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REVISITING OFFSPRING MAXIMA IN BRANCHING PROCESSES

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We present a progress report for studies on maxima related to offspring in branching processes. We summarize and discuss the findings on the subject that appeared in the last ten years. Some of the results are refined and illustrated with new examples.

1. Introduction

There is a significant amount of research in the theory of branching processes devoted to extreme value problems concerning different population characteristics. The history of such studies goes back to the works in 50-ies by Zolotarev [26] and Urbanik [23] (see also [6]) who considered the maximum generation size. Our goal here is to summarize and discuss results on maxima related to the offspring. Papers directly addressing this area of study have begun to appear in the last ten years (though see "hero mothers"example in [7].)

Let $\mathcal{M}_n$ denote the maximum offspring size of all individuals living in the $(n-1)$-st generation of a branching process. This is a maximum of random number of independent and identically distributed (i.i.d.) integer-valued random variables,

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where the random index is the population size of the process. $\mathcal{M}_n$ has two characteristic features: (i) the i.i.d. random variables are integer-valued and (ii) the distribution of the random index is connected to the distribution of the terms involved through the branching mechanism. These two characteristics distinguish the subject matter maxima among those studied in the general extreme value theory.

The study of the sequence $\{\mathcal{M}_n\}$ might be motivated in different ways. It provides a fertility measure characterizing the most prolific individual in one generation. It also measures the maximum litter (or family) size. In the branching tree context, it is the maximum degree of a vertex. The asymptotic behavior of $\mathcal{M}_n$ gives us some information about the influence of the largest families on the size and survival of the entire population.

The paper is organized as follows. Next section deals with maxima in simple branching processes with or without immigration. In Section 3 we derive results about maxima of a triangular array of zero-inflated geometric variables. Later we apply these to branching processes with varying geometric environments. Section 4 begins with limit theorems for the max-domain of attraction of bivariate geometric variables. Then we discuss one application to branching processes with promiscuous matting. The final section considers a different construction in which a random score (a continuous random variable) is associated with each individual in a simple branching process. We present briefly limiting results for the score's order statistics. In the end of the section, we give an extension to two-type processes.

2. Maximum family size in simple branching processes

Define a Bienaymé–Galton–Watson (BGW) branching process and its n-th generation maximum family size by $Z_0 = 1$;

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n) \quad \text{and} \quad \mathcal{M}_n \max_{1\le i\le Z_{n-1}} X_i(n) \quad (n = 1, 2, \ldots),$$

respectively, where the offspring variables $X_i(n)$ are i.i.d. nonnegative and integer-valued.

Along with the BGW process $\{Z_n\}$, we consider the process with immigration $\{Z_n^{im}\}$ and its offspring maximum

$$Z_n^{im} = \sum_{i=1}^{Z_{n-1}^{im}} X_i(n) + Y_n \quad \text{and} \quad \mathcal{M}_n^{im} = \max_{1\le i\le Z_{n-1}^{im}} X_i(n) \quad (n = 1, 2, \ldots),$$

respectively, where $\{Y_n,\ n = 1, 2, ...\}$ are independent of the offspring variables, i.i.d. and integer-valued non-negative random variables.

Finally, let us modify the immigration component such that immigrants may enter the n-th generation only if the $(n-1)$-st generation size is zero. Thus, we have the Foster-Pakes process and its offspring maximum

$$Z_n^0 = \sum_{i=1}^{Z_{n-1}^0} X_i(n) + I_{\{Z_{n-1}^0 = 0\}} Y_n \quad \text{and} \quad \mathcal{M}_n^0 = \max_{1 \le i \le Z_{n-1}^0} X_i(n) \quad (n = 1, 2, \dots),$$

where I_A stands for the indicator of A.

Denote by $F(x) = P(X_i(n) \le x)$ the common distribution function of the offspring variables with mean $0 < m < \infty$ and variance $0 < \sigma^2 \le \infty$. In this section, we deal with the subcritical $(m < 1)$, critical $(m = 1)$, and supercritical $(m > 1)$ processes separately.

2.1. Subcritical processes

Let $\hat{\mathcal{M}}_n$ denote the maximum family size in all three processes defined above: $\{Z_n\}$, $\{Z_n^{im}\}$, and $\{Z_n^0\}$. Let $g(s)$ be the immigration p.g.f.. Also, let $\mathcal{A}_n = \{Z_{n-1} > 0\}$ for processes without immigration, and $\mathcal{A}_n$ be the certain event - otherwise. The following result is true.

Theorem 1. *If* $0 < m < 1$, *then for* $x \ge 0$

$$\lim_{n\to\infty} P(\hat{\mathcal{M}}_n \le x | \mathcal{A}_n) = \gamma(F(x)) \tag{1}$$

and

$$\lim_{n\to\infty} E(\hat{\mathcal{M}}_n | \mathcal{A}_n) = \sum_{k=0}^{\infty} [1 - \gamma(F(k))] \tag{2}$$

where

(i) in case of $\{Z_n\}$, γ *is the unique p.g.f. solution of* $\gamma(f(s)) = m\gamma(s) + 1 - m$ *and (2) holds if, in addition,* $EX_i(n)\log(1 + X_i(n)) < \infty$.

(ii) in case of process $\{Z_n^{im}\}$, *(1) holds provided* $E\log(1 + Y_n) < \infty$ *and* γ *is the unique p.g.f. solution of* $\gamma(s) = g(s)\gamma(f(s))$. *(2) is true if, in addition,* $EY_n < \infty$.

(iii) in case of process $\{Z_n^0\}$ *we assume that* $E\log(1 + Y_n) < \infty$. *Then* $\gamma(s) = 1 - \sum_{n=0}^{\infty}[1 - g(f_n(s))]$ $(0 < s \le 1)$ *and* $\gamma(0) = \{1 + \sum_{n=0}^{\infty}[1 - g(f_n(0))]\}^{-1}$. *Also, (2) holds if, in addition,* $EY_n < \infty$.

Example 1. *Consider* $\{Z_n\}$ *with geometric offspring p.g.f.* $f(s) = p/(1-qs)$, *where* $1/2 < p = 1 - q < 1$. *Then* $m = q/p < 1$ *and it is not difficult to see that* $\gamma(s) = (1-m)s/(1-ms)$. *Hence*

$$\lim_{n\to\infty} P(\mathcal{M}_n \le k \mid Z_{n-1} > 0) = \frac{(p-q)(1-q^{k+1})}{p-q(1-q^{k+1})}.$$

It can also be seen ([20]) that

$$\frac{m}{1-pm} \le \lim_{n\to\infty} E(\mathcal{M}_n | Z_n > 0) \le \frac{m}{1-m}.$$

Example 2. *Consider* $\{Z_n^{im}\}$ *(see [15]) with*

$$f(s) = (1+m-ms)^{-1} \quad (0 < m < 1) \quad \text{and} \quad g(s)f^{\nu}(s) \qquad (\nu > 0).$$

Then $\gamma(s) = ((1-m)/(1-ms))^{\nu}$, *a negative binomial p.g.f., and the above theorem yields*

$$\lim_{n\to\infty} P(\mathcal{M}_n^{im} \le x) \left(\frac{1-m}{1-mF(x)}\right)^{\nu}$$

and

$$\lim_{n\to\infty} E\mathcal{M}_n^{im} = \sum_{j=0}^{\infty} 1 - \left[\frac{1-F(j)}{1-mF(j)}\right]^{\nu} \le \frac{\nu m^2}{1-m}.$$

Example 3. *Let* $\mu = EY_n$. *Consider* $\{Z_n^0\}$ *with*

$$f(s) = (1+m-ms)^{-1} \quad \text{and} \quad g(s) = 1 - (\mu/m)\log(1+m-ms) \quad (0 < m < 1).$$

In this case $\gamma = (m - \mu\log(1-ms))/(m - \mu\log(1-m))$ *and by the theorem*

$$\lim_{n\to\infty} P\{\mathcal{M}_n^0 \le x\} = \frac{m - \mu\log(1-mF(x))}{m-\mu\log(1-m)} \ ,$$

and $\quad \lim_{n\to\infty} E\mathcal{M}_n^0 =$

$$\mu \ \frac{m + \sum_{k=0}^{\infty} \log \dfrac{1 - m[(1+m)^{k+1} - m^{k+1}]}{1-m}}{m - \mu\log(1-m)} \le \frac{\mu m}{m-\mu\log(1-m)} \ \frac{m}{1-m}.$$

2.2. Critical processes

In the rest of this section we need some asymptotic results for the maxima of i.i.d. random variables. Recall that a distribution function $F(x)$ belongs to the max-domain of attraction of a distribution function $H(x, \theta)$ (i.e., $F \in D(H)$) if and only if there exist sequences $a(n) > 0$ and $b(n)$ such that

$$\lim_{n \to \infty} F^n(a(n)x + b(n)) = H(x, \theta) \ , \tag{3}$$

weakly. According to the classical Gnedenko's result, $H(x;\theta)$ has the following (von Mises) form

$$H(x;\theta) = \exp\{-h(x;\theta)\} = \exp\left\{-(1 + x\theta^{-1})^{-\theta}\right\}, \quad 1{+}x\theta^{-1} > 0; \ -\infty < \theta < \infty. \tag{4}$$

Necessary and sufficient conditions for $F \in D(H)$ are well-known. In particular, $F \in D(\exp\{-x^{-a}\})$, $a > 0$ if and only if for $x > 0$ the following regularity condition on the tail probability holds

$$1 - F(x) = x^{-a} L(x) \ , \tag{5}$$

where $L(x)$ is a slowly varying at infinity function (s.v.f.).

A. Processes without immigration. In case of a simple BGW process, the following result holds.

Theorem 2. *Let $m = 1$ and $\sigma^2 < \infty$. (i) If (3) holds, then*

$$\lim_{n \to \infty} P\left(\frac{\mathcal{M}_n - b(n)}{a(n)} \leq x | Z_{n-1} > 0\right) = \frac{1}{1 + \sigma^2 h(x, \theta)/2}. \tag{6}$$

(ii) If (5) holds, then

$$\lim_{n \to \infty} \frac{E(\mathcal{M}_n | Z_{n-1} > 0)}{n^{1/a} L_1\,(n)} \frac{\pi/a}{\sin(\pi/a)} \qquad (a \geq 2), \tag{7}$$

where $L_1(x)$ is certain s.v.f. with known asymptotics.

The theorem implies that if $F \in D(\exp\{-e^{-x}\})$ then the limiting distribution is logistic with c.d.f. $(1 + e^{-x})^{-1}$; and if $F \in D(\exp\{-x^{-a}\})$ then the limiting distribution is log-logistic with c.d.f. $(1 + x^{-a})^{-1}$.

Theorem 3. *Let $m = 1$, $\sigma^2 = \infty$, and (5) holds. Then for $x \geq 0$ and $1 < a \leq 2$*

$$\lim_{n\to\infty} P\left(\frac{\mathcal{M}_n}{n^{1/[a(a-1)]}L_2(n)} \leq x \mid Z_{n-1} > 0\right) = 1 - \frac{1}{\left(1 + x^{a(a-1)}\right)^{1/(a-1)}}, \tag{8}$$

which is a Burr Type XII distribution (e.g. [22]) and

$$\lim_{n\to\infty} \frac{E(\mathcal{M}_n|Z_{n-1} > 0)}{n^{1/[a(a-1)]}L_2(n)} \tag{9}$$

$$= \frac{1}{a-1} B\left(\frac{1}{a-1} - \frac{1}{a(a-1)}, 1 + \frac{1}{a(a-1)}\right) \quad (1 < a \leq 2),$$

where $B(u, v)$ is the Beta function and $L_2(x)$ is certain s.v.f. with known asymptotics.

Note that for $a = 2$ the right-hand sides in (6) (under assumption (5)) and (7) coincide with those in (8) and (9), respectively. The right-hand side in (9) is the expected value of the limit in (8) (see [22]).

Example 4. *Let $1 - F(x) \sim x^{-2}\log x$. In this case one can check (see [20]) that Theorem 3 with $a = 2$ implies*

$$\lim_{n\to\infty} P\left(\frac{\mathcal{M}_n}{n^{1/2}(\log n)^{3/2}} \leq x \mid Z_{n-1} > 0\right) = \frac{4x^2}{1 + 4x^2}\ .$$

for $x \geq 0$ and

$$\lim_{n\to\infty} \frac{E(\mathcal{M}_n|Z_{n-1} > 0)}{n^{1/2}(\log n)^{3/2}} = \frac{\pi}{2}\ .$$

B. Processes with immigration $\{Z_n^{im}\}$. Let $\mu = EY_n$. We have the following theorem.

Theorem 4. *Assume that $m = 1$, $0 < \sigma^2 < \infty$, and $0 < \mu < \infty$. (i) If (3) holds, then*

$$\lim_{n\to\infty} P\left(\frac{\mathcal{M}_n^{im} - b(n)}{a(n)} \leq x\right) = \frac{1}{(1 + \sigma^2 h(x, \theta)/2)^{2\mu/\sigma^2}}. \tag{10}$$

(ii) If (5) is true, then

$$\lim_{n\to\infty} \frac{EM_n^{im}}{n^{1/a}L_2(n)} \frac{2\mu}{\sigma^2} B\left(\frac{2\mu}{\sigma^2}+\frac{1}{a}, 1-\frac{1}{a}\right) \quad (a \geq 2), \tag{11}$$

where $B(u,v)$ is the Beta function and $L_2(x)$ is certain s.v.f. with known asymptotics.

The theorem implies that if $F \in D(\exp\{-e^{-x}\})$ then the limiting distribution is generalized logistic with c.d.f. $\left(1+\sigma^2 e^{-x}/2\right)^{-2\mu/\sigma^2}$; if $F \in D(\exp\{-x^{-a}\})$ then the limiting distribution is a Burr Type III (e.g. [22]) with c.d.f. $\left(1+\sigma^2 x^{-a}/2\right)^{-2\mu/\sigma^2}$. The right-hand side in (11) is the expected value of the limit in (10) (see [22]).

Theorem 5. *Let $m = 1$, $\sigma^2 = \infty$, and (5) holds. In addition, suppose*

$$\Theta(x) := -\int_0^x \log[1-P(Z_t^{im} > 0)]dt = c\log x + d + \varepsilon(x), \tag{12}$$

where $\lim_{x\to\infty}\varepsilon(x) = 0$, $c > 0$, and d are constants. Then for $x \geq 0$,

$$\lim_{n\to\infty} P\left(\frac{\mathcal{M}_n^{im}}{n^{1/[a(a-1)]}L_2(n)} \leq x\right) \frac{1}{(1+x^{-a(a-1)})^c} \quad (1 < a \leq 2), \tag{13}$$

which is a Burr Type III distribution (e.g. [22]) and

$$\lim_{n\to\infty} \frac{E\mathcal{M}_n^{im}}{n^{1/[a(a-1)]}L_2(n)} = cB\left(c+\frac{1}{a(a-1)}, 1-\frac{1}{a(a-1)}\right) \quad (1 < a \leq 2), \tag{14}$$

where $B(u,v)$ is the Beta function and $L_2(x)$ is certain s.v.f. with known asymptotics. The right-hand side in (14) is the expected value of the limit in (13).

Note that for $c = 1$ and $a = 2$ the right-hand sides in (13) and (14) coincide with those in (8) and (9), respectively. The condition (12) holds even when the immigration mean is not finite. Next example illustrates this point.

Example 5. *Following [16], we consider offspring and immigrants generated by*

$$f(s) = 1-(1-s)(1+(a-1)(1-s))^{-1/(a-1)} \text{ and } g(s) = \exp\{-\lambda(1-s)^{a-1}\},$$

respectively. Then (5) holds and (12) yields

$$\Theta(t) = (\lambda/(a-1))[\log t + \log(a-1) + \log(1+(a-1)t)^{-1}].$$

Therefore,

$$\lim_{n\to\infty} P\left(\frac{\mathcal{M}_n^{im}}{n^{1/[a(a-1)]}L_3\,(n)} \le x\right) \frac{1}{(1+x^{-a(a-1)})^{\lambda/(a-1)}} \quad (1 < a \le 2),$$

and

$$\lim_{n\to\infty} \frac{E\mathcal{M}_n^{im}}{n^{1/[a(a-1)]}L_3\,(n)} \frac{\lambda}{a-1} B\left(\frac{\lambda}{a-1} - \frac{1}{a(a-1)}, 1 - \frac{1}{a(a-1)}\right) \quad (1 < a \le 2),$$

where $B(u,v)$ is the Beta function and $L_3(x)$ is certain s.v.f. with known asymptotics.

C. Foster-Pakes processes $\{Z_n^0\}$. The following limit theorem for $\mathcal{M}_n^0$ under a non-linear normalization holds.

Theorem 6. *Assume that $m = 1,\ 0 < \sigma^2 < \infty$, and $0 < \mu < \infty$. If*

$$\lim_{n\to\infty} \frac{P(X_1(1) > n)}{P(X_1(1) > n+1)} = 1 \tag{15}$$

then for $0 < x < 1$,

$$\lim_{n\to\infty} P\left(\frac{\log U(\mathcal{M}_n^0)}{\log n} \le x\right) = x, \tag{16}$$

where $U(y) = 1/(1-F(y))$.

Note that (15) is a necessary condition for $X_1(n)$ to be in a max-domain of attraction.

2.3. Supercritical processes

Denote by $\hat{\mathcal{M}}_n$ (as in the subcritical case above) the maximum family size in all three processes: $\{Z_n\}$, $\{Z_n^{im}\}$, and $\{Z_n^0\}$. The following result is true.

Theorem 7. *Assume that $m > 1$ and $EX_i(n)\log(1 + X_i(n)) < \infty$. If (3) holds, then*

$$\lim_{n\to\infty} P\left(\frac{\hat{\mathcal{M}}_n - b(m^n)}{a(m^n)} \le x\right) = \psi(h(x,\theta))$$

If (5) is true, then

$$\lim_{n\to\infty} \frac{E\hat{\mathcal{M}}_n}{m^{-n/a}L_1\left(m^{-n/a}\right)} = \int_0^\infty 1 - \psi(x^{-a})dx,$$

where $L_1(x)$ is certain s.v.f. with known asymptotics.

(i) in case of $\{Z_n\}$, ψ is the unique, among the Laplace transforms, solution of

(17) $$\psi(u) = f(\psi(um^{-1})), \qquad (u > 0).$$

(ii) in case of $\{Z_n^{im}\}$, we assume in addition that $E\log(1 + Y_n) < \infty$ and

$$\psi(u) = \prod_{k=1}^{\infty} g(\varphi(um^{-k})) \qquad (u > 0)\ ,$$

where $\varphi(u)$ is the unique, among the Laplace transforms, solution of (17).

(iii) in case of $\{Z_n^0\}$, we assume in addition that $EY_n < \infty$ and

$$\psi(u) = g(\varphi(u)) - \sum_{n=0}^{\infty}[1 - f(\varphi(um^{-n}))]P(Z_n^0 = 0) \qquad (u > 0)$$

and $\varphi(u)$ is the unique, among the Laplace transforms, solution of (17).

It is interesting to compare the limiting behavior of the maximum family size in the processes allowing immigration with that when the processes evolve in "isolation", i.e., without immigration. In the supercritical case, as might be expected, the immigration has little effect on the asymptotics of the maximum family size. The limits differ only in the form of the Laplace transform $\psi(u)$. In the subcritical and critical cases the mechanism of immigration eliminates the conditioning on non–extinction. Theorem 6 for the Foster-Pakes process differs from the rest of the results by the non-linear norming of $\mathcal{M}_n$. The study of the limiting behavior of the expectation in this case needs additional efforts.

It is known that some of the most popular discrete distributions, like geometric and Poisson, do not belong to any max-domain of attraction. This restricts the applicability of the results in the critical and supercritical cases above. A general construction of discrete distributions attracted in a max-domain is given in Wilms (1994). As it is proved there, if X is attracted by a Gumbel or Fréchet distributions, then the same holds for the integer part $[X]$. Next we follow a different approach considering triangular arrays of geometric variables which leads to branching processes with varying environments.

The results in this section are published in [9], [12], and [18]-[20]. In [25] an extension for order statistics is considered.

3. Maximum family size in processes with varying environments

It is well-known that the geometric law is not attracted to any max-stable law. Therefore, the limit theorems for maxima in the critical and supercritical cases above do not apply to geometric offspring. In this section we utilize a triangular array of zero-modified geometric (ZMG) offspring distributions, instead.

3.1. Maxima of arrays of zero-modified geometric variables

In this subsection we prove limit theorems for maximum of ZMG with p.m.f.

$$P(X_i(n) = j) = \begin{cases} a_n p_n (1-p_n)^{j-1} & \text{if } j \geq 1, \\ 1 - a_n & \text{if } j = 0, \end{cases} \qquad (n = 1, 2, \dots)$$

For a positive integer ν_n consider the triangular array of variables

$$\begin{array}{lll} X_1(1), X_2(1), & \dots, & X_{\nu_1}(1) \\ X_1(2), X_2(2), & \dots, & \quad X_{\nu_2}(2) \\ & \dots & \\ X_1(n), X_2(n), & \dots, & \qquad\quad X_{\nu_n}(n) \end{array}$$

We prove limit theorems as $\nu_n \to \infty$ for the row maxima

$$\mathcal{M}_n = \max_{1 \leq i \leq \nu_n} X_i(n).$$

Let Λ has the standard Gumbel law with c.d.f. $\exp(-e^{-x})$ for $-\infty < x < \infty$.

Theorem 8. *Assume that for some real c*

$$\lim_{n\to\infty} p_n = 0 \quad \text{and} \quad \lim_{n\to\infty} p_n \log(\nu_n a_n) = 2c.$$

A. If $\lim_{n\to\infty} \log(\nu_n a_n) = \infty$, then $c \geq 0$ and

$$p_n \mathcal{M}_n - \log(\nu_n a_n) \xrightarrow{d} \Lambda - c.$$

B. If $\lim_{n\to\infty} \log(\nu_n a_n) = \alpha$, $(-\infty < \alpha < \infty)$, then

$$p_n \mathcal{M}_n \xrightarrow{d} (\Lambda + \alpha)^+.$$

The idea of the proof is to exploit: (i) the exponential approximation to the zero-modified geometric law when its mean a_n/p_n is large; (ii) the fact that exponential law is attracted by Gumbel distribution.

3.2. Processes with varying geometric environments

Consider a branching process with ZMG offspring law defined over the triangular array above. Thus, we have a simple branching process with geometric varying environments. For this process we prove limit theorems for the offspring maxima in all three classes: subcritical, critical, and supercritical. Define $\mu_0 = 1$,

$$\mu_n = E(Z_n|Z_0 = 1) = \prod_{j=1}^{n} m_j \qquad (n \geq 1).$$

If the environments are weakly varying, i.e., $\mu = \lim_{n\to\infty} \mu_n$ exists, then the processes can be classify (see [11]) as follows.

$$\{Z_n\} \text{ is } \begin{cases} \text{supercritical} & \text{if } \mu = \infty & \text{i.e. } \sum_n (m_n - 1) \to \infty \\ \text{critical} & \text{if } \mu \in (0, \infty) & \text{i.e. } \sum_n (m_n - 1) < \infty \\ \text{subcritical} & \text{if } \mu = 0 & \text{i.e. } \sum_n (m_n - 1) \to -\infty \end{cases}$$

Define the maximum family size for the process with varying geometric environments as

$$\mathcal{M}_n^{ge} = \max_{1\leq i\leq Z_n} X_i(n), \qquad (n = 1, 2, \ldots)$$

In the result below the role played by ν_n before is played by B_{n-1} where

$$B_n = \mu_n \sum_{j=1}^{n} \frac{p_j^{-1} - 1}{\mu_j}.$$

Let $\mathcal{V}$ be a standard logistic random variable with c.d.f. $(1 + e^{-x})^{-1}$ for $-\infty < x < \infty$.

Theorem 9. *Suppose that* $\lim_{n\to\infty} B_n = \infty$ *and for* c *real*

$$\lim_{n\to\infty} p_n = 0 \quad \text{and} \quad \lim_{n\to\infty} p_n \log(B_{n-1}a_n) = 2c.$$

A. If $\lim_{n\to\infty} \log(B_{n-1}a_n) = \infty$, *then*

$$(p_n\mathcal{M}_n^{ge} - \log(B_{n-1}a_n)|Z_{n-1} > 0) \xrightarrow{d} \mathcal{V} - c.$$

B. If $\lim_{n\to\infty} \log(B_{n-1}a_n) = \alpha$, $(-\infty < \alpha < \infty)$, *then*

$$(p_n\mathcal{M}_n^{ge}|Z_{n-1} > 0) \xrightarrow{d} (\mathcal{V} + \alpha)^+.$$

Referring to the above theorem, we can say that the branching mechanism transforms Gumbel to logistic distribution. It is interesting to notice that this is in parallel with results for maximum of i.i.d. random variables with random geometrically distributed index discussed in [5].

Example 6. *Let us sample a linear birth and death process $(\mathcal{B}_t)$ at irregular times. Let $Z_n = \mathcal{B}_{t_n}$ where $0 < t_n < t_{n+1} \to t_\infty \le \infty$. If λ and μ are the birth and death rates, respectively, and $d_n = t_n - t_{n-1}$, then $a_n = m_n p_n$,*

$$p_n = \begin{cases} \dfrac{\lambda-\mu}{\lambda m_n - \mu} & \text{if } \lambda \neq \mu, \\ \dfrac{1}{1+\lambda d_n} & \text{if } \lambda = \mu, \end{cases} \qquad m_n = e^{(\lambda-\mu)d_n}.$$

and

$$B_n = \begin{cases} \dfrac{\lambda(\mu_n - 1)}{\lambda-\mu} & \text{if } \lambda \neq \mu, \\ \lambda t_n & \text{if } \lambda = \mu, \end{cases} \qquad \mu_n = e^{(\lambda-\mu)t_n}.$$

A. If $\lambda > \mu$ and

$$\lim_{n\to\infty} \frac{t_n}{m_n} = \frac{2c}{\lambda-\mu} \in [0,\infty),$$

then

$$\left(\frac{\mathcal{M}_n^{ge}}{m_n} - (\lambda-\mu)t_n) \mid Z_{n-1} > 0 \right) \xrightarrow{d} \mathcal{V} - c.$$

B. If $\lambda = \mu$ and $t_n = n^\delta l(n) \quad (\delta \ge 1)$, then

$$\left(\frac{\mathcal{M}_n^{ge}}{\lambda\delta n^{\delta-1} l(n)} - \log n \mid Z_{n-1} > 0 \right) \xrightarrow{d} \mathcal{V}.$$

The results in this section can be found in [11].

4. Maxima in bisexual processes

In this section we consider maxima of triangular arrays of bivariate geometric random vectors. The obtained results are applied to a class of bisexual branching processes.

4.1. Max-domain of attraction of bivariate geometric arrays

The following construction is due to Marshall and Olkin [8]. Consider a random vector (U, V) having Bernoulli marginals, i.e., it takes on four possible values

(0,0), (0,1), (1,0), and (1,1) with probabilities $p_{00},\ p_{01},\ p_{10}$, and p_{11}, respectively. Thus the marginal probabilities for U and V are

$$P(U=0)=p_{0+}=p_{00}+p_{01}, \qquad P(U=1)=p_{1+}=p_{10}+p_{11}$$
$$P(V=0)=p_{+0}=p_{00}+p_{10}, \qquad P(V=1)=p_{+1}=p_{01}+p_{11}.$$

Consider a sequence $\{(U_n,V_n)\}_{n=1}^{\infty}$ of independent and identically distributed with (U,V) random vectors. Let ξ and η be the number of zeros preceding the first 1 in the sequences $\{U_n\}_{n=1}^{\infty}$ and $\{V_m\}_{n=1}^{\infty}$, respectively. Both ξ and η follow a geometric distribution and, in general, they are dependent variables. The vector (ξ,η) has a bivariate geometric distribution with probability mass function for integer l and k

$$(18) \qquad P(\xi=l,\eta=k)\begin{cases} p_{00}^{l}p_{10}p_{+0}^{k-l-1}p_{+1} & \text{if} \quad 0\le l<k, \\ p_{00}^{l}p_{11} & \text{if} \quad l=k, \\ p_{00}^{k}p_{01}p_{0+}^{l-k-1}p_{1+} & \text{if} \quad 0\le k<l. \end{cases}$$

and

$$(19) \qquad P(\xi>l,\eta>k)\begin{cases} p_{00}^{l+1}p_{+0}^{k-l} & \text{if} \quad 0\le l\le k, \\ p_{00}^{k+1}p_{0+}^{l-k} & \text{if} \quad 0\le k<l. \end{cases}$$

The marginals of ξ and η for integer l and k are $P(\xi=l)=p_{1+}p_{0+}^{l} \quad (l\ge 0)$ and $P(\eta=k)=p_{+1}p_{+0}^{k} \quad (k\ge 0)$, respectively and

$$(20) \bar{F}_{\xi}(l)=P(\xi>l)=p_{0+}^{l+1}\ (l\ge 0), \quad \bar{F}_{\eta}(k)=P(\eta>k)=p_{+0}^{k+1}\ (k\ge 0).$$

For $n=1,2,\ldots$, let ν_n be a positive integer and $\{(\xi_i(n),\eta_i(n)):i=1,2,\ldots,\nu_n\}$ be a triangular array of independent random vectors with the same bivariate geometric distribution (18) where p_{ij} are replaced by $p_{ij}(n) \quad (i,j=0,1)$ for $n=1,2,\ldots$ That is,

$$\begin{array}{lll} (\xi_1(1),\eta_1(1)),(\xi_2(1),\eta_2(1)), & \ldots, & (\xi_{\nu_1}(1),\eta_{\nu_1}(1)) \\ (\xi_1(2),\eta_1(2)),(\xi_2(2),\eta_2(2)), & \ldots, & (\xi_{\nu_2}(2),\eta_{\nu_2}(2)) \\ & \cdots & \\ (\xi_1(n),\eta_1(n)),(\xi_2(n),\eta_2(n)), & \ldots, & (\xi_{\nu_n}(n),\eta_{\nu_n}(n)) \end{array}$$

Below we prove a limit theorem as $\nu_n\to\infty$ for the bivariate row maximum

$$(\mathcal{M}_n^{\xi},\mathcal{M}_n^{\eta})=\left(\max_{1\le i\le\nu_n}\xi_i(n),\ \max_{1\le i\le\nu_n}\eta_i(n)\right).$$

Theorem 10. *Let* $\lim_{n\to\infty}\nu_n=\infty$. *If there are constants* $0\le a,b,c<\infty$, *such that*

$$(21)\lim_{n\to\infty}p_{11}(n)\log\nu_n=2c\quad\lim_{n\to\infty}\frac{p_{10}(n)}{p_{11}(n)}\log\nu_n=a\quad and\quad\lim_{n\to\infty}\frac{p_{01}(n)}{p_{11}(n)}\log\nu_n b,$$

then for $x,y\ge 0$

$$\lim_{n\to\infty}P\left(p_{11}(n)\mathcal{M}_n^{\xi}-\log\nu_n\le x,\quad p_{11}(n)\mathcal{M}_n^{\eta}-\log\nu_n\le y\right)$$
$$\exp\left\{-e^{-x-a-c}-e^{-y-b-c}+e^{-\max\{x,y\}-a-b-c}\right\}.$$

P r o o f. Set $x_n=(x+\log\nu_n)/p_{11}(n)$ and $y_n=(y+\log\nu_n)/p_{11}(n)$.

$$\begin{aligned}P\left(\mathcal{M}_n^{\xi}\le x_n,\mathcal{M}_n^{\eta}\le y_n\right)&=(F(x_n,y_n))^{\nu_n}\\&=\left(1-\bar F_{\xi}(x_n)-\bar F_{\eta}(y_n)+P(\xi_i(n)>x_n,\eta_i(n)>y_n)\right)^{\nu_n}.\end{aligned}$$

Let $x<y$ and thus, $x_n<y_n$. Taking logarithm, expanding in Taylor series, and using (19) and (20), we obtain

$$\begin{aligned}(22)\qquad&\log P\left(\mathcal{M}_n^{\xi}\le x_n,\mathcal{M}_n^{\eta}\le y_n\right)\\&=\nu_n\log\left(1-\bar F_{\xi}(x_n)-\bar F_{\eta}(y_n)+P(\xi_i(n)>x_n,\eta_i(n)>y_n)\right)\\&=-\nu_n\left\{[\bar F_{\xi}(x_n)+\bar F_{\eta}(y_n)-P(\xi_i(n)>x_n,\eta_i(n)>y_n)](1+o(1))\right\}\\&=-\left(\nu_np_{0+}(n)^{[x_n]+1}+\nu_np_{+0}(n)^{[y_n]+1}-\nu_np_{00}(n)^{[x_n]+1}p_{+0}(n)^{[y_n]-[x_n]}\right)(1+o(1)).\end{aligned}$$

Write $[x_n]=x_n-\{x_n\}$, where $0\le\{x_n\}<1$ is the fractional part of x_n. It is easily seen that $\lim_{n\to\infty}(p_{0+}(n))^{[x_n]+1}=\lim_{n\to\infty}(p_{0+}(n))^{x_n+1-\{x_n\}}\lim_{n\to\infty}(p_{0+}(n))^{x_n}$ as $n\to\infty$. Furthermore, taking into account (21), we have

$$\begin{aligned}&\log\left(\nu_np_{0+}^{x_n}(n)\right)\log\nu_n+\frac{x+\log\nu_n}{p_{11}(n)}\log(1-p_{1+}(n))\\&=\log\nu_n-\frac{x+\log\nu_n}{p_{11}(n)}\left(p_{11}(n)+p_{10}(n)+\frac{1}{2}(p_{11}(n)+p_{10}(n))^2+O(p_{1+}^3(n))\right)\\&=-x(1+o(1))-\frac{p_{10}(n)}{p_{11}(n)}\log\nu_n-\frac{(p_{11}(n)+p_{10}(n))^2}{2p_{11}(n)}\log\nu_n+O(p_{11}^2(n))\\&=-x(1+o(1))-\left(\frac{p_{10}(n)}{p_{11}(n)}+\frac{1}{2}p_{11}(n)\right)\log\nu_n(1+o(1))+O(p_{11}^2(n))\\&\to-x-a-c\,.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \nu_n p_{0+}(n)^{[x_n]+1} = e^{-x - a - c} \,. \tag{23}$$

Similarly we arrive at

$$\lim_{n\to\infty} \nu_n p_{+0}(n)^{[y_n]+1} = e^{-y-b-c} \text{ and } \lim_{n\to\infty} \nu_n p_{00}(n)^{[x_n]+1} = e^{-x-a-b-c} \,. \tag{24}$$

Finally,

$$\begin{aligned}
\log p_{+0}^{y_n - x_n}(n) & \frac{(y - \log\nu_n) - (x - \log\nu_n)}{p_{11}(n)} \log(1 - p_{11}(n) - p_{01}(n)) \\
= \; & -\frac{y-x}{p_{11}(n)} \left(p_{11}(n) + p_{01}(n) + \frac{1}{2}(p_{11}(n) + p_{01}(n))^2 + O(p_{+1}^3(n)) \right) \\
= \; & x - y - (y - x)\left(\frac{p_{01}(n)}{p_{11}(n)}(1 + o(1)) + \frac{1}{2}p_{11}(n)(1+o(1)) + O(p_{+1}^2(n)) \right) \\
\to \; & x - y
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} p_{+0}(n)^{[y_n]-[x_n]} = e^{x - y} \,. \tag{25}$$

The assertion of the theorem for $x < y$ follows from (22)-(25). The case $y < x$ is treated similarly. This completes the proof. □

In particular, if $a = b = 0$ then

$$\lim_{n\to\infty} P\left(p_{11}(n)\mathcal{M}_n^{\xi} - \log\nu_n \le x, p_{11}(n)\mathcal{M}_n^{\eta} - \log\nu_n \le y\right) \exp\left\{-e^{-\min\{x,y\}-c}\right\}.$$

Note that in this case the limit is proportional to the upper bound for the possible asymptotic distribution of a multivariate maximum given in [4], Theorem 5.4.1.

For the componentwise maxima, applying Theorem 10, one can obtain the following limiting results. If $p_{1+}(n)\log\nu_n \to 2c_1 < \infty$, then

$$\lim_{n\to\infty} P(p_{11}(n)\mathcal{M}_n^{\xi} - \log\nu_n \le x) \exp\left\{-e^{-x-c_1}\right\}.$$

If $p_{1+}(n)\log\nu_n \to 2c_2 < \infty$, then

$$\lim_{n\to\infty} P\left(p_{11}(n)\mathcal{M}_n^{\eta} - \log\nu_n \le y\right) \exp\left\{-e^{-y-c_2}\right\}.$$

4.2. Bisexual processes with varying geometric environments

Consider the array of bivariate random vectors $\{(\xi_i(n), \eta_i(n)) : \ i = 1, 2, \ldots; \ n = 0, 1, \ldots\}$, which are independent with respect to both indexes. Let $L : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+$ be a mating function. A bisexual process with varying environments is defined (see [14]) by the recurrence: $Z_0 = N > 0$,

$$(Z^F_{n+1}, Z^M_{n+1}) = \sum_{i=1}^{Z_n} (\xi_i(n), \eta_i(n))$$

and

$$Z_{n+1} = L(Z^F_{n+1}, Z^M_{n+1}) \quad (n = 0, 1, \ldots).$$

Define the mean growth rate per mating unit

$$r_{nj} = j^{-1} E(Z_{n+1} | Z_n = j) \quad (j = 1, 2, \ldots)$$

and

$$\mu_n = \prod_{i=0}^{n-1} r_{i1}, \ \mu_0 = 1 \quad (n = 1, 2, \ldots)$$

Lemma 1. *([14]) If*

(26) $$\sum_{n=0}^{\infty} \left(1 - \frac{r_{n1}}{r_n}\right)$$

then

$$\lim_{n \to \infty} \frac{Z_n}{\mu_n} = W \quad a.s.,$$

where W is a nonnegative random variable with $E(W) < \infty$.

If, in addition, there exist constants $A > 0$ and $c > 1$ such that

(27) $$\prod_{i=j}^{n+j-1} r_{i1} \geq A c^n \quad j = 1, 2, \ldots; \ n = 0, 1, \ldots$$

and there exists a random variable X with $E(X \log(1 + X)) < \infty$ such that for any u

(28) $$P(X \leq u) \leq P\left(\frac{L(\xi_i(n), \eta_i(n))}{r_{n1}} \leq u\right) \qquad (n = 0, 1, \ldots),$$

then $P(W > 0) > 0$.

Further on we assume that $(\xi_i(n), \eta_i(n))$ are i.i.d. copies of the bivariate geometric vector (ξ, η) introduced above and that the mating is promiscuous, i.e.,

$$L(\xi(n), \eta(n)) = \xi(n) \min\{1, \eta(n)\}. \tag{29}$$

Theorem 11. *Let $\{Z_n\}$ be a bisexual branching process with varying geometric environments and mating function (29). If*

$$\prod_{j=1}^{\infty} p_{+0}(j) p_{0+}(j) \neq 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} p_{+1}(n) < \infty \ , \tag{30}$$

then

$$\lim_{n\to\infty} \frac{Z_n}{\mu_n} = W \quad a.s., \tag{31}$$

where W is a nonnegative random variable with $E(W) < \infty$ and $P(W > 0) > 0$.

Proof. To prove the theorem it is sufficient to verify the assumptions (26)-(28) in the above lemma. First, we prove that (26) holds. Indeed, for $j \geq 1$

$$\begin{aligned} j r_{nj} &= E(Z^F_{n+1} \min\{1, Z^M_{n+1}\}) \\ &= EE(Z^F_{n+1} \min\{1, Z^M_{n+1}\} \mid Z^M_{n+1}) \\ &= (1 - P(Z^M_{n+1} = 0)) E Z^F_{n+1} \\ &= (1 - p^j_{+1}(n)) \frac{j p_{0+}(n)}{p_{1+}(n)} \ , \end{aligned} \tag{32}$$

where we have used that both Z^M_{n+1} and Z^F_{n+1} are negative binomial with parameters $(j, p_{+1}(n))$ and $(j, p_{1+}(n))$, respectively. Thus,

$$r_n = \lim_{j\to\infty} r_{nj} = \lim_{j\to\infty} (1 - p^j_{+1}(n)) \frac{p_{0+}(n)}{p_{1+}(n)} = \frac{p_{0+}(n)}{p_{1+}(n)} \ . \tag{33}$$

Now, (32) and (33) imply $1 - r_{n1}/r_n = p_{+1}(n)$, which along with (30) leads to (26).

Let us prove (28). Indeed, for $k \geq 1$

$$\begin{aligned} P(L(\xi(n), \eta(n)) = k) &= \sum_{j=1}^{\infty} P(\xi(n) \min\{1, \eta(n)\} = k | \eta(n) = j) P(\eta(n) = j) \\ &= P(\xi(n) = k) \sum_{j=1}^{\infty} P(\eta(n) = j) \end{aligned}$$

$$\begin{aligned} &= p_{1+}(n)p_{0+}^k(n)\sum_{j=1}^{\infty} p_{+1}(n)p_{+0}^j(n) \\ &= p_{+0}(n)p_{1+}(n)p_{0+}^k(n) \ . \end{aligned}$$

Therefore, $P(L(\xi(n),\eta(n))/r_{n1} \geq u) = p_{0+}^{[ur_{n1}]+1}(n)$ and hence, similarly to (23), taking into account (30), we obtain

$$\begin{aligned} \log P\left(\frac{L(\xi(n),\eta(n))}{r_{n1}} \geq u\right) &\sim ur_{n1}\log p_{0+}(n) \\ &= -u\frac{p_{+0}(n)p_{0+}(n)}{p_{1+}(n)}p_{1+}(n)(1+o(1)) \\ &\to -u \end{aligned}$$

Thus, $\lim_{n\to\infty} P(L(\xi(n),\eta(n)/r_{n1}) \geq u) = e^{-u}$, which implies (28).

Finally, to prove (27), observe that (32) implies for any j and n

$$\begin{aligned} \prod_{i=j}^{n+j-1} r_{i1} &= \prod_{i=j}^{n+j-1} p_{+0}(i)p_{0+}(i) \prod_{i=j}^{n+j-1} p_{11}^{-1}(i) \\ &\geq \prod_{i=1}^{\infty} p_{+0}(i)p_{0+}(i) \prod_{i=j}^{n+j-1} p_{11}^{-1}(i) \\ &\geq Ac^n \ , \end{aligned}$$

where $A = \prod_{i=1}^{\infty} p_{+0}(i)p_{0+}(i) > 0$ (provided that the product in (30) is finite) and $c = \min_{i\geq j} p_{11}^{-1}(i) > 1$ ($p_{11}(i) \to 0$ under (30)). (27) also holds if the product in (30) is infinite. Now, referring to the above lemma we complete the proof of the theorem. $\square$

Define offspring maxima in the bisexual process $\{Z_n\}$ by

$$(\mathcal{M}_n^F, \mathcal{M}_n^M) = \left(\max_{1\leq i\leq Z_n} \xi_i(n), \ \max_{1\leq i\leq Z_n} \eta_i(n)\right).$$

Theorem 12. *Assume that* $\mu_n \to \infty$ *and there are constants* $0 \leq a,b,c < \infty$*, such that*

$$\lim_{n\to\infty} p_{11}(n)\log\mu_n = 2c \quad \lim_{n\to\infty}\frac{p_{10}(n)}{p_{11}(n)}\log\mu_n = a \quad \textit{and} \quad \lim_{n\to\infty}\frac{p_{01}(n)}{p_{11}(n)}\log\mu_n = b. \tag{34}$$

Also assume that

$$\prod_{j=1}^{\infty} p_{+0}(j)p_{0+}(j) \neq 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} p_{+1}(n) < \infty \ . \tag{35}$$

Then

$$\lim_{n\to\infty} P\left(p_{11}(n)\mathcal{M}_n^F - \log\mu_n \le x, p_{11}(n)\mathcal{M}_n^M - \log\mu_n \le y\right) =$$

$$\int_0^\infty (G(x,y))^z dP(W \le z),$$

where

$$G(x,y) = \exp\left\{-e^{-x-a-c} - e^{-y-b-c} + e^{-\max\{x,y\}-a-b-c}\right\}.$$

P r o o f. Set $x_n = (x + \log\mu_n)/p_{11}(n)$ and $y_n = (y + \log\mu_n)/p_{11}(n)$. Under assumption (34), Theorem 11 implies

$$P\left(\mathcal{M}_n^F \le x_n, \mathcal{M}_n^M \le y_n \mid Z_n = k\right) = (F(x_n,y_n))^k \to H(x,y). \tag{36}$$

Under (35), Theorem 12 implies

$$\lim_{n\to\infty} P\left(\frac{Z_n}{\mu_n} \le x\right) = P(W \le x). \tag{37}$$

Therefore, by (36) and (37),

$$\begin{aligned} P\left(\mathcal{M}_n^F \le \frac{x+\log\mu_n}{p_{11}(n)}, \mathcal{M}_n^M \le \frac{y+\log\mu_n}{p_{11}(n)}\right) &= \sum_{k=0}^\infty P\left(Z_n = k\right)\left(F(x_n,y_n)\right)^k \\ &= \sum_{k=0}^\infty P\left(\frac{Z_n}{\mu_n} = \frac{k}{\mu_n}\right)(F(x_n,y_n))^{\mu_n k/\mu_n} \\ &= \int_0^\infty (G(x,y))^z dP(W \le z). \end{aligned}$$

□

Next example, adopted from [10], shows that the various conditions in Theorem 13 can be satisfied.

Example 7. *Let $\alpha > 1$ and $\beta > 1$. Set*

$$p_{11}(n) = n^{-\alpha} \qquad \text{and} \qquad p_{01}(n) = p_{10}(n) = n^{-(\alpha+\beta)} \qquad (n \ge 2).$$

It is not difficult to see that with this choice of $p_{ij}(n)$ $(i,j = 0,1)$, we have

$$\log\mu_n \sim \alpha n \log n \qquad \text{as} \qquad n \to \infty$$

and both (34) (with $a = b = c = 0$) and (35) are satisfied.

The exposition in this section follows [10], extending some of the results there.

5. Maximum score

In this section we assume that every individual in a Galton-Watson family tree has a continuous random characteristic which maximum is of interest.

5.1. Maximum scores in Galton-Watson processes

Let us go back to the simple BGW process and attach random scores to each individual in the family tree. More specifically, associate with the j-th individual in the n-th generation a continuous random variable $Y_j(n)$. Arnold and Villaseñor (1996) published the first paper studying the maxima individual scores ("heights"). Pakes (1998) proves more general results concerning the laws of offspring score order statistics. Quoting [17], "these results provide examples of the behavior of extreme order statistics of observations from samples of random size." Define by $M_{(k),n}$ the k-th largest score within the n-th generation and by $\bar{M}_{(k),n}$ the k-th largest among the random variables $\{Y_i(n):\ 1 \leq i \leq Z_\nu, 0 \leq \nu \leq n\}$, i.e., the k-th largest score up to and including the n-th generation. Pakes (1998) studies the limiting behavior of "near maxima", i.e., (upper) extreme order statistics $M_{(k),n}$ and $\bar{M}_{(k),n}$ when $n \to \infty$ and k remains fixed. The two general cases that arise are whether the law of Z_n (or the total progeny $T_n = \sum_{\nu=0}^{n} Z_\nu$), conditional on survival, do not require or do require, normalization to converge to non-degenerate limits.

If no normalization is required then no particular restriction need to be placed on the score distribution function S, but the limit laws are rather complex mixtures of the laws of extreme order statistics. The principal result states that

$$\lim_{n\to\infty} P(M_{(k),n} \leq x|\mathcal{A}_n) = \sum_{j=1}^{\infty}\sum_{i=0}^{k-1}\binom{j}{i}(1-S(x))^i S^{j-i}(x) g_j,$$

where it is assumed that the conditional law $\mathcal{G}_n$ of Z_n given $\mathcal{A}_n$ ($\mathcal{A}_n$ includes non-extinction) converges to a discrete and non-defective limit $\mathcal{G}$ and g_j denote the masses attributed to j by $\mathcal{G}$.

If normalization is required then one must assume that the score distribution function S is attracted to an extremal law, and then the limit laws are mixtures of the classical limiting laws of extreme order statistics. let us assume that there are positive constants $C_n \uparrow \infty$ such that for the conditional law $\mathcal{G}_n$ we have $\mathcal{G}_n(xC_n) \Rightarrow N(x)$, where $N(x)$ is a non-defective but possibly degenerate distribution function. Assume also that the score distribution function S is in the domain of attraction of on extremal law given by (4). The general result in

[17] is

$$(38)\ \lim_{n\to\infty} P\left(\frac{M_{(k),n}-b(C_n)}{a(C_n)} \le x|\mathcal{A}_n\right) = \sum_{i=0}^{k-1}\frac{(h(x,\theta))^i}{i!}\int_0^\infty y^i e^{-yh(x,\theta)}dN(y).$$

Example 8. *Consider an immortal (i.e., $P(X=0)=0$) supercritical process with shifted geometric offspring law given by its p.g.f. $f(s)=s/(1+m-ms)$ $(m>1)$, then (see Pakes (1998)) (38) becomes*

$$\lim_{n\to\infty} P\left(\frac{M_{(k),n}-b(C_n)}{a(C_n)} \le x|\mathcal{A}_n\right) = 1-\left(\frac{h(x,\theta)}{1+h(x,\theta)}\right)^k.$$

Thus the limit has a generalized logistic law when the score law is attracted to Gumbel law, $h(x,\theta)=e^{-x}$; and a Pareto-type law results when S is attracted to the Fréchet law.

Phatarford (see [17]) has raised the question (in the context of horse racing), "What is the probability that the founder of a family tree is better than all its descendants?" The answer turns out to be $E\left(T^{-1}\right)$, where $T=\sum_{n=0}^{\infty} Z_n$ is the total number of individuals in the family tree. More generally, if τ_n is the index of the generation up to the n-th which contains the largest score, Pakes (1998) proves that

$$P(\tau_n=k)=E\left(\frac{Z_k}{T_n}\right), \quad (k=0,1,\ldots,n),$$

as well as limit theorems for τ_n as $n\to\infty$.

This subsection is based on [2] and [17].

5.2. Maximum scores in two-type processes

Let each individual in a two-type branching process be equipped with a nonnegative continuous random variable - individual score. We present limit theorems for the maximum individual score. Consider two independent sets of independent random vectors with integer nonnegative components

$$\{\mathbf{X}^1(n)\}=\{(X^1_{1j}(n),X^1_{2j}(n))\} \text{ and } \{\mathbf{X}^2(n)\}=\{(X^2_{1j}(n),X^2_{2j}(n))\}\ (j\ge 1; n\ge 0).$$

A two-type branching process $\{\mathbf{Z}(n)\}=\{(Z_1(n),Z_2(n))\}$ is defined as follows: $\mathbf{Z}(0)\ne\mathbf{0}$ a.s. and for $n=1,2,\ldots$

$$Z_1(n)=\sum_{j=1}^{Z_1(n-1)} X^1_{1j}(n)+\sum_{j=1}^{Z_2(n-1)} X^2_{1j}(n),$$

$$Z_2(n) = \sum_{j=1}^{Z_1(n-1)} X_{2j}^1(n) + \sum_{j=1}^{Z_2(n-1)} X_{2j}^2(n).$$

Here $X_{kj}^i(n)$ refers to the number of offspring of type k produced by the j-th individual of type i. With the j-th individual of type i living in the n-th generation we associate a non-negative continuous random variable $\zeta_{ij}(n)$, ($i = 1, 2$) "score", say. Assume that the offspring of type 1 and type 2 have scores, which are independent and identically distributed within each type. Define the maximum score within the n-th generation by

$$\mathcal{M}_n^\zeta = \max\{\mathcal{M}_n^{\zeta_1},\ \mathcal{M}_n^{\zeta_2}\}, \qquad \text{where} \quad \mathcal{M}_n^{\zeta_i} = \max_{1\le j\le Z_i(n-1)} \zeta_{ij}(n) \qquad (i = 1, 2).$$

Note that this is maximum of random number, independent but non-identically distributed random variables. Let $F_i(x) = P(\zeta_i \le x)$ $(i = 1, 2)$ be the c.d.f.'s of the scores of type 1 and type 2 individuals, respectively.
Assumption 1 (tail-equivalence) We assume that F_1 and F_2 are tail equivalent, i.e., they have the same right endpoint x_0 and for some $A > 0$

$$\lim_{x\uparrow x_0} \frac{1 - F_1(x)}{1 - F_2(x)} = A.$$

Assumption 2 (max-stability) Suppose F_1 is in a max-domain of attraction, i.e., (3) holds.

We consider the critical branching process $\mathbf{Z}(n)$ with mean matrix $\mathbf{M}$, which is positively regular and nonsingular. Let $\mathbf{M}$ has maximum eigenvalue 1 and associated right and left eigenvectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, normalized such that $\mathbf{u}\cdot\mathbf{v} = 1$ and $\mathbf{u}\cdot\mathbf{1} = 1$.

Theorem 13. *Let $\{\mathbf{Z}(n)\}$ be the above critical two-type branching process. If the offspring variance $2B < \infty$ and both Assumptions 1 and 2 hold, then*

$$\lim_{n\to\infty} P\left(\frac{\mathcal{M}_n^\zeta - b(v_1Bn)}{a(v_1Bn)} \le x|\mathbf{Z}(n) \ne \mathbf{0}\right) = \frac{1}{1 + h(x,\theta) + (v_2/v_1)h(cx+d,\theta)}, \tag{39}$$

where if $-\infty < \theta < \infty$ is fixed, then $c = A^{1/|\theta|}$ and $d = 0$; if $\theta \to \pm\infty$, then $c = 1$ and $d = \ln A$.

Proof. Since $F_1(x)$ and $F_2(x)$ are tail-equivalent, we have (see [21], p.67)

$$\lim_{n\to\infty} \left(F_2(a(n)x+b(n))\right)^n \to H(cx+d,\theta),$$

where the constants c and d are as in (39). On the other hand, it is well-known (see [3], p.191) that for $x>0$ and $y>0$

$$\lim_{n\to\infty} P\left(\frac{Z_1(n)}{v_1Bn} \le x, \frac{Z_2(n)}{v_2Bn} \le y|\mathbf{Z}(n) \neq \mathbf{0}\right) = G(x,y),$$

where the limiting distribution has Laplace transform

$$\psi(\lambda,\mu) = \frac{1}{1+\lambda+\mu} \qquad (\lambda>0,\ \mu>0). \tag{40}$$

Set $x_n = a(v_1Bn)x + b(v_1Bn)$, $s_n = k/v_1Bn$, and $t_n = l/v_2Bn$. Referring to the definition of both $\mathcal{M}_n^\zeta$ and process $\{\mathbf{Z}(n)\}$ we obtain

$$\begin{aligned}
P\left(\mathcal{M}_n^\zeta \le x_n|\mathbf{Z}_n \neq \mathbf{0}\right) &= \sum_{(k,l)=\mathbf{0}}^{\infty} P\left(\mathbf{Z}(n)=(k,l)|\mathbf{Z}(n)\neq\mathbf{0}\right) P\left(\max\left\{\mathcal{M}_n^{\zeta_1},\mathcal{M}_n^{\zeta_2}\right\} \le x_n\right) \\
&= \sum_{(k,l)=\mathbf{0}}^{\infty} P\left(\frac{Z_1(n)}{v_1Bn} = \frac{k}{v_1Bn}, \frac{Z_2(n)}{v_2Bn} = \frac{l}{v_2Bn}|\mathbf{Z}(n)\neq\mathbf{0}\right) \left[F_1(x_n)\right]^k \left[F_2(x_n)\right]^l \\
&= \sum_{(k,l)=\mathbf{0}}^{\infty} P\left(\frac{Z_1(n)}{v_1Bn} = s_n, \frac{Z_2(n)}{v_2Bn} = t_n|\mathbf{Z}(n)\neq\mathbf{0}\right) \left[F_1(x_n)\right]^{(v_1Bn)s_n} \left[F_2(x_n)\right]^{(v_1Bn)t_n(v_2/v_1)} \\
&\to \int_0^\infty\int_0^\infty H(x,\theta)^s H(cx+d,\theta)^{(v_2/v_1)t} dG(s,t) \\
&= \int_0^\infty\int_0^\infty \exp\left\{-sh(x,\theta) - t\frac{v_2}{v_1}h(cx+d,\theta)\right\} dG(s,t) \\
&= \left[1+h(x,\theta)+\frac{v_2}{v_1}h(cx+d,\theta)\right]^{-1},
\end{aligned}$$

where in the last formula we used the Laplace transform of $G(u,v)$ given in (40). The proof is complete. □

The two examples below illustrate the kind of limit laws that can be encountered.

Example 9. *Let F_1 and F_2 be Pareto c.d.f.'s given for $x_i>\theta_i>0$ and $c>0$ by*

$$F_i(x_i) = 1 - \left(\frac{\theta_i}{x_i}\right)^c \qquad (i=1,2).$$

Note that the two distributions share the same value of the parameter c. It is not difficult to check that the limit is log-logistic given by

$$\lim_{n\to\infty} P\left\{\frac{\mathcal{M}_n^{\zeta}}{\theta_1(v_1Bn)^{1/c}} \le x|\mathbf{Z}(n)\neq\mathbf{0}\right\} = \left[1+\left(1+\frac{v_2}{v_1}\left(\frac{\theta_1}{\theta_2}\right)^{-c}\right)x^{-c}\right]^{-1}.$$

Example 10. *Let F_1 and F_2 be logistic and exponential c.d.f.'s given by*

$$F_1(x_1) = 1-e^{-x_1} \quad (0<x_1<\infty) \quad \textit{and} \quad F_2(x_2) = \frac{1}{1+e^{-x_2}} \quad (-\infty<x_2<\infty),$$

respectively. It is known that both are in the max-domain of attraction of $H(x) = \exp\{-\exp\{-x\}\}$ *and share (see [1], p.91) the same normalizing constants* $a(n) = 1$ *and* $b(n) = \ln n$. *This fact, after inspecting the proof of the theorem, allows us to bypass the tail-equivalence assumption and obtain a logistic limiting distribution, i.e, for* $-\infty<x<\infty$

$$\lim_{n\to\infty} P\left\{\mathcal{M}_n^{\zeta} - \log(v_1Bn) \le x \mid \mathbf{Z}(n)\neq\mathbf{0}\right\}\left[1+\left(1+\frac{v_2}{v_1}\right)e^{-x}\right]^{-1}.$$

The results in this subsection are modifications of those in [13].

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