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EARLY DETECTION OF EMERGENT EVENTS BASED ON AN EXTREMAL PROCESS APPROACH

Christine Jacob, Zaher Khraibani, Elisaveta Pancheva

Let $\{S_n\}_{n\geq 0}$ be a real renewal process representing the successive arrival times of some event (ex.: clinical case of an infectious disease). We wish to test that the first observed events are sporadic (the interarrival times $\{\Delta S_k\}_{k\leq n}$ are i.i.d.), and not emergent. For that we build the extremal process $R(\cdot)$ from the point process $\{(T_k, X_k)\}$, where $T_k = S_k - S_1$, $X_k := [\Delta S_k]^{-1}$, $k = 1, 2, \ldots$. Assuming that the $\{\Delta S_k\}$ are i.i.d. according to $\mathcal{E}xp(\lambda)$, we calculate the distribution of $\{R(t)\}$. We also compare this distribution to the one got under the independency of $\{T_k\}$, $\{X_k\}$ (standard setting). We finally illustrate this approach by testing on the first observations of a simulation of a slowly emergent phenomenon that this phenomenon is a sporadic one, and we show that the statistic based on the extremal process is much more efficient and robust than the statistic based on the record values.

1. Introduction

We consider the successive occurrence times $\{S_n\}_{n\geq 0}$ of some phenomenon, such as the occurrence times of the first clinical cases of a new disease, or the occurrence times of the breakdowns of some new machine. In order to control the phenomenon as early as possible, we wish to detect from these first occurrence times if we deal with a sporadic phenomenon or an emergent one which means that the

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mean occurrence rate of the events during the observation period has a tendency to increase as this period increases. So we need to elaborate some fine and exact statistic. Since the smallest values of the interarrival times $\{\Delta S_k\}_{k\geq 1}$ and the interarrival times between successive such small values are particularly significative of a potential emergence, we will consider the extremal process $\{R(t)\}_{t\geq 0}$ built from the point process $\{T_k, X_k\}_{k\geq 1}$, where $T_k := S_k - S_1$ is the occurrence time of the $k+1$th case with $T_1 = 0$, and $X_k = (\Delta S_k)^{-1}$. By construction $\{T_k\}$ and $\{X_k\}$ are completely linked and $\{R(t)\}$ is a jump process that jumps at the successive random times $\{T_{L_m}\}_{m\geq 1}$ of occurrence of the upper record values $\{R_m\}_{m\geq 1}$ of the sequence $\{X_k\}$, where $R_m = X_{L_m}$ with $L_1 = 1$. We will derive the distribution of $\{R(t)\}$. We will also derive this distribution in the usual standard setting where $\{T_k\}$ and $\{X_k\}$ are independent in order to compare the results. Finally we will illustrate on a simulated trajectory of a slowly emergent phenomon, the test of the assumption H_0 of a sporadic phenomenon: the $\{\Delta S_k\}$ are i.i.d., using either the jump values $\{R(t_m)\}_{m\geq 1}$, where $t_m = T^{obs}_{L_m}$ is the mth jump time of $\{R(t)\}$ observed on the simulated trajectory, or the record process $\{R_m\}_{m\geq 1} := \{R(T_{L_m}\}_{m\geq 1}$. We will show that the extremal process is much more efficient and robust than the record process since, contrary to R_m, it contains, at each time t_m, the information of the number of events L^{obs}_m in $(0, t_m]$, but not the information m giving the number of records in $(0, t_m]$. The details of the proofs may be found in [2].

2. Assumptions and notations

We assume that the $\{\Delta S_k\}_{k\geq 1}$ are independent and identically distributed with a continuous c.d.f. $E_\lambda(t)$, where $E_\lambda(t) = 1 - e^{-\lambda t}$ (exponential distribution). We define $T_k := S_k - S_1$, $k = 1, 2, \ldots$. So $T_1 = 0$, which means that the time origin, corresponds to the occurrence time S_1 of the second event. Then $\Delta T_k := T_k - T_{k-1}$ has the same distribution as ΔS_k for $k \geq 2$. For convenience, we also define $\Delta T_1 := \Delta S_1$. We define $X_k := (\Delta T_k)^{-1}$, $k \geq 1$. Therefore the $\{X_k\}$ are i.i.d. according to the distribution:

$$\forall\, x > 0, P(X_k \leq x) = P(\Delta T_k \geq x^{-1}) = e^{-\lambda x^{-1}} =: \Phi_{1,\lambda}(x)$$

implying $d\Phi_{1,\lambda}(x) = \lambda x^{-2} e^{-\lambda x^{-1}} dx$. Moreover since $\lim_{x\to 0} \Phi_{1,\lambda}(x) = 0$ and $\lim_{x\to 0} \Phi'_{1,\lambda}(x) = 0$, where $\Phi'_{1,\lambda}(x) = d\Phi_{1,\lambda}(x)/dx$, then we set $\Phi_{1,\lambda}(0) = 0$ and $\Phi'_{1,\lambda}(0) = 0$.

We will denote for all $t > 0$, $x > 0$, $t_\lambda = \lambda t$, $x_{\lambda^{-1}} = \lambda^{-1} x$. Since λ represents the chosen time scale at which time t is defined, then t_λ and $x_{\lambda^{-1}}$ are invariant by scale change if x represents the inverse of a time. This implies that each distribution $G_\lambda(t,x)$ depending on t or x (inverse of a time) observed at the scale λ should be invariant by scale change, that is it should satisfy $G_\lambda(t,x) = G_1(t_\lambda, x_{\lambda^{-1}})$. This is the case for $\Phi_{1,\lambda}(x)$, that is $\Phi_{1,\lambda}(x) = \Phi_{1,1}(x_{\lambda^{-1}})$, and we will check this property for the distributions that will be derived here. Notice that $\Phi_{1,1}(x_{\lambda^{-1}})$ is a Fréchet distribution.

We define the p.p. (point process) $\mathcal{N} = \{(T_k, X_k)\}_{k\geq 1}$ and the extremal process $\{R(t)\}$ generated by this p.p.: So, for all $t \geq 0$, $R(t) := R_{n(t)+1} := \vee_{k=1}^{n(t)+1} X_k$, where $n(t) := n(0,t] := \#\{k \geq 2 : T_k \leq t\}$, for $t > 0$, and $n(0) = 0$, is the counting process associated to $\{T_k\}$. We denote $U(s,t] := \vee_{k=n(s)+2}^{n(t)+1} X_k$, the max-increment of $\{R(t)\}$ in $(s,t]$. We define τ_m as the mth jump time of $R(\cdot)$ with $\tau_1 = T_1 = 0$. Then defining L_m as the index of the mth record with $L_1 = 1$, we have $\tau_m = T_{L_m}$ and $R(\tau_m) = R_m$ are the mth record time and value of the process $\{(T_k, X_k)\}_{k\geq 1}$.

3. Distribution of $\{R(t)\}$ in the dependent setting

Since $X_k = (\Delta T_k)^{-1}$, for all k, the time and the state components of the point process $\{T_k, X_k\}$, are completely linked contrary to the usual setting. By construction $\mathcal{N} = \{(T_k, X_k)\}$ generates a process $\{R(t)\}$ with CADLAG and nondecreasing step functions. Moreover, for $0 = t_1 < t_2 < \cdots < t_n$,

$$P(R(t_1) \leq x_1, R(t_2) \leq x_2, \ldots, R(t_n) \leq x_n) =$$
$$P(X_1 \leq x_1, X_1 \vee U(t_1,t_2] \leq x_2, \ldots, X_1 \vee U(t_1,t_2] \vee \cdots \vee U(t_{n-1},t_n] \leq x_n),$$

where the $\{U(t_{k-1},t_k]\}$ are homogenous but nonindependent because of the nonindependence of the time and state components. So $\{R(t)\}$ is called generalized extremal process.

Moreover define $C(t) := \inf\{x \geq 0 : P(R(t) \leq x) > 0\}$, the lower endpoint of the c.d.f. of $R(t)$. The curve $C : [0,\infty) \to [0,\infty)$ is the *lower curve* of the process $R(\cdot)$. The c.d.f. of the max-increment $U(\cdot)$ above the lower curve is unique if we impose the condition $U(s,t] \geq C(t)$ a.s., for all $0 \leq s < t$ (see [1]). We will see in proposition 3.1 that here $C(\cdot) = 0$.

Let $P_{t;\lambda}(x) := P\left(\bigvee_{k=2}^{n(t)+1} X_k \leq x\right)$. Then the distribution of $R(t)$ is given by

$$F_{t;\lambda}(x) := P(R(t) \leq x) = P(R(s) \leq x, \forall\, 0 \leq s \leq t) = \Phi_{1,\lambda}(x) P_{t;\lambda}(x). \tag{1}$$

Proposition 3.1. *For a given t, $P_{t;\lambda}(x)$ is continuous in x on $(0,\infty)$ and is infinitely derivable on $(0,t^{-1})$ and (t^{-1},∞). For a given x, $\{P_{t;\lambda}(x)\}$ is infinitely* *derivable on* $(0,\infty)$. *Moreover, for* $t \geq 0$, $x \geq 0$, *defining* $\overline{E_\lambda}(\cdot) = 1 - E_\lambda(\cdot)$,

$$\begin{aligned} P_{t;\lambda}(x) &= \sum_{m=0}^{\infty} \int \cdots \int_{\substack{\sum_{k=1}^{m} u_k \leq t \\ \{u_k \geq x^{-1}\}_{k\leq m}}} \bar{E}_\lambda\left(t - \sum_{k=1}^{m} u_k\right) \Pi_{k=1}^{m} dE_\lambda(u_k) \\ &= e^{-\lambda t} \sum_{m=0}^{[xt]} \frac{(\lambda t)^m}{m!} \left(1 - \frac{m}{xt}\right)^m \qquad (2) \\ P_{t;\lambda}(0) &= e^{-\lambda t},\ P_{0;\lambda}(x) = 1, \end{aligned}$$

where $[xt]$ *is the integer part of* xt, *and by convention* $(t - 0/x)^0/0! = t^0/0! = 1$, $\bigvee_{k=2}^{1} X_k = 0$. *In addition* $P_{t;\lambda}(x) = P_{t_\lambda;1}(x_{\lambda^{-1}})$.

Consequence. Using (1),

$$F_{t;\lambda}(x) = e^{-\lambda(t+x^{-1})} \sum_{m=0}^{[xt]} \frac{(\lambda t)^m}{m!} \left(1 - \frac{m}{xt}\right)^m = F_{t_\lambda;1}(x_{\lambda^{-1}}).$$

Proof. Let $x \geq 0$. Using the full probability principle on $n(t)$, we have

$$P_{t;\lambda}(x) = \sum_{m=0}^{\infty} P\left(\bigvee_{k=2}^{n(t)+1} X_k \leq x, n(t) = m\right)$$

First assume $x = 0$. Then $P_{t;\lambda}(0) = P(\Delta T_2 > t) = e^{-\lambda t}$. Next assume $x > 0$. Then $P_{t;\lambda}(x) = \sum_{m=0}^{\infty} G_{t,m;\lambda}(x)$, where $G_{t,m;\lambda}(x) := P\left(\bigvee_{k=2}^{m+1} X_k \leq x, n(t) = m\right)$, $m \geq 0$, and by convention $\bigvee_{k=2}^{1} X_k = 0$.

Let us calculate $G_{t,m;\lambda}(\cdot)$ stepwise. For $m > 0$,

$$\begin{aligned} G_{t,m;\lambda}(x) &= P\left(\bigvee_{k=2}^{m+1} X_k \leq x, n(t) = m\right) \\ &= P\left(T_{m+2} > t, T_{m+1} \leq t, \Delta T_k \geq x^{-1}, k = 2, \ldots, m+1\right). \end{aligned}$$

This probability is nonnull if $mx^{-1} \leq t$. So assume $x \geq mt^{-1}$.

$$\begin{aligned} G_{t,m;\lambda}(x) = & \int \cdots \int_{\substack{\sum_{k=2}^{m+1} u_k \leq t \\ \{u_k \geq x^{-1}\}_{2\leq k\leq m+1}}} P\left(\Delta T_{m+2} > t - \sum_{k=2}^{m+1} u_k\right) \times \\ & \Pi_{k=2}^{m+1} dP(\Delta T_k = u_k) \\ = & \int \cdots \int_{\substack{\sum_{k=2}^{m+1} u_k \leq t \\ \{u_k \geq x^{-1}\}_{2\leq k\leq m+1}}} \bar{E}_\lambda(t - \sum_{k=2}^{m+1} u_k) \Pi_{k=2}^{m+1} dE_\lambda(u_k) \\ = & \int \cdots \int_{\substack{\sum_{k=2}^{m+1} u_k \leq t \\ \{u_k \geq x^{-1}\}_{2\leq k\leq m+1}}} e^{-\lambda(t - \sum_{k=2}^{m+1} u_k)} \Pi_{k=2}^{m+1} e^{-\lambda u_k} \lambda^m du_2 \ldots du_{m+1}. \end{aligned}$$

In addition to $t_\lambda := \lambda t$, $x_{\lambda^{-1}} := \lambda^{-1} x$, let $v_k := \lambda u_k$, $k = 2, \ldots, m+1$. Then $G_{t,m;\lambda}(x) = e^{-t_\lambda} \int \cdots \int_{\substack{\sum_{k=2}^{m+1} v_k \leq t_\lambda \\ \{v_k \geq x_{\lambda^{-1}}^{-1}\}_{2\leq k\leq m+1}}} dv_2 \ldots dv_{m+1}$. The solution of this integral is given by the change of variable $z_{k-1} = v_k - x_{\lambda^{-1}}^{-1}$, $2 \leq k \leq m+1$. Therefore $0 \leq z_1 + z_2 + \ldots z_m \leq t_\lambda - m x_{\lambda^{-1}}^{-1}$. Let $a_x := t_\lambda - m x_{\lambda^{-1}}^{-1}$, $Z_i = z_1 + \cdots + z_i = Z_{i-1} + z_i$, and assume $x \geq mt^{-1}$. Then

$$\begin{aligned} G_{t,m;\lambda}(x) &= e^{-t_\lambda} \int_0^{a_x} \int_0^{a_x - Z_1} \int_0^{a_x - Z_2} \cdots \int_0^{a_x - Z_{m-1}} dz_m \cdots dz_2 dz_1 \\ &= e^{-t_\lambda} \int_0^{a_x} \cdots \int_0^{a_x - Z_{m-2}} [z_m]_0^{a_x - Z_{m-1}} dz_{m-1} \cdots dz_1 \\ &= e^{-t_\lambda} \int_0^{a_x} \left[- \frac{(a_x - Z_1 - z_2)^{m-1}}{(m-1)!} \right]_0^{a_x - Z_1} dz_1 \\ &= \cdots = e^{-t_\lambda} \frac{a_x^m}{m!}. \end{aligned}$$

So finally

$$G_{t,m;\lambda}(x) = e^{-t_\lambda} \frac{(t_\lambda - m x_{\lambda^{-1}}^{-1})^m}{m!} \mathbb{I}_{\{m \leq xt\}} = G_{t_\lambda,m;1}(x_{\lambda^{-1}}).$$

This leads to $P_{t;\lambda}(x) = \sum_{m\geq 0} G_{t,m;\lambda}(x) = e^{-t_\lambda} \left(\sum_{m=0}^{[xt]} (m!)^{-1} (t_\lambda - m.x_{\lambda^{-1}}^{-1})^m \right)$ implying itself $P_{t;\lambda}(x) = P_{t_\lambda;1}(x_{\lambda^{-1}})$. Moreover $P_{t;\lambda}(0) := P\left(\bigvee_{k=2}^{n(t)+1} X_k \leq 0 \right) = 0$

while $P_{t;\lambda}(0_+) := \lim_{x \searrow 0} P_{t;\lambda}(x) = e^{-t_\lambda}$. $\square$

More generally, for $x_1 \le x_2$, $P(R(0) \le x_1, R(t) \le x_2) = \Phi_{1,\lambda}(x_1) P_{t;\lambda}(x_2)$, and for $0 < x_1 < x_2 \cdots < x_n$, denoting $T_t = \sup\{T_k : T_k \le t\} := T_{n(t)+1}$,

$$\begin{aligned}
&P(R(t_1) \le x_1, R(t_2) \le x_2, \ldots, R(t_n) \le x_n) = \\
&\Phi_{1,\lambda}(x_1) \int \cdots \int_{t_{t_1} \le t_{t_2} \cdots \le t_{t_n}} dP(U(0, t_{t_1}] \le x_1,\ U(t_{t_1}, t_{t_2}] \le x_2, \ldots, \\
&U(t_{t_{n-1}}, t_{t_n}] \le x_n,\ T_{t_1} = t_{t_1},\ T_{t_{t_2}} = t_{t_2}, \ldots, T_{t_n} = t_{t_n}) = \\
&\int \cdots \int_{t_{t_1} \le t_{t_2} \cdots \le t_{t_n}} \overline{E_\lambda}(t_n - t_{t_n}) P(U(0, t_{t_n} - t_{t_{n-1}}] \le x_n, \Delta T_2 > t_{n-1} - t_{t_{n-1}} | \\
&T_{t_n} - T_{t_{n-1}} = t_{t_n} - t_{t_{n-1}}) \cdots \times \\
&P(U(0, t_{t_2} - t_{t_1}] \le x_2, \Delta T_2 > t_1 - t_{t_1} | T_{t_2} - T_{t_1} = t_{t_2} - t_{t_1}) \times \\
&P(U(0, t_{t_1}] \le x_1, \Delta T_2 > 0 | T_{t_1} = t_{t_1}) \times \\
&dP(T_{t_{t_n} - t_{t_{n-1}}} = t_{t_n} - t_{t_{n-1}}) \ldots dP(T_{t_{t_2} - t_{t_1}} = t_{t_2} - t_{t_1}) dP(T_{t_1} = t_1),
\end{aligned}$$

where, for all n, $P(U(0, t_{t_n} - t_{t_{n-1}}] \le x_n, \Delta T_2 > t_{n-1} - t_{t_{n-1}} | T_{t_n} - T_{t_{n-1}} = t_{t_n} - t_{t_{n-1}})$ may be calculated in a similar way as in proposition 3.1 and $dP(T_t = t_t)$ is easily calculated using E_λ (see [2]).

Corollary 3.1. $\{R(t)\}$ *is stochastically continuous, that is* $\lim_{t \to 0} P(R(t+s) - R(s) > \varepsilon) = 0$*, for all* $\varepsilon > 0$.

Proof. We have

$$\begin{aligned}
&P(R(t+s) - R(s) > \varepsilon) = \\
&\int \cdots \int_{u \le s, v \le t+s, y \in (0,\infty)} P(U(s, t+s] > \varepsilon + y | R(s) = y, T_s = u, T_{t+s} = v) \times \\
&dP(R(s) = y, T_s = u, T_{t+s} = v) = \\
&\int \cdots \int_{u \le s, v \le t+s, y \in (0,\infty)} (1 - P(U(s, t+s] \le \varepsilon + y | R(s) = y, T_s = u, T_{t+s} = v)) \times \\
&dP(R(s) = y, T_s = u, T_{t+s} = v) \le \\
&\iint_{u \le s, v \le t+s} (1 - \mathbb{I}_{v=u}) dP(T_s = u, T_{t+s} = v) \le \\
&1 - \int_{u \le s} dP(T_s = u, T_{t+s} = u) = 1 - P(n(s, t+s] = 0)
\end{aligned}$$

which tends to 0 as $t \to 0$. $\square$

4. Distribution of $\{R(t)\}$ in the independent setting

Assume now that $X_k = \xi_k^{-1}$, for all k, where $\{\xi_k\}$ are i.i.d., with the same distribution as $\{\Delta T_k\}$ but independent of $\{\Delta T_k\}$. So the space component and the time component of the point process $\{(T_k, X_k)\}$ are now independent (classical setting) with $X_1 \sim \Phi_{1,\lambda}(\cdot)$ and $T_1 = 0$ is the same origin as in the dependent setting. Define as previously $U(s,t] := \bigvee_{k=n(s)+2}^{n(t)+1} X_k$, for $0 \leq s < t$. We show here that, thanks to the independence of the increments of $\{n(t)\}$, that the max-increments of $\{R(t)\}$ are independent, and thanks to the Poisson distribution of $n(t)$, for all t, and the independence of $\{T_k\}$, $\{X_k\}$, that $\{R(t)\}$ is a generalized G-extremal process.

Lemma 4.1. *The increments $U(0,s]$ and $U(s,t]$ are independent, homogeneous, with c.d.f.*

$$\begin{aligned} P(U(s,t] \leq x) &:= P\left(\bigvee_{k=n(s)+2}^{n(t)+1} X_k \leq x\right) \\ &= \sum_{m\geq 0} \Phi_{1,\lambda}^m(x) P(n(s,t] = m) \\ &= \exp(-\lambda(t-s)[1-\Phi_{1,\lambda}(x)]) := G_\lambda^{t-s}(x), \end{aligned} \tag{3}$$

where $G_\lambda(x) = \exp(-1 + \Phi_{1,\lambda}(x)) = G_1(x_{\lambda^{-1}})$.

Proof. Using the independence of the space and time components $\{X_k\}$, $\{\Delta T_k\}$, and the i.i.d. property of the $\{X_k\}$, we have

$$\begin{aligned} P(U(s,t] \leq x) &= P\left(\bigvee_{k=2}^{n(s,t]+1} X_{k+n(s)} \leq x\right) \\ &= \sum_{m\geq 0}\sum_{n\geq 0} P\left(\bigvee_{k=2}^{m+1} X_{k+n} \leq x) P(n(s,t] = m, n(s) = n\right) \\ &= \sum_{m\geq 0}\sum_{n\geq 0} \Phi_{1,\lambda}^m(x) P(n(s,t] = m, n(s) = n) \end{aligned}$$

The expression of $P(U(s,t] \leq x)$ is then directly deduced from $\sum_n P(n(s,t] = m, n(s) = n) = P(n(s,t] = m) = \exp(-\lambda(t-s))(\lambda(t-s))^m(m!)^{-1}$. The independence follows from the independence of the time and state components and from the independence of the increments of $n(\cdot)$:

$$\begin{aligned}
&P(U(0,s] \le y, U(s,t] \le x) = \\
&\sum_{m,n} P\left(\bigvee_{k=2}^{m+1} X_k \le y, \bigvee_{k=2}^{n+1} X_{k+m} \le x\right) P(n(s)=m, n(s,t]=n) = \\
&\sum_m \Phi_{1,\lambda}^m(y) P(n(s)=m) \sum_n \Phi_{1,\lambda}^n(x) P(n(s,t]=n) = \\
&P(U(0,s] \le y) P(U(s,t] \le x). \qquad \square
\end{aligned}$$

Proposition 4.1. *The process $\{R(t)\}$ defined by $R(t) := \bigvee_{k=1}^{n(t)+1} X_k$, which is generated by the point process $\mathcal{N} = \{(T_k, X_k)\}$ with independent time and space components is a generalized G-extremal process, that is, for $0 < x_1 < x_2 \cdots < x_n$, $0 = t_1 < t_2 \cdots < t_n$, its multidimensional distribution is $F_{t_1,t_2,\dots,t_n;\lambda}(x_1,\dots,x_n) = \Phi_{1,\lambda}(x_1) G_\lambda^{t_2-t_1}(x_2) \dots G_\lambda^{t_n-t_{n-1}}(x_n) = F_{\lambda t_1,\lambda t_2,\dots,\lambda t_n;1}(\lambda^{-1}x_1,\dots,\lambda^{-1}x_n)$. Moreover its lower curve $C(\cdot)$ is constant and equal to* 0.

P r o o f. The proof for the extremal properties is immediate and that concerning the generalized G-extremal process property is a direct consequence of Lemma 4.1. $\square$

Corollary 4.1. *$\{R(t)\}$ is stochastically continuous, that is $\lim_{t\to 0} P(R(t+s) - R(s) > \varepsilon) = 0$.*

P r o o f.

$$\begin{aligned}
P(R(t+s) - R(s) > \varepsilon) &= \int_0^\infty P(U(s,t+s] > \varepsilon + y | R(s) = y) dP(R(s) = y) \\
&= \int_0^\infty (1 - G_\lambda^t(\varepsilon + y)) d[\Phi_{1,\lambda}(y) G_\lambda^s(y)]
\end{aligned}$$

which implies $\lim_{t\to 0} P(R(t+s) - R(s) > \varepsilon) = 0$. $\square$

5. Comparison of $\{R(t)\}$ in the independent and in the dependent settings

The independency of the max-increments is lost in the dependent setting respectively to the independent setting. This implies that the G-extremal property is also lost. Moreover assume for simplification $\lambda = 1$. Then according to (2) and (3)

$$\begin{aligned} P^{dep.}(R(t) \le x) &= \Phi_{1,1}(x) e^{-t} \sum_{m=0}^{[xt]} \frac{t^m}{m!} \Big(1 - \frac{m}{xt}\Big)^m \\ P^{indep.}(R(t) \le x) &= \Phi_{1,1}(x) e^{-t} e^{t\Phi_{1,1}(x)} = \Phi_{1,1}(x) e^{-t} \sum_{m=0}^{\infty} \frac{t^m}{m!} e^{-mx^{-1}} \end{aligned}$$

which is not easy to compare analytically except for large values of m. But simulations show that $P^{dep.}(R(t) \le x) \le P^{indep.}(R(t) \le x)$, that is the extremal process seems more easily larger in the dependent case than in the independent one.

6. Distribution of $\{R_m\}$

The distribution of $\{R_m\}$ only depends on $\{X_k\}$ (and not on $\{T_k\}$). We recall here the well-known distribution of R_m ([3]) and derive quantities such as the distribution of the index of hitting of some given level by the record process. Recall that $\{\tau_m\}$ are the jump times of the extremal process $\{R(t)\}$ with $\tau_m = T_{L_m}$, and that $R(\tau_m) = R_m = X_{L_m}$ is the record value of the mth record, $m \ge 1$.

Proposition 6.1. *The records $\{R_m\}_{m \ge 1}$ are the points of a nonhomogeneous Poisson point process on the state space $(0, \infty)$ with mean measure*

$$\mu^*_\lambda(a, b] = \ln\Big(\frac{1 - \Phi_{1,\lambda}(a)}{1 - \Phi_{1,\lambda}(b)}\Big) = \mu^*_1(a_\lambda, b_\lambda] \quad 0 < a < b < \infty.$$

We set $\mu^*_\lambda(0, b] = \lim_{a \to 0} \mu^*_\lambda(a, b] = \ln(1 - \Phi_{1,\lambda}(b))^{-1}$.

This result is a direct application of Proposition 4.1.[iii] page 166 in Resnick (1987) ([4]). We will denote $P_{X;\lambda}(x)$ and $dP_{X;\lambda}(x)$ for $P(X \le x)$ and $dP(X = x)$ under $\Delta T_1 \sim E_\lambda$.

Corollary 6.1. *Let $N^*(a, b] = \#\{\mathrm{m} \ge 1 : \mathrm{R_m} \in (\mathrm{a}, \mathrm{b}]\}$ (number of records taking values in $(a, b]$). Then $E(N^*(a, b]) = \mu^*_\lambda(a, b] = \mu^*_1(a_\lambda, b_\lambda]$, and*

$$dP_{N^*(0,x];\lambda}(m) := P(N^*(0, x] = m) = \frac{[\ln(1 - \Phi_{1,\lambda}(x))^{-1}]^m}{m!} (1 - \Phi_{1,\lambda}(x))$$

and $dP_{N^*(0,x];\lambda}(m) = dP_{N^*(0,x_{\lambda^{-1}}];1}(m)$.

Let $\eta(x) = \inf\{m \geq 1 : R_m > x\}$ be the hitting index of (x, ∞) by the record values sequence. Then $\eta(b) - \eta(a) = \#\{\mathrm{m : R_m \in (a, b]}\} = \mathrm{N^*(a, b]}$ and moreover we have the following result.

Corollary 6.2. *Let $x > 0$. Then, for $m \geq 1$,*

$$\begin{aligned} P_{R_m;\lambda}(x) &:= P(R_m \leq x) = P(N^*(0,x] \geq m) \\ &= \sum_{k \geq m} \frac{[\ln(1-\Phi_{1,\lambda}(x))^{-1}]^k}{k!}(1-\Phi_{1,\lambda}(x)) = P_{R_m;1}(x_{\lambda^{-1}}) \qquad (4) \end{aligned}$$

$$\begin{aligned} dP_{\eta(x);\lambda}(k) &:= P(\eta(x) = k) = P(N^*(0,x] = k-1) \\ &= \frac{[\ln(1-\Phi_{1,\lambda}(x))^{-1}]^{k-1}}{(k-1)!}(1-\Phi_{1,\lambda}(x)) = dP_{\eta(x_{\lambda^{-1}});1}(k). \qquad (5) \end{aligned}$$

In (5), $P(\eta(x) = k)$ is calculated using the intensity of the p.p. $\{R_m\}$. It may be also calculated using the full probability principle and the independence and equidistribution of the $\{X_k\}$ (Proposition 6.2).

Proposition 6.2. *For all $k \geq 2$,*

$$\begin{aligned} &P(\eta(x) = k) = \\ &(1-\Phi_{1,\lambda}(x)) \sum_{1=j_1<j_2<\cdots<j_k<\infty} \frac{[\Phi_{1,\lambda}(x)]^{j_k-j_1}}{(j_2-j_1)(j_3-j_1)\ldots(j_k-j_1)} \qquad (6) \end{aligned}$$

and for $k = 1$, $P(\eta(x) = 1) = 1 - \Phi_{1,\lambda}(x)$.

Proof.

$$\begin{aligned} &P(\eta(x) = k) = \\ &P(X_{L_1} \leq x, \ldots, X_{L_{k-1}} \leq x, X_{L_k} > x) = \\ &\sum_{1=j_1<j_2<\cdots<j_k} P(X_{j_1} \leq x, \ldots, X_{j_{k-1}} \leq x, X_{j_k} > x, L_1 = j_1, \ldots, L_k = j_k) = \\ &\sum_{1=j_1<j_2<\cdots<j_k} \int_{0<x_1<x_2<\cdots<x_{k-1}\leq x} dP(X_{j_1} = x_1, X_{j_1} \leq x, \{X_l \leq X_{j_1}\}_{j_1+1\leq l\leq j_2-1}, \\ &\quad X_{j_2} = x_2, X_{j_2} > X_{j_1}, X_{j_2} \leq x, \{X_l \leq X_{j_2}\}_{j_2+1\leq l\leq j_3-1}, \ldots, \end{aligned}$$

$$X_{j_{k-1}} = x_{k-1}, X_{j_{k-1}} > X_{j_{k-2}}, X_{j_{k-1}} \leq x, \{X_l \leq X_{j_{k-1}}\}_{j_{k-1}+1 \leq l \leq j_k - 1}, X_{j_k} > x) =$$
$$\sum_{1=j_1<j_2<\cdots<j_k} \int_{0<x_1<x_2<\cdots<x_{k-1}\leq x} dP(X_{j_1} = x_1, \{X_l \leq x_1\}_{j_1+1\leq l\leq j_2-1}, X_{j_2} = x_2,$$
$$\{X_l \leq x_2\}_{j_2+1\leq l \leq j_3-1}, \ldots, X_{j_{k-1}} = x_{k-1}, \{X_l \leq x_{k-1}\}_{j_{k-1}+1\leq l\leq j_k-1}, X_{j_k} > x) =$$
$$\sum_{1=j_1<j_2<\cdots<j_k} [\int_{0<x_{k-1}\leq x} \ldots [\int_{0<x_2<x_3} [\int_{0<x_1<x_2} d\Phi_{1,\lambda}(x_1)[\Phi_{1,\lambda}(x_1)]^{j_2-1-j_1}]$$
$$d\Phi_{1,\lambda}(x_2)[\Phi_{1,\lambda}(x_2)]^{j_3-1-j_2}] \ldots d\Phi_{1,\lambda}(x_{k-1})[\Phi_{1,\lambda}(x_{k-1})]^{j_k-1-j_{k-1}}](1-\Phi_{1,\lambda}(x)) =$$
$$\sum_{1=j_1<j_2<\cdots<j_k} [\int_{0<x_{k-1}\leq x} \ldots [\int_{0<x_2<x_3} \frac{[\Phi_{1,\lambda}(x_2)]^{j_2-j_1}}{j_2-j_1} d\Phi_{1,\lambda}(x_2)[\Phi_{1,\lambda}(x_2)]^{j_3-1-j_2}] \ldots$$
$$d\Phi_{1,\lambda}(x_{k-1})[\Phi_{1,\lambda}(x_{k-1})]^{j_k-1-j_{k-1}}](1-\Phi_{1,\lambda}(x)) =$$
$$\sum_{1=j_1<j_2<\cdots<j_k} [\int_{0<x_{k-1}\leq x} \ldots [\int_{0<x_2<x_3} \frac{[\Phi_{1,\lambda}(x_2)]^{j_3-1-j_1}}{j_2-j_1} d\Phi_{1,\lambda}(x_2)] \ldots$$
$$d\Phi_{1,\lambda}(x_{k-1})[\Phi_{1,\lambda}(x_{k-1})]^{j_k-1-j_{k-1}}](1-\Phi_{1,\lambda}(x)) =$$
$$\sum_{1=j_1<j_2<\cdots<j_k} [\int_{0<x_{k-1}\leq x} \ldots [\int_{0<x_3<x_4} \frac{[\Phi_{1,\lambda}(x_3)]^{j_3-j_1}}{(j_3-j_1)(j_2-j_1)} [\Phi_{1,\lambda}(x_3)]^{j_4-1-j_3} d\Phi_{1,\lambda}(x_3)] \ldots$$
$$d\Phi_{1,\lambda}(x_{k-1})[\Phi_{1,\lambda}(x_{k-1})]^{j_k-1-j_{k-1}}](1-\Phi_{1,\lambda}(x)) =$$
$$(1-\Phi_{1,\lambda}(x)) \sum_{1=j_1<j_2<\cdots<j_k<\infty} \frac{[\Phi_{1,\lambda}(x)]^{j_k-j_1}}{(j_k-j_1)(j_{k-1}-j_1)\ldots(j_2-j_1)}. \qquad \square$$

Remark 6.1. *Compare* (6) *and* (5). *Using Taylor's expansion of* $\ln(1-U)^{-1}$ *at* $U=0$, *we get* $\ln(1-U)^{-1} = \sum_{l\geq 1} U^l l^{-1}$, *which implies that*

$$[\ln(1-\Phi_{1,\lambda}(x))^{-1}]^{k-1} = \sum_{n_1\geq 1,\ldots,n_{k-1}\geq 1} \frac{[\Phi_{1,\lambda}(x)]^{n_1+\cdots+n_{k-1}}}{n_1\ldots n_{k-1}} \tag{7}$$

We define $j_1, \ldots, j_{k-1}, \ldots$ *in the following way:* $j_1 = 1$, $j_2 - j_1 = n_1$, $j_3 - j_2 = n_2, \ldots$, $j_k - j_{k-1} = n_{k-1}$. *Then* (7) *becomes*

$$[\ln(1-\Phi_{1,\lambda}(x))^{-1}]^{k-1} = \sum_{1=j_1<j_2<\cdots<j_k} \frac{[\Phi_{1,\lambda}(x)]^{j_k-j_1}}{(j_2-j_1)(j_3-j_2)\ldots(j_k-j_{k-1})}$$

and therefore (5) *becomes*

$$(8)\quad P(\eta(x) = k) = \frac{(1-\Phi_{1,\lambda}(x))}{(k-1)!} \sum_{1=j_1<j_2<\cdots<j_{k-1}<j_k} \frac{[\Phi_{1,\lambda}(x)]^{j_k-j_1}}{(j_2-j_1)(j_3-j_2)\dots(j_k-j_{k-1})}$$

So finally compare (6) *and* (8). *First they are equal for* $k = 1$ *and* $k = 2$. *Next they are equal for any given* k *and all* $x > 0$, *if and only if, for all* $j_k - j_1$ *with* $j_1 = 1$, *and all* $x > 0$,

$$(9)\qquad \sum_{\mathcal{J}_{k-1}(j_k)} \frac{[\Phi_{1,\lambda}(x)]^{j_k-j_1}}{(k-1)!(j_2-j_1)\dots(j_k-j_{k-1})} = \sum_{\mathcal{J}_{k-1}(j_k)} \frac{[\Phi_{1,\lambda}(x)]^{j_k-j_1}}{(j_2-j_1)\dots(j_k-j_1)}$$

where $\mathcal{J}_{k-1}(j_k) = \{j_2,\dots,j_{k-1} : 1 = j_1 < j_2 < \cdots < j_{k-1} < j_k\}$; (9) *is checked if for all* $j_k > j_1 + (k-1)$,

$$\sum_{\mathcal{J}_{k-1}(j_k)} \left[\frac{1}{(k-1)!(j_2-j_1)(j_3-j_2)\dots(j_k-j_{k-1})} - \frac{1}{(j_2-j_1)(j_3-j_1)\dots(j_k-j_1)}\right] = 0.$$

This formula is easily checked for small values of k *and* j_k *(for example* $k = 3$ *with* $j_3 \le 5$*).*

7. Test of a sporadic phenomenon from a simulated trajectory

From processes $\{R(t)\}$ and $\{R_m\}$, we may derive statistics of the test of H_0: "the $\{X_k\}$ are i.i.d.", that is the observed phenomenon is a sporadic one. We test here this assumption on a given trajectory simulated under the assumption of an emergent phenomenon, that is we assume that the $\{\Delta S_k\}$ are independent with $\Delta S_k \sim E_k$, where $1 - E_k(\cdot) = (1 - E_\lambda(\cdot))^{\rho_k}$, $\rho_k = a^k$, $a \ge 1$. Then $P(X_k \le x) := P(\Delta T_k \ge x^{-1}) = \exp(-\lambda a^k x^{-1})$ implying $E(\Delta T_k) = 1/\lambda_k$, where $\lambda_k = \lambda.\rho_k = \lambda.a^k$ is the rate of occurrence of the $k+1$th case from time T_{k-1}, which is exponentially increasing when $a > 1$ ("epidemic" situation). We choose for the simulation $\lambda = 1$ with $a = 1.1$, which means a slow emergence. We compare the test results using either $\{R_m\}_m := \{R(T_{L_m})\}$ or $\{R(t_m)\}_m$, where $t_m = T^{obs}_{L_m}$ is the observed real time of occurrence of the mth record corresponding to the L_mth event. The statistic $R(t_m)$ contains the additional information of the number of events in $(0, t_m]$ compared to the statistic R_m, and moreover does not depend on the value m itself, while R_m is strongly dependent on m. Consequently the statistic $R(t_m)$ is more informative and should be much more robust than R_m. We will moreover see on this example that it is much easier to reject H_0 with $R(t_m)$ than with R_m and with $(R(0), R(t_m))$ than with $R(t_m)$.

7.1. Test of H_0 using $\{R_m\}_{m\geq 1}$

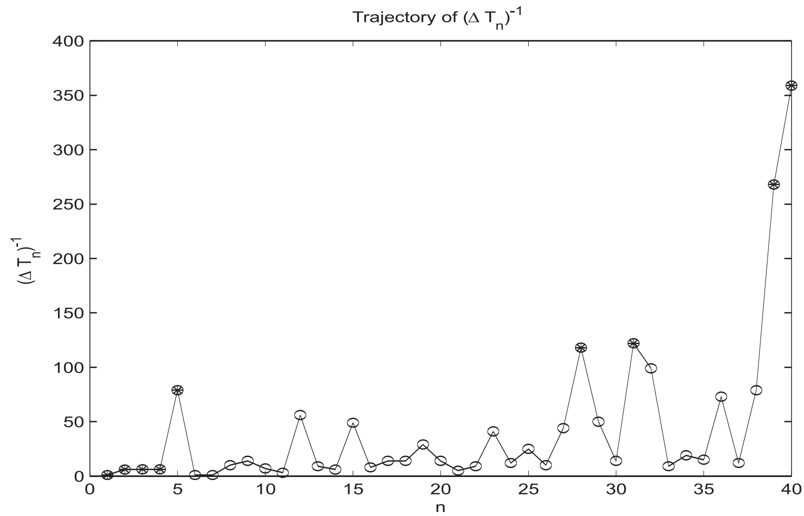

Figure 1: Simulated trajectory: $\{X_n\}_{n\leq 40}$ with the observed records represented by black ovals.

Recall that, for $\lambda = 1$, according to (4),

$$P_{H_0}(R_m > x) = \sum_{0\leq k\leq m-1} \frac{-\ln((1-\Phi_1(x))^k}{k!}(1-\Phi_1(x))$$

where $\Phi_1(x) = \exp(-x^{-1})$. Since $\Phi_1(x)$ is continuous in $x \in (0,\infty)$, we have $P_{H_0}(R_m \geq x) = P_{H_0}(R_m > x)$. For the test of H_0, we choose $x = R_m^{obs} := X_{L_m}^{obs}$ (mth observed record), for different small values of m and will reject H_0 for each value of m, if $P_{H_0}(R_m > x)$ is small enough (table 1).

Notice that since X_2, X_3, X_4 are of the same magnitude order on this trajectory, then we could set $m = 2, 3, 4, 5, 6, 7$ with the same set of values of $\{L_m\}$ as above (for example $L_m = 28$ for $m = 4$) instead of $m = 2, 5, 6, 7, 8, 9$. This implies $P_{H_0}(R_m \geq R_m^{obs}) = 0.4412, 0.1880, 0.2981, 0.4748, 0.5131$ instead of values

m	2	5	6	7	8	9
L_m	2	5	28	31	39	40
$R_m^{obs} = X_{L_m}^{obs}$	6	79	118	122	268	359
$P_{H_0}(R_m \geq R_m^{obs})$	0.4412	0.5558	0.6554	0.7897	0.7979	0.8589

Table 1: Observed levels using $\{R_m\}$

of table 1, which shows the high dependency on m of the statistic. However in both cases, H_0 cannot be rejected.

7.2. Test of H_0 using $\{R(t_m)\}_{m\geq 1}$

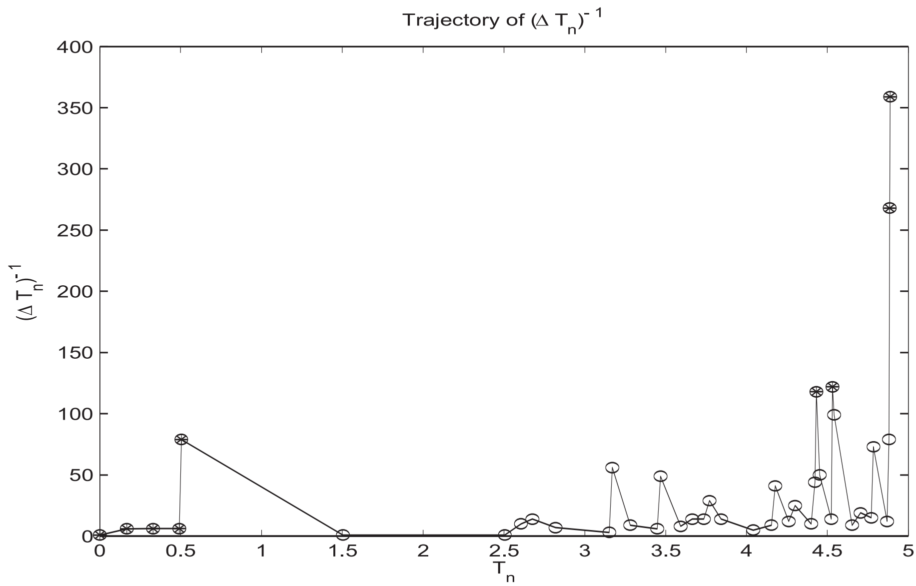

Figure 2: Simulated trajectory: $\{X_n\}$ according to $T_n \leq T_{40} = 4.8940$ with the observed records represented by black ovals.

Recall that, for $\lambda = 1$, according to (1) and (2),

$$P_{H_0}(R(t) \leq x) = \Phi_1(x)P_t(x) = \exp(-(t + x^{-1})) \sum_{m=0}^{[xt]} \frac{t^m}{m!} \Big(1 - \frac{m}{xt}\Big)^m$$

which is continuous in x on $(0,\infty)$. Therefore $P(R(t) \geq x) = P(R(t) > x) = 1-P(R(t) \leq x)$. We choose $t_m := T^{obs}_{L_m} := \sum_{k=2}^{L_m} \Delta T^{obs}_k$, for different values of m and reject H_0 if $P_{H_0}(R(t_m) \geq R^{obs}(t_m))$ is small enough (table 2). We also use the joint statistics $\{(R(0), R(t_m))\}_m$ and reject H_0 if $P_{H_0}(R(0) \geq R^{obs}(0), R(t_m) \geq R^{obs}(t_m))$ is small enough (table 2), where $R^{obs}(0) = 1.$. Notice that $P_{H_i}(R(0) \geq R^{obs}(0), R(t_m) \geq R^{obs}(t_m)) \leq P_{H_i}(R(t_m) \geq R^{obs}(t_m))$, for $i = 0, 1$, where H_1 represents the emergence assumption. Consequently, under H_0, we reject H_0 more easily with $R(0), R(t_m)$ than with $R(t_m)$, but the power of the test under H_1 is also lower using $R(0), R(t_m)$ than using $R(t_m)$. We have

$$\begin{aligned}
&P_{H_0}(R(0) \geq R^{obs}(0), R(t_m) \geq R^{obs}(t_m)) = \\
&P_{H_0}(R(t_m) \geq R^{obs}(t_m)) - P_{H_0}(R(0) \leq R^{obs}(0), R(t_m) \geq R^{obs}(t_m)) = \\
&1 - \Phi_{1,\lambda}(R^{obs}(t_m))P_{t_m;\lambda}(R^{obs}(t_m)) - \Phi_{1,\lambda}(R^{obs}(0))(1 - P_{t_m;\lambda}(R^{obs}(t_m)))
\end{aligned}$$

m	6	7	8	9
L_m	28	31	39	40
$t_m = T^{obs}_{L_m}$	4.4402	4.5398	4.8912	4.8940
$R^{obs}(t_m)$	118.	122.	268.	359.
$P_{H_0}(R(t_m) \geq R^{obs}(t_m))$	0.0526	0.0517	0.0253	0.0190
$P_{H_0}(R(0) \geq R^{obs}(0), R(t_m) \geq R^{obs}(t_m))$	0.0362	0.0355	0.0173	0.0130

Table 2: Observed levels using $\{R(t_m)\}$

So we can reject H_0, as soon as $t_m = 4.4402$ (cooresponding to $L_m = 28$ observed events) at the approximate level 0.05 using $R(t_m)$ and 0.04 using $(R(0), R(t_m))$ (table 2).

In conclusion, we reject easily H_0 with the extremal process which is not the case with the records values. This comes from the fact that the extremal process contains the useful information of the number of events in the observed period.

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REFERENCES

[1] A. A Balkema, E. I. Pancheva. Decomposition for Multivariate Extremal Processes. *Commun. Stat. Theory Meth.*, **25**, No 4 (1996), 737–758.

[2] C. Jacob, Z. Khraibani, E. Pancheva. An extremal process approach for sporadic events. Technical Report, Unité MIA-INRA, Jouy-en-Josas, 2008, 66 pp.

[3] V. B. Nevzorov. Records: mathematical Theory. Translations of Mathematical Monographs, vol. **194**, American Mathematical Society, Providence, RI, 2001.

[4] S. I. Resnick. Extreme Values, Regular Variation and Point Processes. Springer-Verlag, 1987.

Christine Jacob
Mathematics and Informatics unit
UR341, INRA
F-78352 Jouy-en-Josas, France
e-mail: christine.jacob@jouy.inra.fr

Zaher Khraibani
Mathematics and Informatics unit
UR341, INRA
F-78352 Jouy-en-Josas, France
and
Animal Epidemiology unit
UR346, INRA
63122 Saint Genès Champanelle, France
e-mail: khraibani812@hotmail.com

Elisaveta Pancheva
Department of Probability and Statistics
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgarie
e-mail: pancheva@math.bas.bg