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SOME BOUNDS FOR ALMOST ABSORBING BIRTH AND DEATH PROCESSES WITH CATASTROPHES

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We consider nonstationary almost absorbing birth and death processes (BDPs) with catastrophes. The bounds of the rate of convergence to the limit regime and the estimates of the limit probabilities are obtained. We also study the bounds for the mean of the process and consider a queueing example.

1. Introduction

The simplest queueing models with catastrophes have been studied some years ago, see the detailed motivation and some results in [1, 4], see also another approach in [3]. More detailed review and study of the first occurrence of effective catastrophe for general stationary BDPs one can find in [2]. First results for concrete nonstationary BDP have been obtained in [8]. On the other hand, estimates for some classes of almost absorbing nonstationary continuous-time Markov chains (firstly BDPs) have been found in [7]. Here we consider the situation of almost absorbing BDPs with catastrophes and obtain some interesting bounds and approximations.

Our approach is based on the notion of logarithmic norm and special transformations of the intensity matrix, see, for instance, [5].

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Let $X = X(t)$, $t \geq 0$ be a BDP with catastrophes, and let $\lambda_n(t)$, $\mu_n(t)$ and $\xi(t)$ be birth, death, and catastrophe rates, respectively.

Let $p_{ij}(s,t) = Pr\left\{X(t) = j\,|X(s) = i\right\}$ for $i, j \geq 0$, $0 \leq s \leq t$ be the transition probability functions of the process $X = X(t)$ and $p_i(t) = Pr\left\{X(t) = i\right\}$ be the state probabilities.

The probabilistic dynamics of the process is represented by the forward Kolmogorov system of differential equations:

$$
\left\{
\begin{aligned}
&\frac{dp_0}{dt} = -\left(\lambda_0(t) + \xi(t)\right)p_0 + \mu_1(t)p_1 + \xi(t),\\
&\frac{dp_k}{dt} = \lambda_{k-1}(t)p_{k-1} - \left(\lambda_k(t) + \mu_k(t) + \xi(t)\right)p_k + \mu(t)_{k+1}p_{k+1}, k \geq 1.
\end{aligned}
\right. \tag{1.1}
$$

We denote by $\mathbf{p}(t) = \left(p_0(t), p_1(t), \dots\right)^T$, $t > 0$ the column vector of state probabilities and by $\mathbf{A}(t) = \{a_{ij}(t),\ t \geq 0\}$ the matrix related to (1.1) where

$$
a_{ij}(t) = \begin{cases}
\lambda_{i-1}\left(t\right), & \text{if} \quad j = i-1\\
\mu_{i+1}\left(t\right), & \text{if} \quad j = i+1\\
-\left(\lambda_i\left(t\right) + \mu_i\left(t\right) + \xi(t)\right), & \text{if} \quad j = i\\
0, & \text{otherwise.}
\end{cases} \tag{1.2}
$$

We shall restrict ourselves to birth and death processes whose rates have the following form:

$$
\lambda_n\left(t\right) = \nu_n\lambda\left(t\right),\ \mu_n\left(t\right) = \eta_n\mu\left(t\right), \quad t \geq 0, \quad n \in E, \tag{1.3}
$$

with the assumptions that the rates are bounded, $0 \leq \eta_n \leq M,\ 0 \leq \nu_n \leq M$, see [6] for details.

Then we can rewrite the system (1.1) in the form

$$
\frac{d\mathbf{p}}{dt} = \mathbf{A}\left(t\right)\mathbf{p} + \mathbf{g}(t), \quad t \geq 0, \tag{1.4}
$$

as a differential equation in the space of sequences l_1, where $\mathbf{g}(t) = \left(\xi(t), 0, 0, \dots\right)^T$.

Let $\Omega = \{\mathbf{x} :\ \mathbf{x} \geq 0,\ \|\mathbf{x}\|_1 = 1\}$.

Throughout the whole paper we assume that $\lambda(t)$, $\mu(t)$ and $\xi(t)$ are locally integrable for $t \geq 0$. Moreover, we suppose (only for simplicity) that these functions are bounded, namely

$$\lambda(t) + \mu(t) + \xi(t) \leq L < \infty, \tag{1.5}$$

for almost all $t \geq 0$.

Then

$$\|A(t)\|_1 = \sup_j \sum_i |a_{ij}(t)| \leq 2ML, \tag{1.6}$$

for almost all $t \geq 0$.

Hence, the Cauchy problem formed by (1.4) with the initial condition $\mathbf{p}(0)$ has the unique solution

$$\mathbf{p}(t) = U(t)\mathbf{p}(0) + \int_0^t U(t,\tau)\mathbf{g}(\tau)\, d\tau, \tag{1.7}$$

where $U(t,s)$ is the Cauchy operator of equation (1.4).

Moreover, if $\mathbf{p}(s) \in \Omega$ then $\mathbf{p}(t) \in \Omega$ for any $t \geq s$.

We shall study the following mean values

$$E_{\mathbf{p}(0)}(t) = E_{\mathbf{p}(0)}\left\{X(t)\right\} = E\left\{X(t)\,|\mathbf{p}(0)\right\}, \tag{1.8}$$

and particularly

$$E_k(t) = E\left\{X(t)\,|X(0) = k\right\}. \tag{1.9}$$

Definition 1. *Markov chain $X(t)$ has the* limiting mean $\varphi(t)$ *if*

$$\lim_{t\to\infty} \left(\varphi(t) - E_k(t)\right) = 0 \tag{1.10}$$

for any k.

The property $\mathbf{p}(t) \in \Omega$ for any $t \geq s$ allows to put $p_0(t) = 1 - \sum_{i\geq 1} p_i(t)$ (for ordinary BDP see this way of study, for instance, in [6]), then we obtain the following system from (1.4)

$$
\begin{pmatrix} \frac{dp_1}{dt} \\ \frac{dp_2}{dt} \\ \vdots \\ \frac{dp_n}{dt} \\ \vdots \end{pmatrix} \tag{1.11}
$$

$$
= \begin{pmatrix} -(\lambda_0+\lambda_1+\mu_1+\xi) & (\mu_2-\lambda_0) & -\lambda_0 & -\lambda_0 & \cdots & \cdots \\ \lambda_1 & -(\lambda_2+\mu_2+\xi) & \mu_3 & 0 & 0 & \cdots \\ 0 & \lambda_2 & -(\lambda_3+\mu_3+\xi) & \mu_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

$$
\times \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}
$$

or otherwise

$$
\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t). \tag{1.12}
$$

This is a linear non-homogeneous differential system the solution of which can be written as

$$
\mathbf{z}(t) = V(t,0)\mathbf{z}(0) + \int_0^t V(t,z)\mathbf{f}(z)\,dz, \tag{1.13}
$$

where $V(t,z)$ is the Cauchy operator of (1.12).

Consider the matrix

$$
D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{1.14}
$$

and the spaces of sequences

$$
\ell_{1D} = \left\{\mathbf{z} = (p_1, p_2, \dots)^T : \|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|_1 < \infty\right\}, \tag{1.15}
$$

and

$$\mathcal{B} = \left\{ \mathbf{z} = (p_1, p_2, \dots)^T : \|\mathbf{z}\|_{\mathcal{B}} = \sum_{i\geq 1} d_i |p_i| < \infty \right\}, \tag{1.16}$$

where d_i are some positive numbers.

We have

$$D^{-1} = \begin{pmatrix} d_1^{-1} & -d_2^{-1} & 0 & \ddots & \\ 0 & d_2^{-1} & -d_3^{-1} & 0 & \ddots \\ \ddots & 0 & \ddots\, d_3^{-1} & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Applying this transformation to the matrix $B(t)$ in (1.12) , we arrive to the matrix $DB(t)D^{-1}$

(1.17)

$$DB(t)D^{-1} = \begin{pmatrix} -(\lambda_0+\mu_1+\xi) & d_1\cdot d_2^{-1}\cdot\mu_1 & 0 & & \cdots \\ d_2\cdot d_1^{-1}\cdot\lambda_1 & -(\lambda_1+\mu_2+\xi) & d_2\cdot d_3^{-1}\cdot\mu_2 & 0 & \\ 0 & d_3\cdot d_2^{-1}\cdot\lambda_2 & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ & \cdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Now we can study BDP with catastrophes using the logarithmic norm and related bounds.

2. Bounds

Put

$$\alpha_k = \lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+2}}{d_{k+1}}\lambda_{k+1}(t) - \frac{d_k}{d_{k+1}}\mu_k(t), \quad k \geq 0, \tag{2.18}$$

and

$$\beta(t) = \inf_{k\geq 0} \alpha_k(t) \tag{2.19}$$

Let

$$\int_0^\infty \beta(t)\, dt = +\infty \tag{2.20}$$

and

$$\lambda_0(t) \le \varepsilon\beta(t), \quad t \ge 0. \tag{2.21}$$

Theorem 1. *Let a process with rates $\lambda_k(t)$, $\mu_k(t)$, and $\xi(t)$ be given. Let $\{d_i\}$ be a nondecreasing sequence of positive numbers such that $d_1 = 1$, and (2.21), (2.20) be fulfilled. Let ε be sufficiently small. Then (for any $\xi(t)$) $X(t)$ is weakly ergodic, and the following inequalities hold:*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{\mathcal{B}} \le 2e^{-\int\limits_s^t \beta(\tau)\,d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \tag{2.22}$$

for any $s, t,\ 0 \le s \le t$, and for any acceptable initial conditions $\mathbf{p}^*(s),\ \mathbf{p}^{**}(s)$;

$$\sum_{i\ge 1} d_i |p_i(t) - \pi_i(t)| \le 4e^{-\int_0^t \beta(\tau)d\tau} \sum_{i\ge 1} g_i |p_i(0) - \pi_i(0)|, \tag{2.23}$$

for any $\mathbf{p}(0)$,
and

$$\liminf_{t\to\infty} p_0(t) \ge 1 - 2\varepsilon, \tag{2.24}$$

where $g_k = \sum_{i=1}^k d_i$

P r o o f. We have now the following bound of the logarithmic norm $\gamma(B(t))$ in l_{1D}:

$$\gamma(B(t))_{1D} = \sup_{k\ge 0}(-\lambda_k(t) - \mu_{k+1}(t) - \xi(t) + \frac{d_{k+2}}{d_{k+1}}\lambda_{k+1}(t) + \frac{d_k}{d_{k+1}}\mu_k(t)) \le -\beta(t), \tag{2.25}$$

$$\|U(t,s)\|_{1D} \le e^{-\int_s^t \beta(\tau)d\tau} \tag{2.26}$$

for any $\xi(t)$.

Consider $\mathcal{B}$ and l_{1D} norms of a vector $\mathbf{z} = (z_1, z_2, \dots)^T$, then

$$\|\mathbf{z}\|_{\mathcal{B}} = \sum_{i\ge 1} d_i z_i = d_1 \left(\left| \sum_{i\ge 1} z_i + \sum_{i\ge 2} -z_i \right| \right) + d_2 \left(\left| \sum_{i\ge 2} z_i + \sum_{i\ge 3} -z_i \right| \right) + \dots$$

$$\le d_1 \left| \sum_{i\ge 1} z_i \right| + 2d_2 \left| \sum_{i\ge 2} z_i \right| + \dots \le 2\|\mathbf{z}\|_{1D}, \tag{2.27}$$

and we obtain weak ergodicity of BDP $X(t)$ for *any* $\xi(t)$ and inequalities (2.22), (2.23).

Then for any $\mathbf{p}(0)$ in l_{1D} norm we have

$$\|\mathbf{z}(t)\| \le \|U(t,0)\|\|\mathbf{z}(0)\| + \int_0^t \|U(t,\tau)\|\|\mathbf{f}(\tau)\|d\tau \le e^{-\int_0^t \beta(\tau)d\tau}\|\mathbf{z}(0)\|$$

$$+\int_0^t e^{-\int_\tau^t \beta(s)ds}|\lambda_0(\tau)|d\tau \le e^{-\int_0^t \beta(\tau)d\tau}\|\mathbf{z}(0)\| + \varepsilon, \tag{2.28}$$

hence we have

$$\limsup_{t\to\infty} \|\pi(t)\|_{1D} \le \varepsilon, \tag{2.29}$$

and (2.24). □

Corollary 1. *For any* $k \ge 1$

$$\sum_{i\ge1} d_i|p_{0i}(t) - p_{ki}(t)| \le 2e^{-\int_0^t \beta(\tau)d\tau}\sum_{i=1}^k d_i \tag{2.30}$$

Corollary 2. *Let, in addition, the numbers* d_i *grow sufficiently fast so that* $\inf\limits_{k\ge1}\dfrac{d_k}{k} = \omega > 0$. *Then* $X(t)$ *has the limiting mean, say* $\phi^*(t)$, *and the following bounds hold:*

$$|\phi^*(t) - E_k(t)| \le \frac{2}{\omega}e^{-\int\limits_0^t \beta(\tau)\,d\tau}\|\mathbf{p}^*(0) - \mathbf{e_k}\|_{1D}, \tag{2.31}$$

$$\limsup_{t\to\infty} E_{\mathbf{p}(0)}(t) \le \frac{2\varepsilon}{\omega}. \tag{2.32}$$

Remark 1. *We can choose the limiting mean with concrete special properties (for instance, constant or periodic) under the respective special assumptions.*

Theorem 2. *Let under assumptions of the previous Corollary all intensities be* 1*-periodic. Then there exists* 1*-periodic limit regime, say* $\pi(t) = (\pi_0(t),$

$\pi_1(t), \dots)^T$, *and the respective limiting mean* $\phi(t)$. *Moreover, the following bounds hold:*

$$\|\mathbf{p}(t) - \pi(t)\|_{\mathcal{B}} \leq 2e^{-\int\limits_0^t \beta(\tau)\,d\tau} \|\mathbf{p}(0) - \pi(0)\|_{1D}, \tag{2.33}$$

$$|\phi(t) - E_k(t)| \leq \frac{2}{\omega} e^{-\int\limits_0^t \beta(\tau)\,d\tau} \|\pi(0) - \mathbf{e_k}\|_{1D}. \tag{2.34}$$

Unfortunately, our bounds (2.33) and (2.34) have the essential defect, namely, we have no real information about $\|\mathbf{p}(0)-\pi(0)\|_{1D}$. Let $X(0) = \mathbf{e}_k$, then we have $\|\mathbf{e_k}\|_{1D} = \sum_{i=1}^k d_i$ for $k \geq 1$ and $\|\mathbf{e_0}\|_{1D} = 0$. On the other hand, we can obtain the bound for $\|\pi(0)\|_{1D}$ using the approach of [6]. Let instead of (2.21):

$$|\lambda_0(t)| \leq \varepsilon, \quad t \geq 0. \tag{2.35}$$

We have

$$\sup_{|t-s|\leq 1} \int\limits_s^t \alpha(\tau)\,d\tau = K < \infty, \tag{2.36}$$

and

$$\begin{aligned} \limsup_{t\to\infty} \|\pi(t)\|_{1D} &\leq \left\| \int_0^t V(t,\tau)\mathbf{f}(\tau)d\tau \right\|_{1D} \leq \varepsilon \int_0^t e^{-\int_\tau^t \beta(u)\,du} d\tau \\ &\leq e^K \varepsilon \int_0^t e^{-\beta^*(t-\tau)} d\tau \leq \frac{e^K \varepsilon}{\beta^*}, \end{aligned} \tag{2.37}$$

where $\beta^* = \int_0^1 \beta(u)\,du$.

Now, $\|\pi(0)\|_{1D} \leq \limsup_{t\to\infty} \|\pi(t)\|_{1D}$ by 1-periodicity of $\pi(t)$. Hence we obtain the following bound:

$$\|\pi(0) - \mathbf{e_k}\|_{1D} \leq \limsup_{t\to\infty} \|\pi(t)\|_{1D} + \|\mathbf{e_k}\|_{1D}, \tag{2.38}$$

and

Corollary 3. *Let under assumptions of the previous Corollary* $X(t) = k$. *Then the following bounds hold:*

$$\|\mathbf{p}(t) - \pi(t)\|_{\mathcal{B}} \le 2e^{-\int\limits_0^t \beta(\tau)\,d\tau} \left(\sum_{i=1}^{k} d_i + \frac{e^K \varepsilon}{\beta^*} \right), \tag{2.39}$$

and

$$|\phi(t) - E_k(t)| \le \frac{2}{\omega} e^{-\int\limits_0^t \beta(\tau)\,d\tau} \left(\sum_{i=1}^{k} d_i + \frac{e^K \varepsilon}{\beta^*} \right). \tag{2.40}$$

3. Approximations

Consider now the family of truncated processes $X_n(t)$ on the state space $E_n = \{0, 1, \dots, n\}$ with the same intensities for $k \le n$ and intensity matrices $A_n(t)$.

Let $\{h_k\}$ be a sequence of positive numbers, $1 = h_1 \le h_2 \le \dots$, and

$$w_n = \sup_{k \ge n} \frac{h_k}{d_k}. \tag{3.41}$$

We will denote by $\|\mathbf{z}\|_{\mathcal{B}_d}$ and by $\|\mathbf{z}\|_{\mathcal{B}_h}$ the norms in the spaces $\mathcal{B}$ for the sequence $\{d_k\}$ and $\{h_k\}$ respectively.

The following statement can be proved by the way of [8].

Theorem 3. *Let the assumptions of Theorem 2 be fulfilled, and let in addition,* $\lim_{n\to\infty} w_n = 0$*. Let* $X(0) = X_n(0) = 0$*. Then*

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\|_{\mathcal{B}_h} \le \frac{6LMw_n e^K \varepsilon t}{\beta^*} \tag{3.42}$$

for any $t \ge 0$ *and any* n*.*

Remark 2. *Probably the most interesting bounds are obtained if* $h_k = 1$ *or* $h_k = k$ *for all* k*. In the first case we can compute limit 1-periodic sojourn probabilities, and in the second one we can find the limiting mean.*

Corollary 4. *Let the assumptions of Theorem* 3 *be fulfilled. Let* $X(0) = X_n(0) = 0$*. Then*

$$\|\pi(t) - \mathbf{p}_n(t)\|_1 \le 2e^{-\int\limits_0^t \beta(\tau)\,d\tau} \frac{e^K \varepsilon}{\beta^*} + \frac{6LMw_n^1 e^K \varepsilon t}{\beta *}, \tag{3.43}$$

and

$$|\phi(t) - E_{0,n}(t)| \le \frac{2}{\omega} e^{-\int\limits_0^t \beta(\tau)\,d\tau} \frac{e^K \varepsilon}{\beta^*} + \frac{6LMw_n^2 e^K \varepsilon t}{\beta^*}, \tag{3.44}$$

for any $t \ge 0$, n, *where* $w_n^1 = \sup\limits_{k \ge n} \frac{1}{d_k}$, $w_n^2 = \sup\limits_{k \ge n} \frac{k}{d_k}$ *and* $E_{0,n} = E_k(t) = E\{X_n(t)\,|X_n(0) = 0\}$.

We can obtain the essential bounds for the limit characteristics (limit 1-periodic sojourn probabilities, and the limiting mean) of the considered process. Namely, let intensities ($\lambda(t)$, $\mu(t)$ and $\xi(t)$) be 1-periodic. Then under assumptions of previous Corollary there exists 1-periodic limit regime $\pi(t) = (\pi_0(t), \pi_1(t), \dots)^T$ and we have the method of computing of limit 1-periodic sojourn probabilities $J_k(t)$ (this is probability that the length of the queue at the moment t does not exceed k) by the following way.

Let now δ be an arbitrary positive number.

1. Choose integer m such that the first expression in the right-hand side of (3.44) less than $\delta/3$ for any $t \ge m$.

2. Find n such that the second expression in the right-hand side of (3.44) less than $\delta/3$ for any $t \le m+1$.

3. Then the solution of the Cauchy problem for the truncated Kolmogorov system with initial condition $\mathbf{e}_0$ on the interval $[m; m+1]$ (with error $\delta/3$) gives us the limit 1-periodic regime $\pi(t) = (\pi_0(t), \pi_1(t), \dots)^T$ with the error δ.

4. Finally, the limiting behaviour of $J_k(t) = \Pr\{X(t) \le k\}$ can be computed as $\sum\limits_{i=0}^{k} \pi_i(t)$ with the same error δ.

The limiting mean can be found be the same way.

4. Example

Computing of the limiting mean $\phi(t)$ and the sojourn probabilities $J_k(t)$ for some k-s. Particularly, $J_0(t)$ is the probability that the queue is empty at the moment t.

Consider queue-length process for the $M(t)/M(t)/100$ queue with catastrophes and intensities $\lambda(t) = 0.1 + 0.1\sin 2\pi t$, $\mu(t) = 2 + \cos 2\pi t$, $\xi(t) = 2 + \sin 4\pi t$.

Using the way of Section 3, put $d = 2$ and $d_k = d^k$. Then we have $\varepsilon = 0.2$, $L = 6.2$, $M = 100$, $w_n^1 = 2^{-n}$, $w_n^2 = \frac{n}{2^n}$, furthermore , $\beta(t) = \mu(t) - (d-1)\lambda(t) = 1.9 + \cos 2\pi t - 0.1\sin 2\pi t$, $K \le 3$, and $\beta^* = 1.9$.

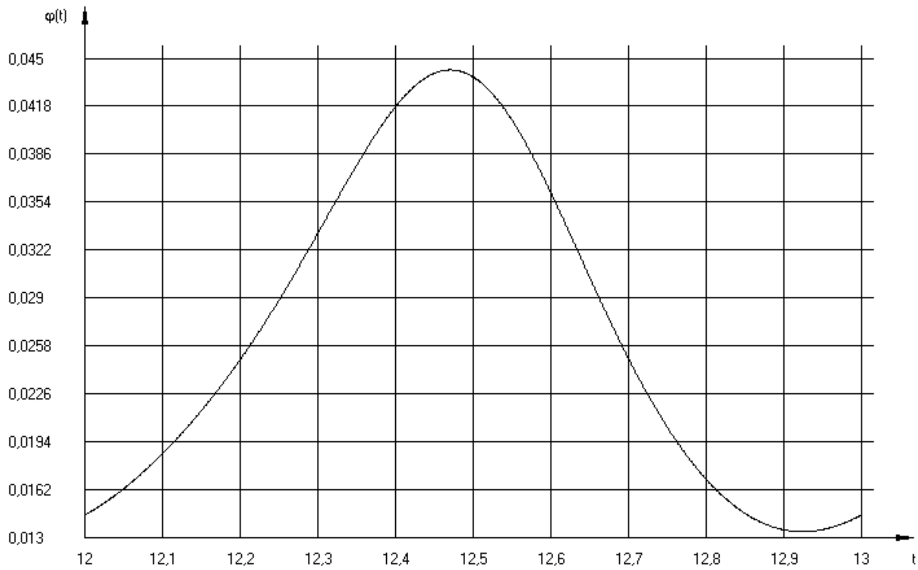

Figure 1: The limiting mean

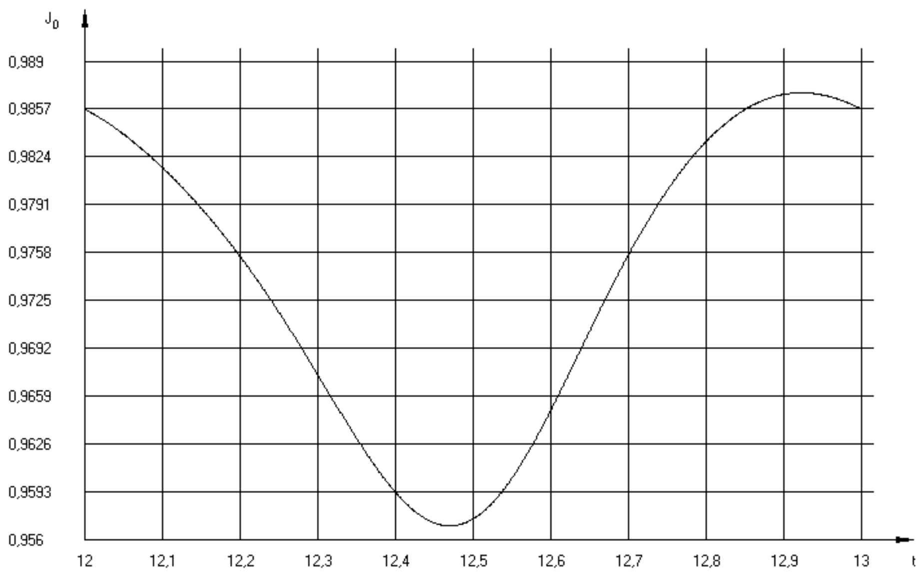

Figure 2: The sojourn probability $Pr\,(X(t)) = 0$

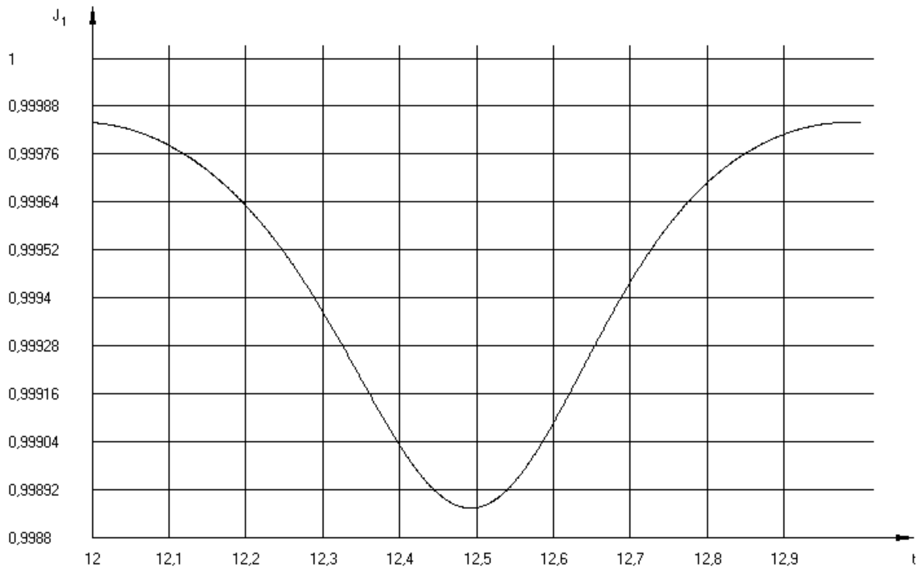

Figure 3: The sojourn probability $Pr\,(X(t)) \leq 1$

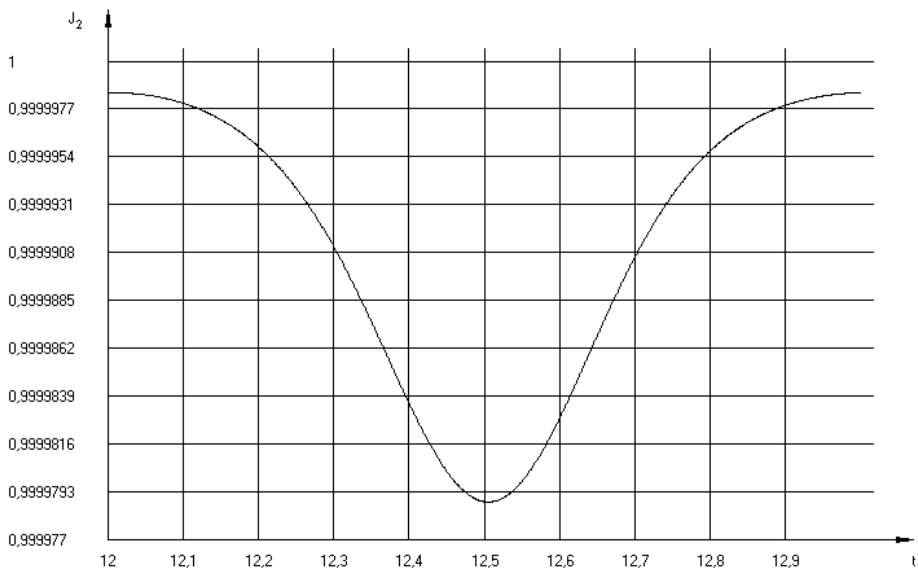

Figure 4: The sojourn probability $Pr\,(X(t)) \leq 2$

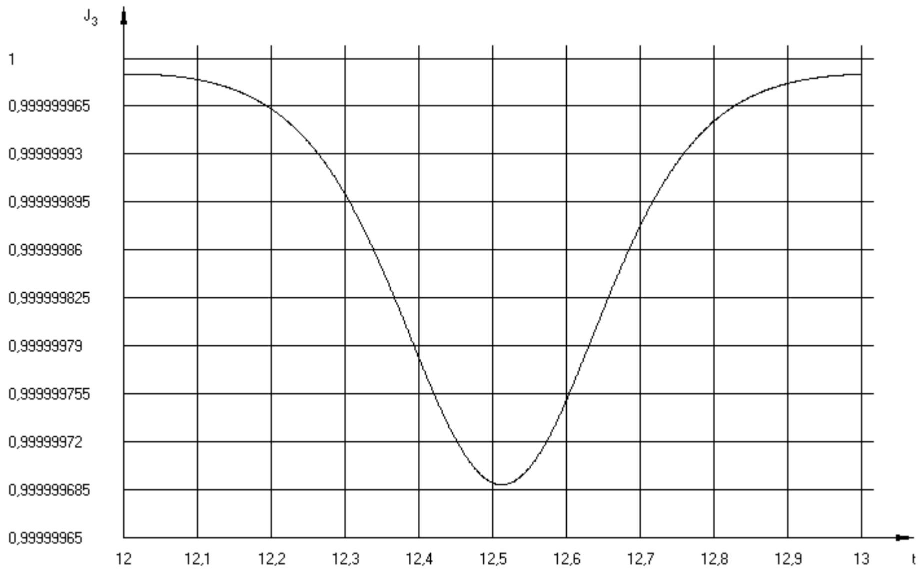

Figure 5: The sojourn probability $Pr\left(X(t)\right) \leq 3$

Estimates of Section 2 (2.24) and (2.32) give us the following (sufficiently rough) bounds: $\liminf_{t\to\infty} p_0(t) \geq 0.72$, and $\limsup_{t\to\infty} E_{\mathbf{p}(0)}(t) \leq 0.14$.

On the other hand, we can obtain essentially more sharp bounds. Let $\delta = 10^{-6}$. Then it is sufficient to choose $m = 12$ and $n = 50$. Then we obtain the limiting mean $\phi(t)$ and all sojourn probabilities $J_k(t)$ with error $\delta = 10^{-6}$ as the respective characteristics of the solution with initial condition $\mathbf{e}_0$ of the Cauchy problem for the respective truncated Kolmogorov system on the interval $[m, m+1]$. Now, figures 1 – 5 shows us the approximations of the limit characteristics $\phi(t)$, and $J_0(t) - J_3(t)$ respectively. Moreover, $J_k(t) \approx 1$ for $k \geq 4$.

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