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A STOCHASTIC CONTROL APPROACH TO A PARABOLIC EQUATION, RECIPROCAL PROCESSES

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A controllability problem for a Fokker-Planck equation is considered. A solution (v^*, Φ^*) to that problem is constructed by a theorem of Jamison, under proper assumptions. We give a sufficiency condition concerning the initial and terminal data for that solution to exist. We show that v^* is an optimal feedback control for a stochastic optimal control problem. Further, we prove that the corresponding optimally controled stochastic process is a reciprocal process which is Markov.

1. Introduction

This work is concerned with a solution to a controllability problem for a Fokker-Planck equation. This problem deserves interest in both the area of the control of parabolic partial differential equations and for the control of Schrödinger's equation which can be written in the form of a system of two equations, one of which is the continuity equation of hydrodynamics and the other one has similitaries with the Hamilton-Jacobi equation of classical mechanics. This problem is connected to a programme initiated by Schrödinger in 1931 [16]. The aim of Schrödinger was to construct some unconventional diffusion processes associated with the classical heat equation, in such a way that their properties are as close as possible to the probabilistic concepts involved in quantum mechanics.

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In his paper Schrödinger has solved the problem:*"knowing the position of a Brownian particle in an Euclidean space at times a and b, b > a ; what is the probability for this particle to have passed through some prescribed domain of the space at some intermediate time?"*

This programme has challenged mathematicians and has been extensively developed and still poses deep questions. On the way from the 1931, Schrödinger's article to modern approaches are the ones of Bernstein [3], Beurling [4], Jamison [13], etc. Their main concept is the one of *reciprocal processes.*

Also, articles of Wakolbinger [19], which rely on Föllmer's approach [10], make a connection between a stochastic variational problem, a problem of minimum entropy distance and a problem of large deviations contained in Schrödinger's article. This new bridge permits a deeper insight into our approach in the area of stochastic optimal control, and exhibits its links with a problem of *minimum entropy distance.*

This work begins with the above mentioned controllability problem for a Fokker-Planck equation, termed (P1). Under proper assumptions on the initial and terminal data a solution (v^*, Φ^*) to that problem is constructed by Theorem1. Theorem 2 gives a sufficient condition concerning the initial and terminal data for this solution to exist. We ought to ask the following question: *as the solution constructed is not unique, what does characterise that solution among the set of all solutions to probem(P1)?*

An answer to this question is provided by Theorem3, which states that $v^*$is an optimal feedback control for a stochastic optimal control problem with contraint on the end-state, termed (P2). Further, v^* corresponds to the minimum of an entropy distance, termed (P3). Finally, problem (P1) is transformed into a controllability problem for a stochastic differential equation, termed (P4). The solution to (P4) corresponding to the one constructed in (P1) appears to be the Markovian process which satisfies the given end conditions in a set of reciprocal processes of Jamison.

2. Notation and assumption

R^n denotes an n-dimensional Euclidean space, $\mathcal{B}$ its $\sigma-$field of Borel sets, $[0, T]$ a compact interval in R_+. Definitions of stochastics processes, Brownian motion, Wiener process, diffusion process, fundamental solution are borrowed from [7], [12] and [14].

Let $U : S = [0, T] \times R^n \rightarrow R_+$ be a measurable function. We need the assumption:

$(H1)$ *U is bounded, continuous on S and satisfies a Hölder condition with respect to $x \in \mathbb{R}^n$.*

Under assumption $(H1)$, the operator $L = \frac{\partial}{\partial t} - \frac{1}{2}\triangle + U$ and its adjoint have a fundamental solution which we shall denote by

$$p(s, x; t, y), \quad 0 \leq s < t \leq T, \; x, \; y \in \mathbb{R}^n.$$

Recall that if L is homogeneous, i.e., $U = 0$, its fundamental solution is the Wiener transition probability density, denoted by $k(s, x; t, y)$.

3. Controllability problem for a Fokker-Planck equation

Let μ_0 and μ_1 be given probability measures on B having densities Φ_0 et Φ_1, respectively, with respect to the Lebesgue's measure, i.e.,

$$\mu_i(B) = \int_B \Phi_i(x)\, dx \qquad i = 0, 1.$$

The problem is:

(P_1) : Find $v : S \to R^n$ such that the equation:

$$\frac{\partial \Phi}{\partial t} = -div(v\Phi) + \frac{1}{2}\triangle\Phi \tag{1}$$

has a solution satifying both given initial and terminal data:

$$\Phi(0, .) = \Phi_0, \tag{2}$$

$$\Phi(T, .) = \Phi_1. \tag{3}$$

A solution to the problem (P_1) is given by the following result.

Theorem 1. *Suppose E is a σ-compact metric space, μ_0 and μ_1 are probability measures on $\mathcal{B}(E)$ and that q is an everywhere continuous, strictly positive function on $E \times E$. Then there is a unique pair (μ, π) of measures on $\mathcal{B}(E) \otimes \mathcal{B}(E)$ for which:*
(a) μ is a probability and π is a σ-finite product measure.
(b) $\mu(B \times E) = \mu_0(B)$, $\mu(E \times B) = \mu_1(B)$, $B \in \mathcal{B}(E)$.
(c) $d\mu = q d\pi$.

Let (H1) hold. Let $E = R^n$ and $q(x,y) = p(0,x;T,y)$. By Theorem 1, there exists measures π_0, π_1 on $\mathcal{B}$ such that :

$$\begin{aligned} \mu_0(B) &= \int_B d\pi_0(x) \int p(0,x;T,y)\, d\pi_1(y) \\ \mu_1(B) &= \int_B d\pi_1(y) \int p(0,x;T,y)\, d\pi_0(x) \end{aligned} \qquad B \in \mathcal{B}.$$

Obviously, the pair (π_0, π_1) is not unique: for any pair $(k_0, k_1) \in R^2$ with $k_0 k_1 = 1$, the pair $(k_0\pi_0,\ k_1\pi_1)$ produces the same measure π.

When $d\mu_i = \Phi_i d\lambda\,, i = 0,1,$ we show that $d\pi_i = \varphi_i d\lambda\,,\ i = 0,1,\ \lambda$ Lebesgue's measure, where the Radon-Nikodym's derived φ_i, $i = 0,1$, are nonnegative and satisfying the Schrödinger system:

$$\Phi_0(x) = \varphi_0(x) \int p(0,x;T,y)\, \varphi_1(y)\, dy, \tag{4}$$

$$\Phi_1(y) = \varphi_1(y) \int p(0,x;T,y)\, \varphi_0(x)\, dx. \tag{5}$$

If assumption (H1) holds and φ_0 and $\varphi_1$are continuous and bounded then the functions ρ and $\overline{\rho}$ given by:

$$\rho(s,x) = \int p(s,x;T,y)\, \varphi_1(y)\, dy\,, \qquad [0,T[\times \mathbb{R}^n, \tag{6}$$

$$\overline{\rho}(s,x) = \int p(0,y;s,x)\, \varphi_0(y)\, dx, \qquad]0,T] \times \mathbb{R}^n \tag{7}$$

with $\rho(T,y) = \varphi_1(y)$ and $\overline{\rho}(0,x) = \varphi_0(x)$, are solutions to the Cauchy problems

$$\begin{cases} \dfrac{\partial\ \rho}{\partial s} = -\dfrac{1}{2}\triangle\ \rho + U\ \rho & [0,T[\times \mathbb{R}^n, \\ \rho(T,x) = \varphi_1(x) & \mathbb{R}^n, \end{cases} \tag{8}$$

$$\begin{cases} \dfrac{\partial\ \overline{\rho}}{\partial s} = \dfrac{1}{2}\triangle\ \overline{\rho} - U\ \overline{\rho} &]0,T] \times \mathbb{R}^n, \\ \overline{\rho}(0,x) = \varphi_0(x) & \mathbb{R}^n \end{cases} \tag{9}$$

respectively. Further, the functions $\rho(s,x)$ in $[0,T[\times R^n$ and $\overline{\rho}(t,y)$ in $]0,T]\times\mathbb{R}^n$ are strictly positive.

One can verify by direct computation that (1), (2) et (3) are satisfied for $(v, \Phi) = (v^*, \Phi^*)$ given by

$$v^* = \frac{\nabla \rho}{\rho} \qquad [0, T[\times \mathbb{R}^n, \tag{10}$$

$$\Phi^* = \overline{\rho}\rho \qquad [0, T] \times \mathbb{R}^n. \tag{11}$$

Therefore v^* is a solution to Problem (P_1).

Note that $\rho(T, y) = \varphi_1(y)$ but $\rho(0, x) \neq \varphi_0(x)$, also, the stochastic representation of the solution to (8) is given by

$$\rho(s, x) = E_0^{s,x}\left[\exp\left(-\int_s^T U(t, x(t))\, dt\right)\varphi_1(x(T))\right],$$

then the logarithmic transformation $W = -\log\rho$ gives us: $v^* = -\nabla W$ where $W = -\log E_0^{s,x}\left[\exp\left(-\int_s^T U(t, x(t))\, dt\right)\varphi_1(x(T))\right]$.

At this point we address the following questions:

1. Under which conditions on Φ_0, Φ_1 can we assert that φ_0 and φ_1 are continuous and bounded?

2. Since the solution we have constructed is not unique, what does characterize that solution among the set of all solutions to Problem (P_1)?

An answer to question 1 is provided by Theorem 2. Let the assumption:

(H2) Φ_i, $i = 0, 1$, are continuous with compact support.

Theorem 2. *If (H.1) and (H.2) hold, then* φ_i, $i = 0, 1$, *are continuous with compact support .*

P r o o f. Let us fix a pair φ_i, $i = 0, 1$, satisfying the system (4) et (5).

a) Since $p(0, \cdot; T, \cdot)$, $g_0(\cdot) = \int p(0, x; T, \cdot)\varphi_0(x)\, dx$ and $g_1(\cdot) = \int p(0, \cdot; T, y)\varphi_1(y)\, dy$ are strictly positive, therefore *Supp* $\varphi_i =$ *Supp* $\Phi_i = K_i$, $i = 0, 1$.

b) Also, since $p(0, \cdot; T, \cdot)$ is bounded away from zero on $K_1 \otimes K_2$, $\varphi_i$is integrable (because otherwise $g_i \equiv \infty$ on K_i, which is impossible by (a)). Hence g_i, $i = 0, 1$, are strictly positive and continuous.

c) By (b), $\varphi_0 = \dfrac{\Phi_0}{g_1}, \varphi_1 = \dfrac{\Phi_1}{g_0}$ are continuous with compact support. □

Question 2 will be studied in subsequent paragraphs. We will see that v^* is the solution to a stochastic optimal control problem, and that this problem is equivalent to a problem of minimum entropy distance for an entropy distance which will be defined below.

4. Stochastic optimal control problem

4.1. Problem statement (P_2)

Let us consider the cost function

$$J(s,x;T,v)=\mathbb{E}^v_{s,x}\left[\int_s^T\left\{\frac{1}{2}v\left(t,\xi\left(t\right)\right)^2+U\left(t,\xi\left(t\right)\right)\right\}dt+W_T\left(\xi\left(T\right)\right)\right] \tag{12}$$

where ξ is a n-dimensional diffusion process in the weak sense with drift v and diffusion coefficient $1/2$. Let W_T be a real valued function defined on the bounded subset $D=\{x\,/\varphi_1(x)>0\}\subset R^n$ by $W_T\left(x\right)=-\log\varphi_1\left(x\right)$.

The problem is to choose v in a set $\mathcal{V}$ of admissible feedback controls with range in $\mathbb{R}^n$, such that the end condition $\xi(T)\in D$ hold $\mathbb{Q}^v_{s,x}-a.s.$ and $J(s,x;T,v)$ is minimised, for all $(s,x)\in[0,T[\times\mathbb{R}^n$.

4.1.1. Definition of the class $\mathcal{V}$

Let us denote by $\mathcal{V}$ the class of functions $v:\ [0,T[\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ which are Borel measurable, such that:

(i) For each (s,x) in $[0,T[\times\mathbb{R}^n$, the system:

$$d\xi(t)=v(t,\xi(t))dt+dw_{sx}(t)\quad s\leq t\leq T \tag{13}$$

$$\xi(s)=x \tag{14}$$

has one and only one weak solution $\left(\Omega,\mathcal{F},\mathcal{F}^s_t,\mathbb{Q}^v_{s,x},w_{sx},\xi\right)$.

(ii)

$$\mathbb{E}^v_{s,x}W_T\left(\xi\left(T\right)\right)<\infty. \tag{15}$$

(iii)

$$\mathbb{E}^v_{s,x}\int v\left(t,\xi\left(t\right)\right)^2dt<\infty\cdot \tag{16}$$

($\mathbb{E}^v_{s,x}$ is the mathematic expectation with respect to $\mathbb{Q}^v_{s,x}$)

4.1.2. Change of probability space

By Friedman [12], if we consider $(\Omega_0, \mathcal{M},\mathcal{M}_t^s, \mathbb{P}_v^{s,x}, X(t))$ where
$\Omega_0 = C^0([0,T];\mathbb{R}^n)$,
$X(t, x(\cdot)) = x(t), \ t \in [0,T]$,
$\mathcal{M}_t^s = \sigma(X(u), \ s \leq u \leq t)$,with
$\mathbb{P}_v^{s,x}\{x(\cdot) \in M\} = \mathbb{Q}_{s,x}^v\{\omega : \xi(\cdot,\omega) \in M\}, \ M \in \mathcal{M}^s$,

then, according to condition (i) in the definition of $\mathcal{V}$, for $v \in \mathcal{V}$ there exists a weak solution $\left(\Omega, \mathcal{F}, \mathcal{F}_t^s, \mathbb{Q}_{s,x}^v, w_{sx}, \xi\right)$ of (13) – (14). Let

$$\mathbb{P}_v^{s,x}\{x(\cdot) \in M\} = \mathbb{Q}_{s,x}^v\{\omega : \xi(\cdot,\omega) \in M\}, \ M \in \mathcal{M}^s, \tag{17}$$

($\mathbb{P}_0^{s,x}$ be the Wiener measure starting from(s, x));
then the cost function(12) becomes:

$$J(s, x; T, v) = \mathbb{E}_v^{s,x}\left[\int_s^T \left\{\frac{1}{2}v(t, X(t))^2 + U(t, X(t))\right\} dt + W_T(X(T))\right] \tag{18}$$

($\mathbb{E}_v^{s,x}$is the mathematic expectation with respect to $\mathbb{P}_v^{s,x}$).
Therefore the problem becomes:

$$(P_2): \quad \min_{v\in\mathcal{V}} J(s, x; T, v)\cdot$$

Lemma 1. *Assume assumptions (H.1) and (H.2) hold. Then v^* given by (10), i.e., $v^*(t, x) = \frac{1}{\rho(t,x)}\nabla\rho(t, x) = -\nabla W(t, x)$ belongs to $\mathcal{V}$, and*

$$\frac{d\mathbb{P}_{v^*}^{s,x}}{d\mathbb{P}_0^{s,x}}(x(\cdot)) = \frac{\rho(T, X(T))}{\rho(s, x)} \exp\left(-\int_s^T U(t, X(t))\, dt\right) \tag{19}$$

Proof. We have to prove (i), (ii) and (iii) of the definition of $\mathcal{V}$.
(i) Existence and uniqueness of a weak solution to

$$d\xi(t) = v^*(t, \xi(t))\, dt + dw_{sx}(t), \qquad s \leq t \leq T, \tag{20}$$

$$\xi(s) = x. \tag{21}$$

Existence: Let

$$z_{sx}^{*}(t)=z_{sx}^{*}(t,x(\cdot))=\frac{\rho(t,X(t))}{\rho(s,x)}\exp\left(-\int_{s}^{t}U(r,X(r))\,dr\right),s\leq t\leq T$$

and $X(t,x(.))=x(t)$.
Under assumptions (H.1) and (H.2), ρ is continuous and bounded on $[s,T]\times R^n$ and, for $\alpha>0$, is of class $C^{1,2}$ on$[s,T-\alpha]\times R^n$. By an application of It ô's formula to ρ, one can easily prove that z_{sx}^{*} is a $(\mathbb{P}_0^{s,x},\mathcal{M}_t^s)$-martingale on $[s,T-\alpha]$, and since z_{sx}^{*} is uniformly bounded, by the Lebesgue's theorem of dominated convergence,

$$\mathbb{E}_0^{s,x}\left[z_{sx}^{*}(T)\,|\mathcal{M}_t^s\right]=z_{sx}^{*}(t),$$

i.e., z_{sx}^{*} is a $(\mathbb{P}_0^{s,x},\mathcal{M}_t^s)$ -martingale on $[s,T]$.

We recall that $v^{*}(t,\xi(t))$ needs not be defined for $t=T$, so that we don't know for now that the mapping $t\longrightarrow v^{*}(t,\xi(t))$ is continuous on $[s,T]$. For that reason, we shall rely first on Theorem 5.7 [14], then on their generalised theorem of Girsanov. Since W *is* $C^{1,2}$ on $[0,T-\alpha]\times R^n$ for any $\alpha>0$, an application of Itô formula to W, up to $T-\alpha$, and of the Hamilton-Jacobi-Bellman equation $\frac{\partial W}{\partial t}=-\frac{1}{2}\Delta W+\frac{1}{2}(\nabla W)^2-U$ on $[0,T-\alpha]\times R^n$, yields $\exp\left(\int_{s}^{t}v^{*}(r,X(r))\,dw(r)-\frac{1}{2}\int_{s}^{t}|v^{*}(r,X(r))|^2\,dr\right)=z_{sx}^{*}(t)$, $s\leq t\leq T-\alpha$, $\mathbb{P}_0^{s,x}-a.s.$

Then, since z_{sx}^{*} is continuous and non-negative, according to Theorem 5.7 [14], for any $(s,x)\in[0,T[\times R^n$, there is an M_t^s -adapted process$\gamma_{sx}(t,x(\cdot))$, $s\leq t\leq T$, such that $\mathbb{E}_0^{s,x}\left[\int_{0}^{T}\gamma_{sx}^2(t,x(\cdot))\,dt<\infty\right]=1$ and such that, for all $s\leq t<T, z_{sx}^{*}(t)=1+\int_{s}^{t}\gamma_{sx}(r,x(\cdot))\,dx(r)$ .That representation is unique.
Then we obtain $\gamma_{sx}(t)=z_{sx}^{*}(t).v^{*}(t,x(t))$, $s\leq t\leq T$, $P_0^{sx}-p.s.$

Since z_{sx}^{*} is a $(\mathbb{P}_0^{s,x},\mathcal{M}_t^s)-$ martingale on $[s,T]$and $z_{sx}^{*}(s)=1$, it satisfies $E_0^{s,x}z_{sx}^{*}(T)=1$, then on the measurable space $(\Omega_0,\mathcal{M}_T^s)$ there exists a probability measure $P_{v^*}^{s,x}$ with

$$d\mathbb{P}_{v^*}^{s,x}=z_{sx}^{*}(T,x(\cdot))\,d\mathbb{P}_0^{s,x}$$

and the random process

$$w_{sx}^{*}(t)=X(t)-X(s)-\int_{s}^{t}z_{sx}^{+}(r)\,\gamma_{sx}(r)\,dr$$

where $z_{sx}^{+}(r) = \dfrac{1}{z_{sx}^{*}(r)}$ if $z_{sx}^{*}(r) > 0$, and 0 si $z_{sx}^{*}(r) = 0$, is a Wiener process with respect to measure $P_{v^*}^{s,x}$.

Therefore, if we set $\xi^* = X$, we find that

$$\xi^*(t) = x - \int_s^t v^*(r, \xi^*(r))\, dr + w_{sx}^*(t), \quad s \le t \le T.$$

Since ξ^* et w_{sx}^* are continuous on $[s, T]$,

$$\lim_{t\to T} \int_s^t v^*(r, \xi^*(r))\, dr = \int_s^T v^*(r, \xi^*(r))\, dr \qquad \mathbb{P}_{v^*}^{s,x} - \text{p.s.}$$

and ξ^* is a strong solution to the system

$$\begin{aligned} d\xi(t) &= v^*(t, \xi(t))\, dt + dw_{sx}^*(t), \quad s \le t \le T \\ \xi(s) &= x. \end{aligned}$$

Therefore, there is a weak solution.

Uniqueness: Suppose η is a strong solution to

$$\begin{aligned} d\eta(t) &= v^*(t, \eta(t))\, dt + dw_{sx}^*(t), \quad s \le t \le T \\ \eta(s) &= x. \end{aligned}$$

in some probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^s, \mathbb{Q}_{s,x})$.

Let $\mathbb{P}^{s,x}(M) = \mathbb{Q}_{s,x}\{\omega : \eta(., \omega) \in M\}, M \in \mathcal{M}_T^s$.

Denote by $\mathbb{Q}_{s,x} \left| \mathcal{M}_{T-\alpha}^s \right.$ and $\mathbb{P}_{v^*}^{s,x} \left| \mathcal{M}_{T-\alpha}^s \right.$ the restrictions of $\mathbb{Q}_{s,x}$ and $\mathbb{P}_{v^*}^{s,x}$, respectively, with respect to $\mathcal{M}_{T-\alpha}^s$, for some $\alpha > 0$.For $\alpha > 0$, we have $\int_s^{T-\alpha} |v^*(r, x(r))|^2\, dr < \infty$ for any $x(.) \in \Omega_0$, since v^* is continuous on $[0, T-\alpha] \times R^n$,
and $\mathbb{E}_0^{s,x} \exp\left(\int_s^{T-\alpha} v^*(r, X(r))\, dw(r) - \frac{1}{2} \int_s^{T-\alpha} |v^*(r, X(r))|^2\, dr \right) = 1$ because of the martingale property of z_{sx}^*.Therefore the probability measures $\mathbb{Q}_{s,x}$ and $\mathbb{P}_{v^*}^{s,x}$ coincide on $\mathcal{M}_{T-\alpha}^s$, for each $\alpha > 0$. Hence, they also coincide on the $\sigma-$algebra $\mathcal{M}_{T-}^s$ generated by $\cup \mathcal{M}_{T-\alpha}^s, 0 \le \alpha \le T - s$. Since, by the continuity of X, $\mathcal{M}_{T-}^s = \mathcal{M}_T^s$, il follows that $\mathbb{Q}_{s,x} = \mathbb{P}_{v^*}^{s,x}$, which ends the proof of (i).

(ii) $\mathbb{E}_{s,x}^{v^*} W_T(\xi(T)) < \infty$.

From the definition of W_T, by making use of the transformation formula for integrals, we obtain
$\int\limits_{\Omega} W_T\left(\xi\left(T\right)\right) d\mathbb{P}_{s,x}^{v^*} = \int\limits_{\Omega_0} W_T\left(X\left(T\right)\right) d\mathbb{P}_{v^*}^{s,x} = \int\limits_{R^n} W_T\left(y\right) \mathbb{P}_{v^*}^{s,x}(X(T) \in dy) =$

$\frac{1}{\rho(s,x)} \int\limits_{R^n} |\varphi_1(y) \log \varphi_1(y)| \, E_0^{s,x}[\exp(-\int\limits_s^T U(t, X(t))dt) \mid X(T) = y]k(s,x;T,y)dy$
where k is the transition density of Brownian motion. By assumption $(H1)$, and from the fact that the function $\varphi_1 \log \varphi_1$ is bounded on the compact support of φ_1, and that $k\left(s,x;T,y\right)$ is bounded, we obtain (ii).

(iii) $\mathbb{E}_{s,x}^{v^*} \int\limits_s^T \left(v^*\left(t,\xi\left(t\right)\right)^2 dt < \infty.$

Let $\left(\Omega, \mathcal{F}, \mathcal{F}_t^s, \mathbb{Q}_{s,x}^{v^*}, w_{sx}, \xi\right)$ be a solution of $(13^*), (14^*)$.Take a sequence of bounded open sets $\mathcal{O}_n, n \geq 1$,with $\mathcal{O} =]0,T[\times R^n \supset \mathcal{O}_{n+1} \supset \overline{\mathcal{O}}_n, n \geq 1$. For $(s,x) \in \mathcal{O}_n$, let be the exit time of $(t, \xi\left(t\right))$ from $\mathcal{O}_n$. Then τ_n is a increasing sequence, which tends to Tas $n \to \infty$, since ξ is a continuous process. Therefore, by monotonic convergence theorem,

$0 \leq \int\limits_s^{\tau_n} |v^*(t,\xi\left(t\right)|^2 dt \nearrow \int\limits_s^T |v^*(t,\xi\left(t\right)|^2 dt, \mathbb{Q}_{s,x}^{v^*} - a.s.$

An application of LemmaV.5.1 of [8] to W,together with the HJB equation on $\mathcal{O}$, yields

$$\mathbb{E}_{s,x}^{v^*} \int\limits_s^{\tau_n} \frac{1}{2} |v^*(t,\xi\left(t\right)|^2 dt = W\left(s,x\right) - \mathbb{E}_{s,x}^{v^*} W\left(\tau_n, \xi\left(\tau_n\right)\right) - \mathbb{E}_{s,x}^{v^*} \int\limits_s^{\tau_n} U\left(t, \xi\left(t\right)\right) dt$$

Since ρ is non negative and bounded on $[0,T] \times R^n, W = -\log \rho$ is bounded below and, accordingly, $-\mathbb{E}_{s,x}^{v^*} W\left(\tau_n, \xi\left(\tau_n\right)\right)$ is bounded above by a constant not depending of n.

Likewise, since U is bounded, $-\mathbb{E}_{s,x}^{v^*} \int\limits_s^{\tau_n} U\left(t, \xi\left(t\right)\right) dt$ is bounded above by a constant not depending of n. Therefore, $\mathbb{E}_{s,x}^{v^*} \int\limits_s^{\tau_n} \frac{1}{2} |v^*(t,\xi\left(t\right)|^2 dt$ is bounded above by a constant not depending of n.It then follows from the monotone convergence theorem that

$\lim \mathbb{E}_{s,x}^{v^*} \int\limits_s^{\tau_n} \frac{1}{2} |v^*(t,\xi\left(t\right)|^2 dt = \mathbb{E}_{s,x}^{v^*} \int\limits_s^T \frac{1}{2} |v^*(t,\xi\left(t\right)|^2 dt < \infty,$

which ends the proof of (iii). Finally, it follows that $v^* \in \mathcal{V}$ $\square$

4.2. Existence of an optimal feedback control to problem (P_2) reformulated as a problem of minimum entropy distance

Theorem 3. *Let assumtions (H1) and (H2) hold. Then v^* is an optimal feedback control for the stochastic optimal control problem (P_2), i.e.,*

$$\begin{aligned}\min_{v\in\mathcal{V}} J(s,x;T,v) &= J(s,x;T,v^*)\\ &= W(s,x) = -\log\rho(s,\,x)\cdot\end{aligned}$$

Furthermore, problem (P_2) can be reformulated as a problem of minimum entropy distance.

Proof. Since $\xi(t)$ is a diffusion process and that the Novikov condition (16) ,i.e.,

$$\mathbb{E}^v_{s,x}\left[\int_s^T |v(t,\xi(t))|^2\,dt\right] < \infty$$

is satisfied for $v\in\mathcal{V}$, we deduce from Theorem 7.6 [14] that

$$\text{(22)}\quad \frac{d\mathbb{Q}^v_{s,x}}{d\mathbb{Q}^0_{s,x}}(\xi) = \exp\left[\frac{1}{2}\int_s^T |v(t,\xi(t))|^2\,dt + \int_s^T v(t,\xi(t))\,dw_{sx}(t)\right]\quad \mathbb{Q}^v_{s,x}\text{- p.s.}$$

from which we deduce, by taking logarithms and expectations on both sides

$$\text{(23)}\qquad \mathbb{E}^v_{s,x}\log\frac{d\mathbb{Q}^v_{s,x}}{d\mathbb{Q}^0_{s,x}}(\xi) = \mathbb{E}^v_{s,x}\left[\frac{1}{2}\int_s^T |v(t,\xi(t))|^2\,dt\right].$$

Since $\mathbb{E}^v_{s,x}\left[\int_s^T v(t,\xi(t))\,dw_{sx}(t)\right] = 0$, and using[12] , we obtain:

$$\text{(24)}\qquad \mathbb{E}^{sx}_v\log\frac{d\mathbb{P}^{sx}_v}{d\mathbb{P}^{sx}_0}(x(.)) = \mathbb{E}^{sx}_v\left[\frac{1}{2}\int_s^T |v(t,X(t))|^2\,dt\right].$$

Recall the Feynman-Kac formula

$$\text{(25)}\qquad \frac{d\mathbb{P}^{s,x}}{d\mathbb{P}^{s,x}_0} = \exp\left(-\int_s^T U(t,X(t))\,dt\right),$$

where $\mathbb{P}^{s,x}$ is a positive measure ≤ 1, defined from the transition function having the density $p(s,x;t,y)$ as the fundamental solution of (8) and (9), given by $P^{s,x}\{X(t+h)\in B\,|\mathcal{M}_t^s\} = P(t,X(t);t+h,B)\,,\quad B\in B\cdot$
The cost function (18) becomes

$$J(s,x;T,v) = \mathbb{E}_v^{s,x}\left[\log\frac{d\mathbb{P}_v^{s,x}}{d\mathbb{P}_0^{s,x}} - \log\frac{d\mathbb{P}^{s,x}}{d\mathbb{P}_0^{s,x}} + W_T(X(T))\right] \tag{26}$$

$$= \mathbb{E}_v^{s,x}\left[\log\frac{d\mathbb{P}_v^{s,x}}{d\mathbb{P}^{s,x}} + W_T(X(T))\right], \tag{27}$$

from (22) and (23). Using (19), we obtain

$$J(s,x;T,v) = \mathbb{E}_v^{s,x}\left[\log\frac{d\mathbb{P}_v^{s,x}}{d\mathbb{P}_{v^*}^{s,x}}\right] - \log(\rho(s,x)) \tag{28}$$

and, for $v = v^*$

$$J(s,x;T,v^*) = -\log\rho(s,x) = W(s,x)\,. \tag{29}$$

Therefore, since $E_v^{s,x}\left[\log\frac{d\mathbb{P}_v^{s,x}}{d\mathbb{P}_{v^*}^{s,x}}\right] \geq 0$ from Jensen's inequality, (28) and (29) give

$$J(s,x;T,v^*) \leq J(s,x;T,v) \quad \forall v\in\vartheta,\quad \forall(s,x)\in[0,T[\times\mathbb{R}^n,$$

i.e., v^* is an optimal feedback control for the stochastic optimal control problem (P_2), further, v^* corresponds, up to an additive constant, to the minimum of an entropy distance of $\mathbb{P}_v^{s,x}$ *from* $\mathbb{P}^{s,x}$, then it follows that problem (P_2) is reformulated as a problem of minimum entropy distance (P_3) given by

$$(P_3): \qquad \min_{P_v^{sx},\,v\in\vartheta}\mathbb{E}_v^{s,x}\left[\log\frac{d\mathbb{P}_v^{s,x}}{d\mathbb{P}^{s,x}} + W_T(X(T))\right]$$

If moreover, we define

$$\begin{aligned} \mathbb{P}_v^{0,\mu_0}(M) &= \int \mathbb{P}_v^{0,x}(M)\,\Phi_0(x)\,dx, \\ \mathbb{P}^{0,\mu_0}(M) &= \int \mathbb{P}^{0,x}(M)\,\Phi_0(x)\,dx \qquad\qquad M\in\mathcal{M}. \end{aligned} \tag{30}$$

Note that

$$\mathbb{P}_v^{0,\mu_0}(X(0)\in B) = \mathbb{P}^{0,\mu_0}(X(0)\in B) = \mu_0(B)\,.$$

Then the problem (P_3) becomes

$$\widetilde{(P_3)}: \qquad \min_{\mathbb{P}_v^{0,\mu_0},\, v\in\vartheta} \mathbb{E}_v^{0,\mu_0}\left[\log\frac{d\mathbb{P}_v^{0,\mu_0}}{d\mathbb{P}^{0,\mu_0}} + W_T(X(T))\right].$$

Finally, if we suppose that

$$\mathbb{P}_v^{0,\mu_0}\left(X(T)\in B\right) = \int_B \Phi_1\left(x\right)dx = \mu_1\left(B\right) \qquad B\in\mathcal{B},$$

we obtain the Wakolbinger formulation

$$\widetilde{\widetilde{(P_3)}}: \qquad \min_{\mathbb{P}_v^{0,\mu_0}\in D} \mathbb{E}_v^{0,\mu_0}\left[\log\frac{d\mathbb{P}_v^{0,\mu_0}}{d\mathbb{P}^{0,\mu_0}}\right]$$

where

$$D(\Phi_0,\Phi_1) = \left\{\mathbb{P}_v^{0,\mu_0}, v\in\vartheta : \mathbb{P}_v^{0,\mu_0}\left(X\left(0\right)\in B\right) = \mu_0\left(B\right), \mathbb{P}_v^{0,\mu_0}\left(X(T)\in B\right) = \mu_1\left(B\right)\right\},$$

since $\mathbb{E}_v^{0,\mu_0}\left[W_T(X(T))\right] = \int W(0,x)\Phi_0\left(x\right)dx = Cte, \forall v\in\vartheta.$ $\square$

5. Controllability problem for a stochastic differential equation (P_4)

Let μ_0 and μ_1 be given probability measures on B having densities Φ_0 et Φ_1, respectively, with respect to the Lebesgue's measure, i.e.,

$$\mu_i\left(B\right) = \int_B \Phi_i\left(x\right)dx \qquad i=0,1.$$

(P_4): Find $v:[0,T]\times R^n\to R^n$ such that the stochastic differential equation

$$d\xi\left(t\right) = v\left(t,\xi\left(t\right)\right)dt + dw\left(t\right), \quad 0\le t\le T,$$

has a weak solution $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P},w(t),\xi\left(t\right))$ satisfying

$$\mathbb{P}\left(\xi\left(0\right)\in B\right) = \int_B \Phi_0\left(x\right)dx = \mu_0\left(B\right)$$

$$\mathbb{P}\left(\xi\left(T\right)\in B\right) = \int_B \Phi_1\left(x\right)dx = \mu_1\left(B\right), \; B\in\mathcal{B}.$$

Proposition 1. *If assumptions (H1) and (H2) hold, then v^* is also a solution to problem (P_4).*

P r o o f. Let $X(t)$ the Wiener process with initial distribution μ_0, i.e., $\left(\Omega_0, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_0^{0,\mu_0}, X(t)\right)$, $0 \leq t \leq T$,

$$\mathbb{P}_0^{0,\mu_0}(X(0) \in B) = \mu_0(B) = \int_B \Phi_0(x)\, dx\cdot$$

Recall that

$$\begin{aligned}\mathbb{P}_0^{0,\mu_0}(X(T) \in B) &= \int_B \left[\int k(0, x; T, y)\, \Phi_0(x)\, dx\right] dy \\ &\neq \mu_1(B).\end{aligned}$$

Then v^* solution to (P_1) gives

1. $\left(\Omega_0, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_{v^*}^{0,\mu_0}, X^*(t), 0 \leq t \leq T\right)$, $0 \leq t \leq T$, is a Wiener process with initial distribution μ_0,

$$X^*(t) = X(t) - \int_0^t v^*(s, X(s))\, ds \qquad \mathbb{P}_{v^*}^{0,\mu_0} - p.s.,$$

   and then $X(t)$ satisfies

$$dX(t) = v^*(t, X(t))\, dt + dX^*(t)\,, \quad 0 \leq t \leq T, \ in \ \left(\Omega_0, \mathcal{M}_t, \mathbb{P}_{v^*}^{0,\mu_0}\right),$$

   with

$$\begin{aligned}d\mathbb{P}_{v^*}^{0,\mu_0}(x(\cdot)) &= \frac{\rho(T, X(T))}{\rho(0, X(0))} \exp\left(-\int_0^T U(t, X(t))\, dt\right) d\mathbb{P}_0^{0,\mu_0} \\ &= \frac{\rho(T, X(T))}{\rho(0, X(0))}\, d\mathbb{P}^{0,\mu_0}\end{aligned}$$

2. $\left(\Omega_0, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_{v^*}^{0,\mu_0}, X(t), 0 \leq t \leq T\right)$ is a reciprocal process (Markovian) having a transition density given by

$$p^*(s, x; t, y) = \frac{\rho(t, y)}{\rho(s, x)}\, p(s, x; t, y) \qquad 0 \leq s \prec t \leq T,$$

with the (joint) endpoint distribution density $\varphi_0(x)p(0,x;T,y)\varphi_1(y)$, *i.e.*, $\overline{\rho}_0 p(0,x;T,y)\rho_T(y)$.

Therefore, the distribution measure of $X(T)$ is μ_1, i.e.,

$$P_{v^*}^{0,\mu_0}(X(T)\in B)=\int_B\left(\int p^*(0,x;T,y)\,\Phi_0(x)\,dx\right)dy=\mu_1(B)$$

□

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