \mathbb{R}$, we have $\{X(t+h) - X(h)\}_{t\in\mathbb{R}} =_d \{X(t) - X(0)\}_{t\in\mathbb{R}}$. The *linear fractional stable motion* (LFSM) process, $X_{H,\alpha} = \{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$, is an α-stable process $(0 < \alpha < 2)$ which has stationary increments and is H-self-similar with $0 < H < 1$. It can be viewed as the α-stable counterpart of the well-known fractional Brownian motion (FBM) process (for a precise definition of the FBM process and its role in the probability theory and its applications see, for example, Taqqu (2003) and the references therein). The LFSM process $X_{H,\alpha} = \{X_{H,\alpha}(t)\}_{t\in\mathbb{R}}$ is defined as

$$
\begin{aligned}
X_{H,\alpha}(t) = \int_{\mathbb{R}} \Big\{ a^+ \Big((t+s)_+^{H-1/\alpha} - (s)_+^{H-1/\alpha} \Big) \\
+ a^- \Big((t+s)_-^{H-1/\alpha} - (s)_-^{H-1/\alpha} \Big) \Big\} M_{\alpha,\beta}(ds),
\end{aligned}
\tag{1.1}
$$

where $a^+, a^- \in \mathbb{R},\ |a^+| + |a^-| > 0$ and where $x_+ := \max\{0, x\},\ x_- := (-x)_+$. Here $M_{\alpha,\beta}(ds),\ s \in \mathbb{R}$ denotes an *independently scattered* strictly α-stable, $0 < \alpha < 2$, random measure on $\mathbb{R}$ with the Lebesgue control measure ds and *constant* skewness intensity $\beta(s) \equiv \beta \in [-1, 1],\ s \in \mathbb{R}$.

Definition 1.1. The measure $M_{\alpha,\beta}(ds)$ is said to be an α-stable measure with Lebesgue control measure and skewness intensity function $\beta(s) \in [-1, 1]$, if:

1. For any collection of disjoint Borel sets $A_1, \ldots, A_n \subset \mathbb{R}$, the random variables $M_{\alpha,\beta}(A_1), \ldots, M_{\alpha,\beta}(A_n)$ are independent and strictly α-stable, and

2. Their characteristic functions are given by

$$
\mathbb{E}e^{i\theta M_{\alpha,\beta}(A_j)} = \begin{cases} \exp\{-|A_j||\theta|^\alpha (1 - i\beta_{A_j}\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2})\} & ,\ \text{if } \alpha \neq 1 \\ \exp\{-|A_j||\theta|\} & ,\ \text{if } \alpha = 1. \end{cases}
$$

where $\theta \in \mathbb{R}$, $|A_j| = \int_{A_j} ds$ and $\beta_{A_j} = \int_{A_j} \beta(s)ds/|A_j|$, $j = 1, \dots, n$. (Here, for simplicity, we suppose the distribution of $M_{\alpha,\beta}(A_j)$ to be symmetric in the case $\alpha = 1$.)

The stochastic integral in (1.2) can be interpreted as an integral with respect to an infinite variance process. It is well defined because the integrand belongs to the space $L^\alpha(ds) = \{g(s) : \int_{\mathbb{R}} |g(s)|^\alpha ds < \infty\}$, for all $t \in \mathbb{R}$ and $0 < H < 1$. For more details on α-stable integrals and the class of LFSM processes see, for example, Ch. 3 and 7 in Samorodnitsky and Taqqu (1994).

The *linear multifractional stable motion* process $Y(t)$ is obtained from a stochastic integral representation of the LFSM process $X_{H,\alpha}(t)$ by replacing the self-similarity parameter H with a function of time $H(t)$, $t \in \mathbb{R}$. More precisely,

$$\begin{aligned} Y(t) \; = \; & \int_{\mathbb{R}} \Big\{ a^+ \Big((t+s)_+^{H(t)-1/\alpha} - (s)_+^{H(t)-1/\alpha} \Big) \\ & \qquad + a^- \Big((t+s)_-^{H(t)-1/\alpha} - (s)_-^{H(t)-1/\alpha} \Big) \Big\} M_{\alpha,\beta}(ds), \end{aligned} \tag{1.2}$$

where now the α-stable measure $M_{\alpha,\beta}(ds)$ may have an arbitrary, *non–constant* skewness intensity $\beta(s) \in [-1, 1]$, $s \in \mathbb{R}$.

When $\alpha = 2$, then the measure $M_{\alpha,\beta}(ds)$ becomes the Gaussian measure $M_2(ds)$ and then the LMSM process $Y(t)$ in (1.2) becomes the multifractional Brownian motion process[1]

$$\begin{aligned} & Y(t) = \textstyle\int_{\mathbb{R}} \Big\{ a^+ \Big((t+s)_+^{H(t)-1/2} - (s)_+^{H(t)-1/2} \Big) \\ & + a^- \Big((t+s)_-^{H(t)-1/2} - (s)_-^{H(t)-1/2} \Big) \Big\} M_2(ds). \end{aligned} \tag{1.3}$$

As shown in Stoev and Taqqu (2004*c*), when the function $H(t)$ has sufficient Hölder regularity, this multifractional Brownian motion satisfies

$$\Big\{ \frac{1}{d(\lambda)} \Big(Y(\lambda t + t_0) - Y(t_0) \Big) \Big\}_{t \in \mathbb{R}} \overset{f.d.d.}{\longrightarrow} \{Z(t)\}_{t \in \mathbb{R}}, \quad \text{as } \lambda \downarrow 0, \tag{1.4}$$

where $d(\lambda) = \lambda^{H(t_0)}$ and where the limit process $Z = \{Z(t)\}_{t \in \mathbb{R}}$ is the fractional Brownian motion process $B_{H(t_0)} = \{B_{H(t_0)}(t)\}_{t \in \mathbb{R}}$ (see also Theorem 1.7 in Benassi *et al.* (1997)). Because of (1.4), the process $Y(t)$ is said to be *locally*

[1] When $(a^+, a^-) = (1, 0)$, the multifractional Brownian motion in (1.3) coincides with the "multifractional Brownian motion" of Peltier and Lévy-Vehel (1995). Typically, different values of (a^+, a^-) correspond to different processes as shown in Stoev and Taqqu (2004*a*).

self-similar at t_0 with self-similarity exponent $H(t_0)$. The limit process in (1.4) can be also viewed as a *tangent process*, at t_0, to the process $Y(t)$.

We established in Stoev and Taqqu (2004*c*), that many of the stochastic properties of the Gaussian multifractional Brownian motion processes (1.3) extend for the class of α-stable LMSM processes (1.2). In particular, the LMSM process $Y(t)$ given in (1.2) is continuous in probability, if and only if the function $H(t) \in (0,1)$ is continuous. Furthermore, if the function $H(t)$ is sufficiently Hölder regular and if the skewness intensity $\beta(s)$ is continuous, then Relation (1.4) continues to hold for the LMSM process Y. In this case, the limit (tangent) process Z coincides with the LFSM process $\{X_{H(t_0),\alpha}(t)\}_{t\in\mathbb{R}}$ (see Theorem 5.1 therein).

The path properties of the LMSM processes are quite different from those of the multifractional Brownian motion processes. If the local scaling exponent function $H(t),\ t \in \mathbb{R}$ is continuous, then the multifractional Brownian motion process has a version with continuous paths (see Corollary 5.1 in Stoev and Taqqu (2004*b*), and also Proposition 3 in Peltier and Lévy-Vehel (1995) and Theorem 2.1 in Ayache and Taqqu (2003)). On the other hand, if $0 < \alpha < 2$, then the paths of the LMSM process can be bounded on finite intervals only if $1/\alpha \le H(t) < 1,\ t \in \mathbb{R}$ (Theorem 3.1 in Stoev and Taqqu (2004*b*)). In particular, the LMSM processes can have continuous paths only if $1 < \alpha < 2$. However, if the function $H(t)$ is sufficiently Hölder regular and if $1/\alpha < H(t) < 1,\ t \in \mathbb{R}$, then the LMSM process $Y(t)$ has a version with continuous paths (Theorem 3.2 in Stoev and Taqqu (2004*c*)).

Let $1 < \alpha \le 2$ and suppose that the time index t takes values in a bounded interval, for example $[-1,1]$. When the paths of X are continuous, then the process $X = \{X(t)\}_{t\in[-1,1]}$ can be viewed as a random variable, taking values in the normed space $\mathcal{C}[-1,1]$ of continuous functions on the interval $[-1,1]$ equipped with the uniform norm $\|f\| := \max_{x\in[-1,1]} |f(x)|,\ f \in \mathcal{C}[-1,1]$. The random process $\{X(t)\}_{t\in[-1,1]}$ then induces a probability distribution on the path space $\mathcal{C}[-1,1]$. Our goal, in this paper, is to show that the finite-dimensional distributions (f.d.d.) convergence in (1.4) of a LMSM process with continuous paths to a tangent process $Z(t),\ t \in [-1,1]$ extends also to a weak convergence of the corresponding distributions on the path space $\mathcal{C}[-1,1]$. This is done in Section 3. where we study two separate cases:

(i) when the function $H(t)$ is less than its Hölder exponent, and

(ii) when $H(t)$ is greater than its Hölder exponent but varies regularly.

In case *(i)* $H(t)$ is well–behaved, the tangent process Z in (1.4) is a LFSM process with continuous paths, and $d(\lambda) = \lambda^{H(t_0)}$. In the case *(ii)* the tangent process is degenerate, that is, for example, $Z(t) = t^{\rho}\xi,\ t > 0$, and $Z(t) = 0,\ t \le 0$, where

$0 < \rho < H(t_0)$, ξ is an α-stable random variable; in this case $d(\lambda) = \lambda^\rho L(\lambda)$, for some slowly varying function L.

The paper is organized as follows. In Section 2., we present some technical results from Stoev and Taqqu (2004*b*). In Section 3., we establish the main results of the paper, which give conditions for the weak convergence of a linear multifractional stable motion process to the tangent process, when $1 < \alpha < 2$. In Section 4., we consider the case $\alpha = 2$ and extend the results of Section 3. to the case of Gaussian multifractional Brownian motion process. Section 5. contains some auxilliary lemmas.

2. Preliminaries

In this section, we state some results about the LMSM prosess Y, defined in (1.2) and its associated α-stable field. These results are valid for all $\alpha \in (0, 2)$ and for $\alpha = 2$, when the LMSM process Y coincides with the multifractional Brownian motion process in (1.3). Their proofs and more details can be found in Stoev and Taqqu (2004*c*, 2004*b*).

Consider the (strictly) α-stable field $X = \{X(u, v),\ u \in \mathbb{R},\ v \in (0, 1)\}$,

$$X(u, v) := \int_{\mathbb{R}} f(u, v, s) M_{\alpha,\beta}(ds),$$

where

$$\begin{aligned} f(u, v, s) &:= \sum_{\kappa \in \{+,-\}} a^\kappa \Big((u+s)_\kappa^{v-1/\alpha} - (s)_\kappa^{v-1/\alpha} \Big) \\ &= a^+ \Big((u+s)_+^{v-1/\alpha} - (s)_+^{v-1/\alpha} \Big) + a^- \Big((u+s)_-^{v-1/\alpha} - (s)_-^{v-1/\alpha} \Big), \end{aligned}$$

and where $M_{\alpha,\beta}(ds)$ is the α-stable measure in (1.2). Relation (1.2) implies that for all $t \in \mathbb{R}$,

$$Y(t) = X(t, H(t)), \quad \text{almost surely.} \tag{2.1}$$

The field $X(u, v)$ has all partial derivatives, $\partial_v^n X(u, v),\ n \in \mathbb{N}$, with respect to $v \in (0, 1)$, in the sense of convergence in probability, and in fact,

$$\partial_v^n X(u, v) = \int_{\mathbb{R}} \partial_v^n f(u, v, s) M_{\alpha,\beta}(ds), \tag{2.2}$$

where $\partial_v^n f(u, v, s)$ denotes the nth partial derivative, in v, of the kernel function $f(u, v, s)$, that is,

$$\partial_v^n f(u, v, s) = \sum_{\kappa \in \{+,-\}} a^\kappa \Big(\ln^n (u+s)_\kappa (u+s)_\kappa^{v-1/\alpha} - \ln(s)_\kappa^n (s)_\kappa^{v-1/\alpha} \Big)$$

(see Lemma 2.1 in Stoev and Taqqu (2004*c*)). One also has the following Taylor series-type expansion of the field $X(u,v)$, in the variable v,

$$X(u,v) = X(u,v_0) + (v - v_0)\partial_v X(u,v_0) + \mathcal{O}_P((v - v_0)^2),\ v \to v_0, \tag{2.3}$$

where $u \in \mathbb{R}$ and $v,\ v_0 \in (0,1)$ (Theorem 2.2 in Stoev and Taqqu (2004*c*)). Using Relation (2.3) and the scaling properties of the field $X(u,v)$ one obtains that for all $1 < \alpha \le 2$ and for any compact $K \subset \mathbb{R} \times (0,1)$,

$$\|X(u',v') - X(u'',v'')\|_\alpha \le C_K\Big(|u' - u''|^{v'} + |v' - v''|\Big), \tag{2.4}$$

for all $(u',v'),\ (u'',v'') \in K$, where C_K is some constant (Theorem 2.1 in Stoev and Taqqu (2004*b*)).

Here $\|\xi\|_\alpha,\ 0 < \alpha \le 2$, denotes the scale coefficient in the characteristic function of the strictlty α-stable rv ξ, that is, $\mathbb{E}e^{i\theta\xi} = \exp\{-\|\xi\|_\alpha^\alpha|\theta|^\alpha(1 - i\beta_\xi \operatorname{sign}(\theta)\tan\frac{\pi\alpha}{2})\}$, where $\theta \in \mathbb{R}$ and where $\beta_\xi \in [-1,1]$ denotes the skewness coefficient of ξ. The scale coefficient $\|\cdot\|_\alpha$ metrizes the convergence in probability of jointly (strictly) α-stable random variables. Namely, let ξ and $\xi_n,\ n \in \mathbb{N}$ be jointly α-stable, then

$$\xi_n \xrightarrow{P} \xi,\ n \to \infty \iff \|\xi_n - \xi\|_\alpha \longrightarrow 0,\ n \to \infty. \tag{2.5}$$

In fact, $\|\cdot\|_\alpha$ is a norm in the linear spaces of strictly α-stable rv's, for $1 < \alpha \le 2$.

In view of (2.5), the inequality (2.4) implies the continuity in probability of the field $X(u,v)$, with respect to $(u,v) \in \mathbb{R} \times (0,1)$. Relation (2.4) can be interpreted as a Hölder regularity condition in the norm $\|\cdot\|_\alpha$ for the field $X(u,v)$, with respect to the variables (u,v). By using (2.1) and (2.4), for any continuous function $H(t) \in (0,1),\ t \in \mathbb{R}$, we obtain that

$$\|Y(t') - Y(t'')\|_\alpha \le C_K\Big(|t' - t''|^{H(t')} + |H(t') - H(t'')|\Big), \tag{2.6}$$

for all $t', t'' \in [a,b]$, where $[a,b] \subset \mathbb{R}$ is an arbitrary closed interval. Here, the set $K := \{(t, H(t))\}_{t\in[a,b]}$ is compact because $H(t)$ is continuous. Relation (2.6) will be used extensively in the sequel.

We also need the following definition. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. The exponent

$$\rho_g^{\mathrm{unif}}(t_0) := \sup\Big\{\rho \ge 0 : \lim_{t',t''\to t_0} \frac{g(t') - g(t'')}{|t' - t''|^\rho} = 0\Big\} \tag{2.7}$$

is called the *uniform pointwise Hölder* exponent, at t_0, of the function g.

Observe that (2.7) implies that for all $0 < \rho < \rho_g^{\mathrm{unif}}(t_0)$, there exists $\epsilon > 0$, such that

$$|g(t') - g(t'')| \leq C|t' - t''|^{\rho}, \tag{2.8}$$

for all $t', t'' \in [t_0 - \epsilon, t_0 + \epsilon]$, where $C > 0$ is some constant.

3. Weak convergence to the tangent process

Here, we consider non-Gaussian LMSM processes Y with continuous paths, defined in (1.2), with $\alpha < 2$. We want to establish sufficient conditions, in terms of the function $H(t)$, which allow us to replace the convergence of the finite-dimensional distributions in (1.4) by the stronger convergence of the corresponding probability measures induced on the path space $\mathcal{C}[-1,1]$. As indicated in the introduction, this can be done only when the function $H(t)$ is continuous, $1 < \alpha < 2$, and $1/\alpha \leq H(t) < 1$, $t \in \mathbb{R}$ (see e.g. Theorem 3.1 in Stoev and Taqqu (2004*b*)). Therefore, we shall supose in this section that

$$1 < \alpha < 2.$$

The following theorem establishes the weak convergence to the tangent process of LMSM in the case when the function $H(t)$, $t \in \mathbb{R}$ has sufficiently large uniform pointwise Hölder exponent. In this case, the corresponding tangent process is the LFSM in (1.1).

Theorem 1. *Let $1 < \alpha < 2$ and let $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be a LMSM process defined as in* (1.2) *with continuous function $H(t) \in (0,1)$, $t \in \mathbb{R}$. Assume that the skewness intensity, $\beta(s)$, is continuous and that*

$$1/\alpha < H(t_0) < \rho_H^{\mathrm{unif}}(t_0), \tag{3.1}$$

for some $t_0 \in \mathbb{R}$, $t_0 \neq 0$.

Then, the process $Y(t)$ has a version $\widetilde{Y}(t)$ with continuous paths in a neighborhood of $t = t_0$ and, as $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^{H(t_0)}}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{X_{H(t_0),\alpha}(t)\}_{t\in[-1,1]}, \tag{3.2}$$

where $\Rightarrow$ denotes the weak convergence of probability distributions on the space $\mathcal{C}[-1,1]$. Here $X_{H(t_0),\alpha}(t) = X_{H(t_0),\alpha,\beta(-t_0)}(t)$ denotes a continuous-path version of the LFSM process defined in (1.1), *with self-similarity parameter $H = H(t_0)$ with constant skewness intensity*[2] *$\beta = \beta(-t_0) \in [-1,1]$.*

[2] If s is replaced by $-s$ in (1.2), then the tangent process would become $X_{H(t_0),\alpha,\beta(t_0)}$, $t \in \mathbb{R}$.

Remarks

1. The assumption $H(t_0) < \rho_H^{\rm unif}(t_0)$ in (3.1) includes situations where $H(t)$ is particularly well–behaved at $t = t_0$; for example $H(t)$ will be differentiable at t_0 if $\rho_H^{\rm unif}(t_0) > 1$.

2. Observe that Relation (3.1) implies $\rho_H^{\rm unif}(t_0) > 1/\alpha$. This condition and the condition $H(t_0) > 1/\alpha$ guarantee that the LMSM process Y has a version with continuous paths (Theorem 3.2 in Stoev and Taqqu (2004*b*)). As shown in Theorem 3.1 therein, the condition $H(t_0) > 1/\alpha$ is essentially necessary for the paths of LMSM to be bounded.

Proof of Theorem 1: In view of (2.8), Relation (3.1) implies that

$$|H(t') - H(t'')| \le C|t' - t''|^\rho, \quad \text{for all } t', t'' \in (t_0 - 2\epsilon, t_0 + 2\epsilon), \tag{3.3}$$

for some $\epsilon > 0$ and $C > 0$, where

$$1/\alpha < H(t_0) < \rho < \rho_H^{\rm unif}(t_0). \tag{3.4}$$

Relations (3.3) and (3.4) imply the assumptions of Theorem 3.2 in Stoev and Taqqu (2004*b*) and hence the process $Y(t)$ has a version $\widetilde{Y}(t)$, with continuous paths on the interval $(t_0 - 2\epsilon, t_0 + 2\epsilon)$.

To prove (3.2), observe first that, as $\lambda \downarrow 0$,

$$\begin{aligned} \{Z_\lambda(t)\}_{t\in[-1,1]} &:= \Big\{\tfrac{1}{\lambda^{H(t_0)}}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \\ &\stackrel{f.d.d.}{\longrightarrow} \{X_{H(t_0),\alpha,\beta(-t_0)}(t)\}_{t\in[-1,1]}. \end{aligned} \tag{3.5}$$

Indeed, this follows by Theorem 5.1 *(a)* in Stoev and Taqqu (2004*c*), since the skewness intensity $\beta(s)$ is continuous and since Relations (3.3) and (3.4), above, imply the condition $H(t) - H(t_0) = o(|t - t_0|^{H(t_0)}),\ t \to t_0$.

It remains to prove (uniform) tightness of the laws of $Z_{\lambda_n},\ n \in \mathbb{N}$ in $\mathcal{C}[-1,1]$, for any sequence $\lambda_n \downarrow 0,\ n \to 0$. In view of Theorem 15.6 in Billingsley (1968), it suffices to prove that

$$\mathbb{E}|Z_\lambda(t) - Z_\lambda(s)|^\gamma \le C|t - s|^{1+\eta}, \quad \forall\ t,\, s \in [-1,1], \tag{3.6}$$

and for all sufficiently small $\lambda > 0$, with some constants $\gamma > 0,\ \eta > 0$ and $C > 0$.

We will now show that (3.6) holds. By Lemma 5.1 we obtain that, for all $0 < \gamma < \alpha$,

$$\mathbb{E}|Z_\lambda(t) - Z_\lambda(s)|^\gamma \le C_{\alpha,\gamma}\|Z_\lambda(t) - Z_\lambda(s)\|_\alpha^\gamma. \tag{3.7}$$

Let $0 < h < H(t_0)$ be arbitrary. We will show that, for all $t, s \in [-1, 1]$ and $\lambda \in (0, \delta)$,

$$\|Z_\lambda(t) - Z_\lambda(s)\|_\alpha = \frac{1}{\lambda^{H(t_0)}} \|\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(\lambda s + t_0)\|_\alpha \leq \widetilde{C}|t - s|^h, \tag{3.8}$$

for some constants $\delta > 0$ and $\widetilde{C} > 0$. Relation (2.6) implies that for all $\lambda \in (0, \epsilon]$ and $t, s \in [-1, 1]$,

$$\begin{aligned} &\|Z_\lambda(t) - Z_\lambda(s)\|_\alpha \\ &\leq \frac{C_{K_\epsilon}}{\lambda^{H(t_0)}}\Big(|\lambda t - \lambda s|^{H(\lambda t + t_0)} + |H(\lambda t + t_0) - H(\lambda s + t_0)|\Big) && (3.9) \\ &\leq C_{K_\epsilon}\Big(\lambda^{H(\lambda t + t_0) - H(t_0)}|t - s|^{H(\lambda t + t_0)} + C\lambda^{\rho - H(t_0)}|t - s|^\rho\Big), && (3.10) \end{aligned}$$

where the set $K_\epsilon := \{(t, H(\epsilon t + t_0))\}_{t \in [-1,1]}$ is compact because H is continuous on the interval $[t_0 - \epsilon, t_0 + \epsilon]$. The last inequality in (3.10) follows from (3.3).

Consider the first term in (3.10) and let $h < H(t_0)$. By the continuity of H, there exists $0 < \delta \leq \epsilon$, such that $h < H(\lambda t + t_0)$, for all $\lambda \in (0, \delta)$ and $t \in [-1, 1]$. Observe that $|t - s|/2 < 1$ for all $t, s \in [-1, 1]$. Hence, for all $t, s \in [-1, 1]$ and $\lambda \in (0, \delta)$, we obtain

$$\begin{aligned} \lambda^{H(\lambda t + t_0) - H(t_0)}|t - s|^{H(\lambda t + t_0)} &= 2^{H(\lambda t + t_0)}\lambda^{H(\lambda t + t_0) - H(t_0)}\Big(\frac{|t - s|}{2}\Big)^{H(\lambda t + t_0)} \\ &\leq 2^{H(\lambda t + t_0)}e^{\ln(\lambda)(H(\lambda t + t_0) - H(t_0))}\Big(\frac{|t - s|}{2}\Big)^h \\ &\leq 2^{H(\lambda t + t_0) - h}e^{C|\ln(\lambda)|\lambda^\rho t^\rho}|t - s|^h && (3.11) \\ &\leq C_\delta|t - s|^h, && (3.12) \end{aligned}$$

for some constant $C_\delta > 0$. The inequality in (3.11) follows by using (3.3), and the inequality in (3.12) follows from the fact that the function $f(x) := |\ln(x)|x^\rho$, $\rho > 0$ is bounded on the interval $(0, \delta)$.

We now turn to the second term in (3.10). We have that $C\lambda^{\rho - H(t_0)}|t - s|^\rho$ equals

$$2^\rho C\lambda^{\rho - H(t_0)}\Big|\frac{t - s}{2}\Big|^\rho \leq 2^\rho C\delta^{\rho - H(t_0)}\Big|\frac{t - s}{2}\Big|^h = 2^{\rho - h}C\delta^{\rho - H(t_0)}|t - s|^h, \tag{3.13}$$

for all $t, s \in [-1, 1]$ and $\lambda \in (0, \delta)$, since $h < H(t_0) < \rho$. Relations (3.11) and (3.13) imply (3.8).

By using (3.7) and (3.8), we get that

$$\mathbb{E}|Z_\lambda(t) - Z_\lambda(s)|^\gamma \le C_{\alpha,\gamma}\widetilde{C}^\gamma(t-s)^{\gamma h} \le C_{\alpha,\gamma,h}(t-s)^{\gamma h}, \tag{3.14}$$

for all $t < s,\ t, s \in [-1, 1]$ and for all sufficiently small $\lambda > 0$. Since $h > 1/\alpha$, we can choose $\gamma < \alpha$ sufficiently close to α, so that $\gamma h > 1$. This implies the inequality (3.6), which concludes the proof of the theorem. □

The following result deals with the case of a *less regular* function $H(t)$, which, however, is *regularly varying*. It corresponds to the case covered in Theorem 5.1 *(b)* of Stoev and Taqqu (2004*c*), where the tangent process Z in (1.4) turns out to be degenerate. Although we will suppose

$$1/\alpha < H(t_0) \quad \text{and} \quad 1/\alpha < \rho_H^{\text{unif}}(t_0), \tag{3.15}$$

to ensure path continuity, we do not assume that $H(t_0) < \rho_H^{\text{unif}}(t_0)$. We shall suppose that $H(t)$ is regularly varying at t_0, that is,

$$H(t) - H(t_0) = (t-t_0)^\rho L(t-t_0),\ t > t_0, \quad \text{with} \quad 0 < \rho < H(t_0), \tag{3.16}$$

where the function $L(x),\ x > 0$ is *normalized* slowly varying to the right of zero. Normalized slowly varying functions are defined in Relation (5.3) below.

Observe that (3.16) implies

$$H(t_0) > \rho \ge \rho_H^{\text{unif}}(t_0). \tag{3.17}$$

Indeed, if $\rho < \rho_H^{\text{unif}}(t_0)$, then by (2.8), we would have that $H(t) - H(t_0) = \mathcal{O}(|t-t_0|^{\rho'})$, as $t \downarrow t_0$, with $\rho < \rho' < \rho_H^{\text{unif}}(t_0)$. But this contradicts

$$\frac{H(t) - H(t_0)}{(t-t_0)^{\rho'}} = (t-t_0)^{\rho-\rho'}L(t-t_0) \longrightarrow \infty, \quad \text{as } t \downarrow t_0.$$

We used here the fact that for any slowly varying function $L(x)$ (to the right of 0), and for any $\delta > 0$,

$$x^\delta L(x) \longrightarrow 0, \quad \text{and} \quad x^{-\delta}L(x) \longrightarrow \infty, \ \text{ as } x \downarrow 0.$$

(see, e.g. Proposition 1.3.6 in Bingham, Goldie and Teugels (1987)).

Theorem 2. *Let* $1 < \alpha < 2$ *and let* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *be a LMSM process defined as in* (1.2) *with continuous function* $H(t) \in (0,1),\ t \in \mathbb{R}$. *Assume that* (3.15) *and* (3.16) *hold, for some* $t_0 \in \mathbb{R},\ t_0 \ne 0$.

Then, the process $Y(t)$ has a version $\widetilde{Y}(t)$ with continuous paths in a neighborhood of $t = t_0$ and, as $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^\rho L(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[0,1]} \Longrightarrow \{t^\rho \xi\}_{t\in[0,1]}, \tag{3.18}$$

where $\Rightarrow$ denotes the weak convergence of probability distributions on the space $\mathcal{C}[0,1]$ and where $\xi =_d \partial_v X(t_0, H(t_0))$ is a non-trivial α-stable rv (see (2.2)).

P r o o f. As shown in the proof of Theorem 1, using (3.15) and Theorem 3.2 in Stoev and Taqqu (2004*b*), we obtain that $Y(t)$ has a version $\widetilde{Y}(t)$ with continuous paths on the interval $[t_0 - \epsilon, t_0 + \epsilon]$, for some $\epsilon > 0$.

By Theorem 5.1 *(b)* in Stoev and Taqqu (2004*c*), Condition (3.16) implies that, as $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^\rho L(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[0,1]} \stackrel{f.d.d.}{\longrightarrow} \{t^\rho \xi\}_{t\in[0,1]}, \tag{3.19}$$

where $\xi =_d \partial_v X(t_0, H(t_0))$ is a non-trivial α-stable rv. Therefore, to prove (3.18), it suffices to show that the processes

$$Z_\lambda = \{Z_\lambda(t)\}_{t\in[0,1]} := \Big\{\frac{1}{\lambda^\rho L(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[0,1]}$$

have (uniformly) tight distributions on the path-space $\mathcal{C}[0,1]$, for all sufficienlty small $\lambda > 0$.

As in the proof of Theorem 1, we get that, for all $\lambda \in (0, \epsilon)$,

$$\begin{aligned} &\|Z_\lambda(t) - Z_\lambda(s)\|_\alpha \le \\ &\tfrac{C_{K_\epsilon}}{\lambda^\rho L(\lambda)}\Big(|\lambda t - \lambda s|^{H(\lambda t + t_0)} + |H(\lambda t + t_0) - H(\lambda s + t_0)|\Big),\ t, s \in [0,1]. \end{aligned} \tag{3.20}$$

The function $f(x) := H(x + t_0) - H(t_0) = x^\rho L(x),\ x > 0$ satisfies the assumptions of Lemma 5.2 and therefore, for any fixed $0 < \delta < \rho$, there exists $\eta \in (0, \epsilon)$, such that for all $\lambda \in (0, \eta)$,

$$\begin{aligned} &\Big|\tfrac{H(\lambda t + t_0) - H(\lambda s + t_0)}{\lambda^\rho L(\lambda)}\Big| = \\ &\Big|\tfrac{f(\lambda t) - f(\lambda s)}{f(\lambda)}\Big| \le K|t - s|^{\rho - \delta},\ t, s \in [0,1], \end{aligned} \tag{3.21}$$

where $K > 0$ is some constant.

The first term in the right–hand side of (3.20) is negligible compared to the second term. Indeed, since $\rho < H(t_0)$, by using the continuity of the function $H(t)$ at t_0, as in the proof of Theorem 1, we obtain that for all sufficiently small $\lambda > 0$,

$$\frac{|\lambda t - \lambda s|^{H(\lambda t + t_0)}}{\lambda^\rho L(\lambda)} \le \frac{\lambda^{h-\rho}}{L(\lambda)}|t-s|^h \le C|t-s|^\rho, \quad t,s \in [0,1], \tag{3.22}$$

where $\rho < h < H(t_0)$ and where $C > 0$ is some constant.

By using the bounds (3.21) and (3.22) in (3.20), we obtain that for all sufficiently small $\lambda > 0$,

$$\|Z_\lambda(t) - Z_\lambda(s)\|_\alpha \le \widetilde{C}|t-s|^{\rho-\delta}, \quad t,s \in [0,1], \tag{3.23}$$

where $\widetilde{C}$ is some constant. By Relations (3.23) and (3.8), for any $\gamma \in (0,\alpha)$,

$$\mathbb{E}|Z_\lambda(t) - Z_\lambda(s)|^\gamma \le \text{const}\,|t-s|^{\gamma(\rho-\delta)}, \quad t,s \in [0,1].$$

Observe that (3.15) and (3.17) imply $1/\alpha < \rho$. By choosing $\delta > 0$ arbitrarily small and γ arbitrarily close to α, we can make the exponent $\gamma(\rho - \delta)$ greater than 1 and hence obtain (3.7). This implies the tightness of the laws of Z_{λ_n}, for any $\lambda_n > 0$, $\lambda_n \downarrow 0$, as $n \to \infty$, and completes the proof of the theorem. □

Theorem 2 establishes the *one-sided* weak convergence to a degenerate tangent process. Observe that we impose conditions on $H(t) - H(t_0)$ only for $t > t_0$. It is interesting to determine the two-sided weak limit in (3.18). In this case, we need to specify $H(t)$ for $t < t_0$ as well as for $t \ge t_0$. Namely,

$$H(t) - H(t_0) = \begin{cases} c^+(t-t_0)^{\rho^+}L^+(t-t_0) & , \text{ if } t_0 < t \\ c^-|t-t_0|^{\rho^-}L^-(t_0-t) & , \text{ if } t < t_0, \end{cases} \tag{3.24}$$

where $t_0 \neq 0$, $0 < \rho^+$, $\rho^- < H(t_0)$ and $(c^+, c^-) \neq (0,0)$. Here $L^+(x)$ and $L^-(x)$, $x > 0$, are *normalized* slowly varying functions to the right of zero.

The following two results extend the one-sided limit in (3.18) to a two-sided one. The first one deals with the situation when $\rho^+ \neq \rho^-$.

Corollary 3.1. *Let* $1 < \alpha < 2$ *and let* $H(t) \in (0,1)$, $t \in \mathbb{R}$ *be a continuous function, which satisfies* (3.15) *and Relation* (3.24), *for all* t *in a neighborhood of* $t_0 \neq 0$. *Assume that*

$$0 < \rho^+ < \rho^- < H(t_0),$$

that $c^+ \neq 0$ *and that the functions* $L^+(x)$ *and* $L^-(x)$, $x > 0$ *are normalized slowly varying to the right of zero. Then the process* $Y(t)$ *has a version with continuous paths in a neighborhood of* t_0 *and, as* $\lambda \downarrow 0$,

$$(3.25)\qquad \Big\{\frac{1}{\lambda^{\rho^+}L^+(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{Z(t)\}_{t\in[-1,1]},$$

where

$$Z(t) := \begin{cases} c^+ t^{\rho^+}\xi & , \text{ if } t > 0 \\ 0 & , \text{ if } t \leq 0. \end{cases}$$

Here $\xi =_d \partial_v X(t_0, H(t_0))$ *is a non-trivial* α*-stable rv and* $\Rightarrow$ *denotes the weak convergence of probability distributions on* $\mathcal{C}[-1,1]$.

Proof. By Theorem 2, we have that, as $\lambda \downarrow 0$,

$$(3.26)\qquad \Big\{\frac{1}{\lambda^{\rho^+}L^+(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[0,1]} \Longrightarrow \{c^+ t^{\rho^+}\xi\}_{t\in[0,1]},$$

where $\widetilde{Y}(t),\ t \in \mathbb{R}$ is a continuous path version of the process $Y(t),\ t \in \mathbb{R}$. Similarly, by applying Theorem 2 to the process $Y_-(t) := Y(t_0 - (t - t_0)),\ t \in \mathbb{R}$, and since $\rho^+ < \rho^-$, we get in view of Lemma 5.3 that, as $\lambda \downarrow 0$,

$$(3.27)\qquad \Big\{\frac{1}{\lambda^{\rho^+}L^+(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,0]} \stackrel{P}{\longrightarrow} \mathbf{0},$$

where $\to^P$ denotes convergence in probability in the metric space $\mathcal{C}[-1,0]$ and where $\mathbf{0}$ denotes the zero function.

By Theorem 4.4 in Billingsley (1968), the "glued" process consisting of the left–hand sides of (3.26) and (3.27) converges in the finite-dimensional distributions to $\{Z(t)\}_{t\in[-1,1]}$. The tightness of the laws of this glued process for any sequence $\{\lambda_n\}_{n\in\mathbb{N}},\ \lambda_n \downarrow 0$ follows from Lemma 5.4. This establishes the weak convergence in (3.25). □

Remark. If we had $0 < \rho^- < \rho^+ < H(t_0)$ in Corollary 3.1, then a limit result similar to (3.25) would be valid with

$$Z(t) := \begin{cases} 0 & , \text{ if } t > 0 \\ c^- |t|^{\rho^-}\xi & , \text{ if } t \leq 0. \end{cases}$$

Corollary 3.2. *Let* $1 < \alpha < 2$ *and let* $H(t) \in (0,1),\ t \in \mathbb{R}$ *be a continuous function, which satisfies* (3.15) *and Relation* (3.24), *for all* t *in a neighborhood of* $t_0 \neq 0$. *Assume that*

$$0 < \rho \equiv \rho^+ = \rho^- < H(t_0),$$

that the functions $L^+(x)$ *and* $L^-(x)$, $x > 0$ *are normalized slowly varying to the right of zero and that* $L^+(x)/L^-(x) \to 1$, $x \downarrow 0$. *Then the process* $Y(t)$ *has a version with continuous paths in a neighborhood of* t_0 *and, as* $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^\rho L^+(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{Z(t)\}_{t\in[-1,1]}, \tag{3.28}$$

where

$$Z(t) := \begin{cases} c^+ t^\rho \xi & , \text{ if } t > 0 \\ c^- |t|^\rho \xi & , \text{ if } t \le 0. \end{cases}$$

Here $\xi =_d \partial_v X(t_0, H(t_0))$ *is a non-trivial* α*-stable rv and* $\Rightarrow$ *denotes the weak convergence of probability distributions on* $\mathcal{C}[-1,1]$.

Proof. Let

$$\xi_\lambda(t) := \frac{\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)}{\lambda^\rho L^+(\lambda)},\ t \in [-1, 0],$$

and

$$\eta_\lambda(t) := \frac{\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)}{\lambda^\rho L^+(\lambda)},\ t \in [0, 1].$$

Following the proof of Corollary 3.1, by using Lemma 5.4, one can establish the uniform tightness of the distributions of

$$\{\xi_{\lambda_n}(t)1_{[-1,0]}(t) + \eta_{\lambda_n}(t)1_{[0,1]}(t)\}_{t\in[-1,1]}$$

in the space $\mathcal{C}[-1,1]$, for any $\lambda_n \downarrow 0$, $n \to \infty$.

We will now show that the convergence in (3.28) holds in the sense of the finite–dimensional distributions. Theorem 5.1 *(b)*, Relation (5.2) in Stoev and Taqqu (2004c) imply that, as $\lambda \downarrow 0$,

$$\eta_\lambda(t) \to^P c^+ t^\rho \xi,\ \forall t \in [0,1], \quad \text{and} \quad \xi_\lambda(t) \to^P c^- |t|^\rho \xi,\ \forall t \in [-1, 0],$$

where $\xi = \partial_v X(t_0, H(t_0))$. Observe that the last two limits involve the *same* random variable ξ. Therefore, the process in the right–hand side of (3.28) converges pointwise in probability to $\{Z(t)\}_{t\in[-1,1]}$, and hence in finite–dimensional distributions. □

Remarks

1. Taking the limit $\lambda \downarrow 0$ with respect to the continuous parameter $\lambda > 0$ in Relations (3.2) and (3.18), is equivalent to taking limits with respect to all subsequences $\lambda_n \downarrow 0$, $n \to \infty$.

2. The regularity conditions on the function $H(t)$ in Theorems 1 and 2 are slightly stronger than the corresponding conditions on $H(t)$ imposed in Theorem 5.1 *(a)* and *(b)* of Stoev and Taqqu (2004*c*), where we establish merely convergence of the finite-dimensional distributions.

3. Falconer (2003) studies the local structure for general stochastic process with paths in the space $\mathcal{D}(\mathbb{R})$. The space $\mathcal{D}(\mathbb{R})$ consists of all continuous to the right functions which have limits to the left and it is endowed with the Skorokhod J_1 topology. Tangent processes therein are defined as the weak limits, in $\mathcal{D}(\mathbb{R})$, of the rescaled increments $\{c_n^{-1}(Y(r_n t + t_0) - Y(t_0))\}_{t\in\mathbb{R}}$, $n \to \infty$, where $r_n \downarrow 0$ and $c_n \downarrow 0$.

   Suppose that at all $t_0 \in \mathcal{A} \subset \mathbb{R}$, the tangent process $\{Z(t;t_0)\}_{t\in\mathbb{R}}$ is *unique* (up to a multiplicative constant). Theorem 3.8 in Falconer (2003) states that for almost all $t_0 \in \mathcal{A}$, the process $\{Z(t;t_0)\}_{t\in\mathbb{R}}$ will be *self-similar* and it will have *stationary increments.*

   These two properties hold for the tangent process in (3.2). However, the tangent process appearing in Theorem 2 is only self-similar and it does not have stationary increments, since $0 < \rho < 1$. This observation shows that there are no functions $H(t)$, which satisfy the conditions of Theorem 2, for almost all $t_0 \in \mathcal{A}$, where $\mathcal{A}$ is a Borel set of positive Lebesgue measure.

4. The Gaussian case $\alpha = 2$

As indicated in the introduction, when $\alpha = 2$ the measure $M_{\alpha,\beta}(ds)$ in (1.2) becomes Gaussian and one recovers the Gaussian multifractional Brownian motion (MBM) process $\{Y(t)\}_{t\in\mathbb{R}}$. In this section, we show that Theorems 1 and 2 can be extended to the Gaussian case. In fact, the conditions $1/\alpha < H(t_0)$ and $1/\alpha < \rho_H^{\mathrm{unif}}(t_0)$ in Relations (3.1) and (3.15) can be omitted. The following result corresponds to Theorem 1 when $\alpha = 2$.

Theorem 3. *Let* $\alpha = 2$ *and let* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *be a LMSM (ie MBM) defined in* (1.3) *with continuous function* $H(t) \in (0,1),\ t \in \mathbb{R}$. *Suppose that*

$$H(t_0) < \rho_H^{\mathrm{unif}}(t_0) \tag{4.1}$$

for some $t_0 \in \mathbb{R},\ t_0 \neq 0$.

Then, the process $Y(t)$ *has a version* $\widetilde{Y}(t)$ *with continuous paths and, as* $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^{H(t_0)}}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{B_{H(t_0)}(t)\}_{t\in[-1,1]}, \tag{4.2}$$

where ⇒ denotes weak convergence of probability measures on $\mathcal{C}[-1,1]$ *and where the tangent process,* $B_{H(t_0)} = \{B_{H(t_0)}(t)\}_{t\in[-1,1]}$*, is a continuous-path version of the fractional Brownian motion process with self-similarity parameter* $H(t_0)$*. The process* $B_{H(t_0)}(t)$ *is defined in* (1.3) *where* $H(t)$ *replaced by the constant* $H(t_0)$*.*

Proof. Since $H(t),\ t\in\mathbb{R}$ is continuous, the MBM process Y has a version $\widetilde{Y}$ with continuous paths (see, for example, Corollary 5.1 in Stoev and Taqqu (2004*b*)). As in the proof of Theorem 1, above, the Condition (4.1) implies that for any ρ, such that

$$H(t_0) < \rho < \rho_H^{\mathrm{unif}}(t_0),$$

Relation (3.3) holds with some $\epsilon > 0$ and $C > 0$. (In the Gaussian case one does not need the assumption $1/\alpha < H(t_0)$ to guarantee the continuity of the paths of the process $\widetilde{Y}$.) One also has that (3.5) holds, where now the right-hand side of this relation is the fractional Brownian motion process $\{B_{H(t_0)}(t)\}_{t\in[-1,1]}$. Therefore, to prove (4.2) it suffices to establish that the moment inequality in (3.6) holds, for some $\gamma > 0$, $\eta > 0$ and $C > 0$.

For any Gaussian random variable X with mean zero and variance $\sigma^2 > 0$, we have that, for all $\gamma > 0$,

$$\mathbb{E}|X|^\gamma = C_\gamma \sigma^\gamma,$$

where $C_\gamma > 0$ is some constant. Thus, Relation (3.7) now holds for arbitrary $\gamma > 0$ (not necessarily for $\gamma\in(0,\alpha)$). Hence, by following the argument in the proof of Theorem 1, above, we obtain that for any $0 < h < H(t_0)$ and for any $\gamma > 0$,

$$\mathbb{E}|Z_\lambda(t) - Z_\lambda(s)|^\gamma \le C|t-s|^{\gamma h},\quad t\le s,\ t,s\in[-1,1], \tag{4.3}$$

for all sufficiently small $\lambda > 0$, where C is some constant, which depends on γ and h.

Since now the exponent γ in (4.3) can be chosen arbitrarily large, (4.3) implies (3.6). This completes the proof of the theorem. □

The following result is the analog of Theorem 2 in the Gaussian case.

Theorem 4. *Let* $\alpha = 2$ *and let* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *be a LMSM (ie MBM) process defined in* (1.3) *with continuous function* $H(t)\in(0,1),\ t\in\mathbb{R}$*. Assume that* $\rho_H^{\mathrm{unif}}(t_0) > 0$*, for some* $t_0\in\mathbb{R},\ t_0\neq 0$ *and that the Condition* (3.16) *is satisfied.*

Then, the process $Y(t)$ *has a version* $\widetilde{Y}(t)$ *with continuous paths and, as* $\lambda\downarrow 0$*, the Relation* (3.18) *holds, where now* $\xi =_d \partial_v X(t_0, H(t_0))$ *is a Gaussian random variable, defined as in* (2.2) *with* $M_{\alpha,\beta}(ds)$ *replaced by the Wiener measure.*

P r o o f. As in the proof of Theorem 3, the continuity of $H(t)$ implies that there exists a continuous-path version $\widetilde{Y}$ of the LMSM process Y. The rest of the proof is essentially identical to the proof of Theorem 2 above. Indeed, one has the convergence of the finite-dimensional distributions in (3.19) (Theorem 5.1 *(b)* in Stoev and Taqqu (2004*c*)). Condition (3.16) and Lemma 5.2 imply Relation (3.21). Also, the inequalities $\rho_H^{\rm unif}(t_0) > 0$, $0 < \rho < H(t_0)$ and the continuity of the function $H(t)$ imply (3.22) and consequently (3.23).

Now, as in the proof of Theorem 3, we obtain that for all $t_1 \le t \le t_2,\ t_1, t_2 \in [0,1]$ and for all $\gamma > 0$,

$$\mathbb{E}|Z_\lambda(t) - Z_\lambda(s)|^\gamma \le C|t-s|^{\gamma(\rho-\delta)},$$

for all sufficiently small $\lambda > 0$ and for some constant $C > 0$. Since $\gamma > 0$ can be chosen arbitrarily large, the last relation implies (3.6) and hence the tightness of the distributions of the processes Z_{λ_n}, for any sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n \downarrow 0$, as $n \to \infty$. This concludes the proof of the theorem. □

The following two results are the analogs of Corollaries 3.1 and 3.2, in the Gasussian case.

Corollary 4.1. *Let $\alpha = 2$ and let $H(t) \in (0,1),\ t \in \mathbb{R}$ be a continuous function, with $\rho_H^{\rm unif}(t_0) > 0$, which satisfies Relation* (3.24)*, for all t in a neighborhood of $t_0 \neq 0$. Assume that*

$$0 < \rho^+ < \rho^- < H(t_0),$$

that $c^+ \neq 0$ and that the functions $L^+(x)$ and $L^-(x),\ x > 0$ are normalized slowly varying to the right of zero. Then the process $Y(t)$ has a version with continuous paths in a neighborhood of t_0 and, as $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^{\rho^+}L^+(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{Z(t)\}_{t\in[-1,1]}, \tag{4.4}$$

where

$$Z(t) := \begin{cases} c^+ t^{\rho^+}\xi & , \ \text{if } t > 0 \\ 0 & , \ \text{if } t \le 0. \end{cases}$$

Here $\xi =_d \partial_v X(t_0, H(t_0))$ is a non-trivial Gaussian ($2-$stable) random variable and $\Rightarrow$ denotes the weak convergence of probability distributions on $\mathcal{C}[-1,1]$.

Corollary 4.2. *Let $\alpha = 2$ and let $H(t) \in (0,1),\ t \in \mathbb{R}$ be a continuous function, with $\rho_H^{\rm unif}(t_0) > 0$, which satisfies Relation* (3.24)*, for all t in a neighborhood of $t_0 \neq 0$. Assume that*

$$0 < \rho \equiv \rho^+ = \rho^- < H(t_0),$$

that the functions $L^+(x)$ and $L^-(x)$, $x > 0$ are normalized slowly varying to the right of zero and that $L^+(x)/L^-(x) \to 1$, $x \downarrow 0$. Then the process $Y(t)$ has a version with continuous paths in a neighborhood of t_0 and, as $\lambda \downarrow 0$,

$$\Big\{\frac{1}{\lambda^\rho L^+(\lambda)}\Big(\widetilde{Y}(\lambda t + t_0) - \widetilde{Y}(t_0)\Big)\Big\}_{t\in[-1,1]} \Longrightarrow \{Z(t)\}_{t\in[-1,1]}, \tag{4.5}$$

where

$$Z(t) := \begin{cases} c^+ t^\rho \xi & , \text{ if } t > 0 \\ c^- |t|^\rho \xi & , \text{ if } t \le 0. \end{cases}$$

Here $\xi =_d \partial_v X(t_0, H(t_0))$ is a non-trivial Gaussian (2−stable) random variable and $\Rightarrow$ denotes the weak convergence of probability distributions on $\mathcal{C}[-1, 1]$.

The proofs of the last two results are similar to the proofs of Corollaries 3.1 and 3.2, respectively. Observe now however, that we do not necessarily assume that (3.15) holds, because in the Gaussian case, the LMSM process Y always has a version with continuous path, provided that the function $H(t)$ is continuous, independently of the value of its Hölder regularity exponent (compare the assumptions of Theorems 2 and 4).

5. Auxiliary lemmas

The following lemma is used in the proofs of Theorems 1 and 2.

Lemma 5.1. *Let $1 < \alpha < 2$ and let ξ be a strictly α-stable rv with characteristic function*

$$\mathbb{E}e^{i\theta\xi} = \exp\{-\|\xi\|_\alpha^\alpha |\theta|^\alpha (1 - i\,\mathrm{sign}(\theta)\beta_\xi \tan(\pi\alpha/2))\} \tag{5.1}$$

where $\theta \in \mathbb{R}$ and $\beta_\xi \in [-1, 1]$ is the skewness coefficient of ξ. Then, for all $\gamma \in (0, \alpha)$,

$$\mathbb{E}|\xi|^\gamma \le C_{\alpha,\gamma} \|\xi\|_\alpha^\gamma,$$

where $C_{\alpha,\gamma}$ is a constant which does not depend on the skewness coefficient β_ξ.

P r o o f. By Property 1.2.13 in Samorodnitsky and Taqqu (1994), we have that

$$\xi =_d \|\xi\|_\alpha \Big(\Big(\frac{1+\beta_\xi}{2}\Big)^{1/\alpha} \xi_1 - \Big(\frac{1-\beta_\xi}{2}\Big)^{1/\alpha} \xi_2\Big), \tag{5.2}$$

where ξ_1 and ξ_2 are independent, identically distributed strictly α-stable random variable with unit scale and skewness coefficients, that is, $\|\xi_j\|_\alpha = 1$ and ξ_j, $j =$

$1, 2$ are totally skewed to the right. By using the inequality $|x+y|^\gamma \le 2|x|^\gamma + 2|y|^\gamma$, valid for all $x, y \in \mathbb{R}$ and $\gamma \in (0, 2]$, and Relation (5.2), we get

$$\mathbb{E}|\xi|^\gamma \le 4\|\xi\|_\alpha^\gamma \Big(\frac{1+|\beta_\xi|}{2}\Big)^{\gamma/\alpha} \mathbb{E}|\xi_1|^\gamma \le 4\mathbb{E}|\xi_1|^\gamma \|\xi\|_\alpha^\gamma =: C_{\alpha,\gamma}\|\xi\|_\alpha^\gamma,$$

because $|\beta_\xi| \le 1$. This completes the proof of the lemma. □

The next result is used in the proof of Theorem 2. It involves the notion of normalized slowly varying function. A function $L(x),\ x > 0$ is called *normalized slowly varying* to the right of zero, if

$$L(x) = c \exp\Big\{ \int_x^1 \epsilon(u)du/u \Big\},\ x \in (0, 1), \tag{5.3}$$

where $c \ne 0$ is a *constant* and where $\epsilon(x) \to 0$, as $x \downarrow 0$ (see, Relation (1.3.4) in Bingham, Goldie and Teugels (1987)).

A function $L(x),\ x > 0$ is called *slowly varying* (to the right of zero), if for all $x > 0$,

$$L(\lambda x)/L(\lambda) \longrightarrow 1, \quad \text{as } \lambda \downarrow 0.$$

Any normalized slowly varying function is also slowly varying. In fact, any (measurable) slowly varying function has the representation (5.3), where the constant c is replaced by a function $c(x),\ x > 0$ such that $c(x) \to c_0 \ne 0$, as $x \downarrow 0$ (see, for example, Theorem 1.3.1 in Bingham *et al* (1987)). As shown by Bojanic and Karamata, the class of normalized slowly varying functions coincides with the Zygmund class of functions (see, for example, p. 24 in Bingham *et al* (1987)). A function $L(x)$ is said to belong to the Zygmund class of slowly varying functions (to the right of zero), if for any $\delta > 0$ the functions $x^\delta L(x)$ and $x^{-\delta} L(x)$ are monotone increasing and decreasing, respectively, for all sufficiently small $x > 0$.

Example. The function $L(x) := \ln(1/x),\ x > 0$ is *normalized* slowly varying to the right of 0. Indeed, it belongs to the Zygmund class of slowly varying functions because, for example, for any $\delta > 0$ and $f_\delta(x) := x^\delta \ln(1/x)$, we have

$$\frac{d}{dx} f_\delta(x) = x^{\delta-1}(\delta \ln(1/x) - 1) > 0, \quad \forall x \in (0, e^{1/\delta}).$$

One also has that, for all $x \in (0, 1/e)$,

$$\ln(1/x) = \exp\{\ln|\ln(x)|\} = \exp\Big\{ \int_x^1 \epsilon(u)du/u \Big\},$$

that is, Relation (5.3) holds, with

$$\epsilon(u) := \begin{cases} 1/|\ln(u)| & , x \in (0, 1/e) \\ 0 & , x \in [1/e, 1]. \end{cases}$$

Lemma 5.2. *Consider the function* $f(x) = x^\rho L(x),\ x > 0$, *where* $0 < \rho < 1$ *and where* $L(\cdot)$ *is a normalized slowly varying function to the right of zero (see* (5.3)*, above). Then, for any* $0 < \delta < \rho$, *there exist* $\eta > 0$ *and* $K > 0$, *such that, for all* $\lambda \in (0, \eta)$,

$$\Big|\frac{f(\lambda t) - f(\lambda s)}{f(\lambda)}\Big| \le K|t - s|^{\rho-\delta}, \quad t, s \in [0, 1]. \tag{5.4}$$

Proof. Without loss of generality, assume that $0 < s < t \le 1$. By using (5.3), we get that for all $0 < s < t \le 1$,

$$\Big|\frac{f(\lambda t) - f(\lambda s)}{f(\lambda)}\Big| = \frac{L(\lambda t)}{L(\lambda)}\Big|t^\rho - s^\rho \exp\Big\{\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\}\Big|. \tag{5.5}$$

The Potter bounds (see, for example, Theorem 1.5.6 in Bingham, Goldie and Teugels (1987)), imply that for any $A > 1$, there exists $\eta > 0$, such that for all $t \in (0, 1]$ and $\lambda \in (0, \eta)$,

$$\frac{L(\lambda t)}{L(\lambda)} \le A\Big((\lambda t/\lambda)^\delta + (\lambda t/\lambda)^{-\delta}\Big) \le \frac{2A}{t^\delta}. \tag{5.6}$$

Consider now the second term in the right-hand side of (5.5):

$$t^\rho - s^\rho \exp\Big\{\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\} = s^\rho\Big((t/s)^\rho - \exp\Big\{\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\}\Big). \tag{5.7}$$

Observe that

$$\begin{aligned} &\exp\Big\{\textstyle\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\} \le \exp\Big\{\sup_{0<u\le\lambda}|\epsilon(u)|\textstyle\int_{\lambda s}^{\lambda t} du/u\Big\} \\ &= (t/s)^{\sup_{0<u\le\lambda}|\epsilon(u)|} \le (t/s)^\delta. \end{aligned} \tag{5.8}$$

The last inequality follows from the facts that $t/s > 1$ and $\epsilon(u) \to 0$, as $u \to 0+$, since by eventually picking a smaller $\eta > 0$, one has $\sup_{0<u\le\lambda}|\epsilon(u)| \le \delta$, for all $\lambda \in (0, \eta)$. Focusing on the right–hand side of (5.7), we note that Relation (5.8) implies

$$(t/s)^\rho - \exp\Big\{\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\} \ge (t/s)^\rho - (t/s)^\delta > 0,$$

since $\delta < \rho$ and $t/s > 1$. Hence the left-hand side of (5.7) is positive, for all $\lambda \in (0, \eta)$ and $0 < s < t \le 1$.

Arguing as in (5.8), for any $\gamma > 0$, we can derive the bound

$$\exp\Big\{\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\} \ge (t/s)^{-\sup_{0<u\le\lambda}|\epsilon(u)|} \ge (s/t)^{\gamma}, \tag{5.9}$$

valid for all $0 < s < t \le 1$, and $\lambda \in (0, \eta)$, with small enough $\eta > 0$. Therefore, for all $\lambda \in (0, \eta)$ and $0 < s < t \le 1$,

$$\begin{aligned} 0 \le t^{\rho} - s^{\rho} \exp\Big\{\int_{\lambda s}^{\lambda t} \epsilon(u)du/u\Big\} &\le t^{\rho} - s^{\rho}(s/t)^{\gamma} = \frac{t^{\rho+\gamma} - s^{\rho+\gamma}}{t^{\gamma}} \\ &\le \frac{(t-s)^{\rho+\gamma}}{t^{\gamma}} = \frac{(t-s)^{\gamma}}{t^{\gamma}}(t-s)^{\rho} \le (t-s)^{\rho}. \end{aligned} \tag{5.10}$$

The first inequality was shown earlier. To get the third inequality, we used that $|a|^{\rho+\gamma} - |b|^{\rho+\gamma} \le |a-b|^{\rho+\gamma}$, valid for all $a, b \in \mathbb{R}$ and $0 < \rho + \gamma \le 1$. By using the bounds in (5.6) and (5.10) and Relation (5.5), we obtain that for all $\lambda \in (0, \eta)$ and $0 < s < t \le 1$,

$$\Big|\frac{f(\lambda t) - f(\lambda s)}{f(\lambda)}\Big| \le \frac{2A}{t^{\delta}}(t-s)^{\rho} = 2A\Big(\frac{t-s}{t}\Big)^{\delta}(t-s)^{\rho-\delta}.$$

Since $(t-s)^{\delta}/t^{\delta} \le 1$, the last inequality implies (5.4), which completes the proof of the lemma. □

The following two lemmas are used in the proofs of Corollaries 3.1 and 3.2.

Lemma 5.3. *Let ξ_n, $n \in \mathbb{N}$ be random elements taking values in a linear metric space (E, ρ) and defined on a common probablity space. Suppose that the laws of ξ_n, $n \in \mathbb{R}$ are uniformly tight. Then, for any $d(n) \to \infty$,*

$$\xi_n/d(n) \longrightarrow^{P} 0, \quad \textit{as } n \to \infty, \tag{5.11}$$

where $\to^{P}$ denotes convergence in probability in the space (E, ρ).

Proof. Let $B_0(r) = \{x \in E,\ \rho(x, 0) < r\}$ denote a ball of radius $r > 0$ and center 0. One needs to show that for all $\delta > 0$ and $\epsilon > 0$,

$$\mathbb{P}\{\xi_n/d(n) \in B_0(\delta)\} \ge 1 - \epsilon, \tag{5.12}$$

for all sufficiently large $n \in \mathbb{N}$.

The tightness of $\xi_n,\ n \in \mathbb{N}$ implies that for all $\epsilon > 0$, there is a compact set $K \subset E$, such that $\mathbb{P}\{\xi_n \in K\} \geq 1-\epsilon,\ \ n \in \mathbb{N}$. Since K is a compact in the linear metric space (E, ρ), there exist $R > 0$, such that $K \subset B_0(R)$. Therefore,

$$1 - \epsilon \leq \mathbb{P}\{\xi_n \in K\} \leq \mathbb{P}\{\xi_n/d(n) \in B_0(R/d(n))\}.$$

The last relation implies that (5.12) holds for all sufficiently large n, because $R/d(n) \to 0,\ n \to \infty$. □

Then next lemma shows that two weakly convergent processes glued together are tight.

Lemma 5.4. *Let $\xi_n = \{\xi_n(t)\}_{t\in[a,c]}$ and $\eta_n = \{\eta_n(t)\}_{t\in[c,b]}$, $n \in \mathbb{N}$, be two sequences of processes with continuous paths, defined on the intervals $[a,c]$ and $[c,b]$ $(a < c < b)$, respectively. Assume that ξ_n and η_n are defined on the same probability space and suppose that, as $n \to \infty$,*

$$\xi_n \Rightarrow_{[a,c]} \xi \quad \text{and} \quad \eta_n \Rightarrow_{[c,b]} \eta, \tag{5.13}$$

where $\Rightarrow_{[a,c]}$ denotes the weak convergence in the space $\mathcal{C}[a,c]$ equipped with the sup–norm.

If $\xi_n(c) = \eta_n(c)$, almost surely, for all $n \in \mathbb{N}$, then the laws of $\zeta_n := \{\zeta_n(t)\}_{t\in[a,b]}$, with

$$\zeta_n(t) := \begin{cases} \xi_n(t) & ,\ t \in [a,c] \\ \eta_n(t) & ,\ t \in [c,b] \end{cases}$$

are tight in the space $\mathcal{C}[a,b]$.

P r o o f. Observe that since $\xi_n(c) = \eta_n(c)$, a.s. for all $n \in \mathbb{N}$, and since $\{\xi_n(t)\}_{t\in[a,c]}$ and $\{\eta_n(t)\}_{t\in[c,b]}$ have continuous paths, the processes $\{\zeta_n(t)\}_{t\in[a,b]}$ have continous paths, with probability one.

In view of Theorem 8.2 in Billingsley (1968), it is sufficient to show that, for all $\epsilon > 0$, there exist $\delta > 0$ and n_0, such that

$$\mathbb{P}\{\omega_\delta(\zeta_n;[a,b]) \geq \epsilon\} := \mathbb{P}\Big\{ \sup_{t,s\in[a,b],\ |t-s|\leq\delta} |\zeta_n(t) - \zeta_n(s)| \geq \epsilon \Big\} \leq \theta, \tag{5.14}$$

for all $n \geq n_0,\ n \in \mathbb{N}$. For all $a \leq t < c < s \leq b$, by the triangle inequality, we have

$$|\zeta_n(t) - \zeta_n(s)| \leq |\xi_n(t) - \xi_n(c)| + |\eta_n(c) - \eta_n(s)|,$$

with probability one, since $\xi_n(c) = \eta_n(c)$, a.s. Therefore,

$$\omega_\delta(\zeta_n;[a,b]) \leq \omega_\delta(\xi_n;[a,c]) + \omega_\delta(\eta_n;[c,b]),$$

and hence the left–hand side of (5.14) can be bounded above as follows

$$\begin{aligned}&\mathbb{P}\{\omega_\delta(\zeta_n;[a,b]) \geq \epsilon\}\\ &\leq \mathbb{P}\{\omega_\delta(\xi_n;[a,c]) \geq \epsilon/2\} + \mathbb{P}\{\omega_\delta(\eta_n;[c,b]) \geq \epsilon/2\}.\end{aligned} \tag{5.15}$$

By Relation (5.13) the laws of $\xi_n,\ n \in \mathbb{N}$ and $\eta_n,\ n \in \mathbb{N}$ are *uniformly tight* in $\mathcal{C}[a,c]$ and $\mathcal{C}[c,b]$, respectively. Thus, by applying Theorem 8.2 in Billingsley (1968) to the right–hand side of (5.15), we obtain that for any $\epsilon > 0$ and $\theta > 0$, Relation (5.14) holds, for some $\delta > 0$ and for all sufficiently large n. □

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