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ESTIMATION OF THE OFFSPRING AND IMMIGRATION MEAN VECTORS FOR BISEXUAL BRANCHING PROCESSES WITH IMMIGRATION OF FEMALES AND MALES*

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This paper concerns with the bisexual branching model with immigration of females and males in each generation introduced in [4]. For this model the problem of estimating the offspring and immigration mean vectors is dealt. The estimation is considered in two sample situations, depending on the ability to observe the number of female and male immigrants or the impossibility to do it. The asymptotic properties of the proposed estimators are investigated in the supercritical case. The behaviour of the estimators is illustrated through a simulated example.

1. Introduction

Bisexual Galton-Watson branching processes are appropriate mathematical models to describe the evolution of two-sex populations where females and males coexist, form couples, and after that, reproduce. The first bisexual process was introduced by D.J. Daley in 1968 (see [2]), since then, this class of branching

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processes has aroused a lot of interest being numerous the modifications introduced to the original model. Surveys of the literature associated with bisexual branching proceses are those of D.M. Hull (see [12]), G. Alsmeyer (see [10]), and M. Molina (see [13]). In particular, bisexual branching processes allowing, in each generation, immigration of females and males, or the immigration of mating units in the population were investigated in [4], [5], [6], [7] and [16]. Also, a class of bisexual processes with immigration depending on the number of mating units in the population was studied in [14] and [15]. In this work, we are interested in the bisexual process with immigration of females and males introduced in [4]. Some probabilistic results about such a model were derived in [5] and [7]. The aim of this paper is to develop its inferential theory, providing several estimators for the offspring and immigration mean vectors by considering the moment and least squares methods.

The bisexual branching process with immigration of females and males, by simplicity BPI, is a bitype sequence $\{(F_n, M_n)\}_{n\geq 1}$ defined in the form, for $n = 0, 1, \ldots$:

$$(1) \qquad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (\gamma_{ni}^1, \gamma_{ni}^2) + (F_{n+1}^I, M_{n+1}^I),\ Z_{n+1} = L(F_{n+1}, M_{n+1}),$$

where the process starts with a positive number N of mating units, i.e. $Z_0 = N$, and the empty sum is considered to be $(0, 0)$. $\{(\gamma_{ni}^1, \gamma_{ni}^2),\ i = 1, 2, \ldots; n = 0, 1, \ldots\}$ and $\{(F_n^I, M_n^I),\ n = 1, 2, \ldots\}$ are independent sequences of i.i.d. non-negative integer valued random variables. $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the mating function, assumed to be non-decreasing in each argument, integer valued for integer arguments and such that $L(x, y) \leq xy$. We will consider superadditive mating functions, i.e., satisfying for every positive n that

$$L\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) \geq \sum_{i=1}^n L(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+, \quad i = 1, \ldots, n.$$

This is not a serious restriction, as was pointed out in [11] the vast majority of mating functions used in the literature on two-sex population models are superadditive.

Intuitively, $(\gamma_{ni}^1, \gamma_{ni}^2)$ represents the number of females and males produced by the ith mating unit in generation n and (F_n^I, M_n^I) is the number of immigrant females and males in this generation. Thus, (F_n, M_n) is the total number of females and males in the nth generation, which form $Z_n = L(F_n, M_n)$ mating units. These reproduce independently through the same probability distribution for each generation.

The distributions of $(\gamma_{ni}^1, \gamma_{ni}^2)$ and (F_n^I, M_n^I) are called offspring and immigration distributions, respectively. We denote by $\mu = (\mu_1, \mu_2)$ and $\mu^I = (\mu_1^I, \mu_2^I)$

their respective mean vectors, and by $\Sigma = (\sigma_{ij})$ and $\Sigma^I = (\sigma^I_{ij})$ their respective covariance matrices. In order to avoid trivialities, we will assume all these parameters positive and finite.

It can be shown that $\{Z_n\}_{n\geq 0}$ and $\{(F_n, M_n)\}_{n\geq 1}$ are homogeneous Markov chains. We will use the simplified notation, for $n = 1, 2, \ldots$:

$$\Gamma_n = (\Gamma_{1n}, \Gamma_{2n}) \text{ where } \Gamma_{1n} = F_n \text{ and } \Gamma_{2n} = M_n,$$

$$\Gamma^I_n = (\Gamma^I_{1n}, \Gamma^I_{2n}) \text{ where } \Gamma^I_{1n} = F^I_n \text{ and } \Gamma^I_{2n} = M^I_n.$$

Analogously to the classification established for the classical Galton-Watson process, the BPI can be classified attending to the value of its *asymptotic mean growth rate* (see [7]), that is, $r = \lim_{k\to\infty} k^{-1}E[Z_{n+1} \mid Z_n = k]$. The BPI will be called subcritical, critical or supercritical, depending on whether $r < 1$, $r = 1$ or $r > 1$, respectively.

In order to investigate the asymptotic properties of the estimators, we will suppose that $r > 1$. Under this framework it has been proved in [7] that, on $[Z_n \to \infty]$:

i) $\{Z^{-1}_{n-1}\Gamma_n\}_{n\geq 1}$ and $\{Z^{-1}_{n-1}\Gamma^I_n\}_{n\geq 1}$ converge almost surely to μ and 0, respectively, as $n \to \infty$.

ii) $\{r^{-n}Z_n\}_{n\geq 1}$ and $\{r^{-n}\Gamma_n\}_{n\geq 1}$ converge almost surely to W and $r^{-1}W\mu$, respectively, as $n \to \infty$, where W is a random variable satisfying $P[0 < W < \infty] = 1$.

The paper is organized as follows: In Section 2. the problem of estimating the offspring and immigration vectors is dealt. The estimation is considered in two situations. First, it is assumed that the sample available is the number of mating units and of female and male immigrants in each generation. On the other hand, we also consider that we can only observe the number of the mating units in each generation. In order to obtain the estimators, it is used for the first sample scheme the method of the moments and for the second one the least squares method. Asymptotic properties of the estimators are investigated in both schemes and, as illustration, a simulated example is provided. Finally, Section 3. is devoted to proving the results previously established.

2. Estimation of the offspring and immigration vectors

From now on, we will denote by $Q = [W > 0]$, $P_Q[\cdot] = P[\cdot \mid Q]$, $F_Q(y) = P_Q[W \leq y]$, $\phi(x)$ the distribution function of the standard normal distribution and $\phi^*(x) = \int_0^\infty \phi(xy^{1/2})dF_Q(y)$.

2.1. Estimation when observations about the immigration are available

In this section we consider the situation in which the number of immigrants (females and males) per generation may be observed.

If for some $k \geq 1$ the sample $\{Z_{k-1},\ \Gamma_k,\ \Gamma_k^I\}$ is available then, from (1), we deduce that

$$E[\Gamma_k \mid Z_{k-1}, \Gamma_k^I] = Z_{k-1}\mu + \Gamma_k^I \quad \text{a.s.}$$

Thus, providing that $Z_{k-1} > 0$, the method of the moments suggests the following estimator for μ :

$$\overline{\mu}_k = (\overline{\mu}_{1k}, \overline{\mu}_{2k}) = Z_{k-1}^{-1}(\Gamma_k - \Gamma_k^I). \tag{2}$$

Remark 2.1. For convention, when $Z_{k-1} = 0$, we will assume that $(1,1)$ is a razonable estimation for μ.

Theorem 2.1. *For a supercritical BPI one has:*

i) $E[\overline{\mu}_k \mid Z_{k-1} > 0] = \mu$.

ii) $\overline{\mu}_k$ *converges almost surely to* μ*, as* $k \to \infty$*, on* $[Z_n \to \infty]$.

iii) For every real number x*,* $P_Q[(\sigma_{ii}^{-1} Z_{k-1})^{1/2}(\overline{\mu}_{ik} - \mu_i) \leq x]$ *converges to* $\phi(x)$*, as* $k \to \infty$*,* $i = 1, 2$.

iv) For every real number x*,* $P_Q[(\sigma_{ii}^{-1} r^{k-1})^{1/2}(\overline{\mu}_{ik} - \mu_i) \leq x]$ *converges to* $\phi^*(x)$*, as* $k \to \infty$*,* $i = 1, 2$.

If the sample $\{Z_{k-1},\ \Gamma_k,\ \Gamma_k^I,\ k = 1, 2, \ldots, n\}$ is available then it seems reasonable to find the best linear convex combination of the estimators $\overline{\mu}_k$, given in (2), for $k = 1, \ldots, n$, i.e., an estimator of the form

$$\sum_{k=1}^{n} \beta_k \overline{\mu}_k.$$

Taking into account that, for $i = 1, 2$, $Var[\Gamma_{ik} \mid Z_{k-1}, \Gamma_k^I] = Z_{k-1}\sigma_{ii}$ almost surely, it is justifiable to consider $\beta_k \propto Z_{k-1}, k = 1, \ldots, n.$ Imposing that $\sum_{k=1}^{n} \beta_k = 1$, we obtain the following estimator for μ :

$$\widetilde{\mu}_n = (\widetilde{\mu}_{1n}, \widetilde{\mu}_{2n}) = \left(\sum_{k=1}^{n} Z_{k-1}\right)^{-1} \sum_{k=1}^{n} (\Gamma_k - \Gamma_k^I). \tag{3}$$

Obviously, the estimator proposed for μ^I will be

(4) $$\widetilde{\mu}_n^I = (\widetilde{\mu}_{1n}^I, \widetilde{\mu}_{2n}^I) = n^{-1} \sum_{k=1}^{n} \Gamma_k^I.$$

It is clear that, from strong law of large numbers, $\widetilde{\mu}_n^I$ converges almost surely to μ^I as $n \to \infty$. Moreover

Theorem 2.2. *For a supercritical BPI one has:*

i) $$E[\widetilde{\mu}_n] = E\left[\left(\sum_{k=1}^{n} Z_{k-1}\right)^{-1} \sum_{k=1}^{n-1} (\Gamma_k - \Gamma_k^I)\right] + E\left[\left(\sum_{k=1}^{n} Z_{k-1}\right)^{-1} Z_{n-1}\right] \mu.$$

ii) $\widetilde{\mu}_n$ *converges almost surely to* μ, *as* $n \to \infty$, *on* $[Z_n \to \infty]$.

iii) *For every real number* x, $P_Q\left[(\sigma_{ii}^{-1} \sum_{k=1}^{n} Z_{k-1})^{1/2} (\widetilde{\mu}_{in} - \mu_i) \leq x\right]$ *converges to* $\phi(x)$, *as* $n \to \infty$, $i = 1, 2$.

iv) *For every real number* x, $P_Q[((\sigma_{ii}(r-1))^{-1}(r^n - 1))^{1/2}(\widetilde{\mu}_{in} - \mu_i) \leq x]$ *converges to* $\phi^*(x)$, *as* $n \to \infty$, $i = 1, 2$.

Example 2.1. Consider a BPI with offspring and immigration laws given respectively by:

$$P[f_{01} = i, m_{01} = j] = \frac{6}{(3-i-j)!i!j!}(0.5)^i(0.35)^j(0.15)^{3-i-j}$$

$$\text{for } i, j = 0, 1, 2, 3;\ i + j \leq 3$$

and

$$P[F_1^I = i, M_1^I = j] = e^{-2}(i!j!)^{-1}; i, j = 0, 1, \ldots,$$

i.e. we take a trinomial distribution with parameters 3, 0.5 and 0.35 as offspring distribution and considering that F_n^I and M_n^I are i.i.d. random variables according to a Poisson distribution with mean 1. We also suppose that the process is governed by the mating function $L(x, y) = \min\{x, y\}$. Under these conditions, it can be derived that $\mu = (1.5, 1.05)$, $\mu^I = (1, 1)$ and $r = 1.05$.

Starting with $Z_0 = 1$ we simulated 200 generations for such a model (see Table 1). From (2), (3) and (4) we calculated the corresponding estimates for μ and μ^I. Figures 1, 2 and 3 show the evolution of the estimates obtained.

n	F_n	M_n	Z_{n-1}	F_n^I	M_n^I
1	3	0	1	0	0
25	1	3	0	1	3
50	46	24	28	0	0
75	472	329	311	3	1
100	1622	1187	1090	2	1
125	5866	4177	3946	1	1
150	19563	13943	13108	3	1
175	68117	47649	45453	1	0
200	227197	158917	151345	2	2

Table 1: Simulated data

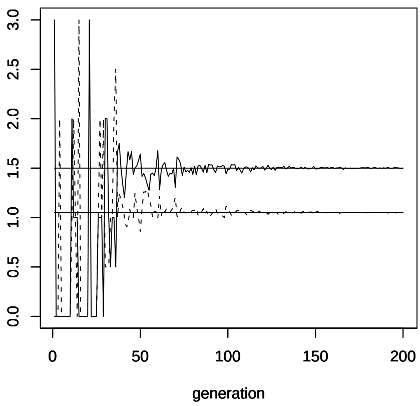

Figure 1: Estimates obtained from $\bar{\mu}_{1n}$ (solid line) and $\bar{\mu}_{2n}$ (dashed line)

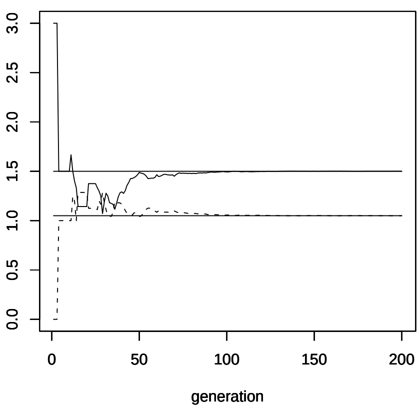

Figure 2: Estimates obtained from $\widetilde{\mu}_{1n}$ (solid line) and $\widetilde{\mu}_{2n}$ (dashed line)

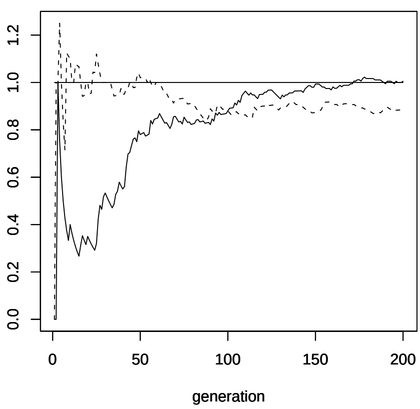

Figure 3: Estimates obtained from $\widetilde{\mu}_{1n}^I$ (solid line) and $\widetilde{\mu}_{2n}^I$ (dashed line)

2.2. Estimation when observations about the immigration are not available

Next, we consider the situation for which the number of immigrants in each generation can not be observed.

Denote by $\mathcal{F}_n = \sigma(Z_0, \Gamma_1, \ldots, \Gamma_n)$, $n = 1, 2, \ldots$, ($\mathcal{F}_0 = \sigma(Z_0)$). Then, for $i = 1, 2, \ldots$,

$$E[\Gamma_i \mid \mathcal{F}_{i-1}] = Z_{i-1}\mu + \mu^I \quad \text{a.s.} \tag{5}$$

If, for $n \geq 2$, the sample $\{Z_{k-1},\ \Gamma_k,\ k = 1, \ldots, n\}$ is available, using the conditional least squares method and (5), estimators for μ and μ^I are obtained by minimizing the expression:

$$\varphi(\mu, \mu^I) = \sum_{i=1}^{2}\sum_{k=1}^{n}(\Gamma_{ik} - Z_{k-1}\mu_i - \mu_i^I)^2. \tag{6}$$

It can be verified that the values of μ and μ^I that minimize (6) are respectively:

$$\widehat{\mu}_n = (\widehat{\mu}_{1n}, \widehat{\mu}_{2n}) = \frac{n\sum\limits_{k=1}^{n} Z_{k-1}\Gamma_k - \sum\limits_{k=1}^{n} Z_{k-1}\sum\limits_{k=1}^{n}\Gamma_k}{n\sum\limits_{k=1}^{n} Z_{k-1}^2 - \left(\sum\limits_{k=1}^{n} Z_{k-1}\right)^2} \tag{7}$$

and

$$\widehat{\mu}_n^I = (\widehat{\mu}_{1n}^I, \widehat{\mu}_{2n}^I) = \frac{\sum\limits_{k=1}^{n} Z_{k-1}^2\sum\limits_{k=1}^{n}\Gamma_k - \sum\limits_{k=1}^{n} Z_{k-1}\sum\limits_{k=1}^{n} Z_{k-1}\Gamma_k}{n\sum\limits_{k=1}^{n} Z_{k-1}^2 - \left(\sum\limits_{k=1}^{n} Z_{k-1}\right)^2}. \tag{8}$$

These estimators verify the following properties:

Theorem 2.3. *For a supercritical BPI, on $[Z_n \to \infty]$, one has that $\widehat{\mu}_n$ converges almost surely to μ as $n \to \infty$.*

Theorem 2.4. *Consider a supercritical BPI. Then, for every real number x, one has that*

$$P_Q\left[\left(\left(\sum_{k=1}^{n} Z_{k-1} + n\right)(r^2 + r + 1)(r+1)^{-2}\sigma_{jj}^{-1}\right)^{1/2}(\widehat{\mu}_{jn} - \mu_j) \leq x\right]$$

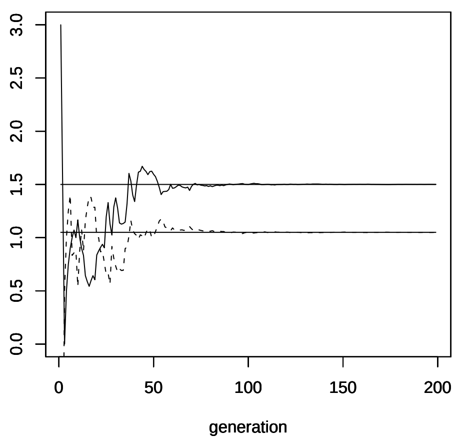

Figure 4: Estimates obtained from $\widehat{\mu}_{1n}$ (solid line) and $\widehat{\mu}_{2n}$ (dashed line)

converges to $\phi(x)$*, as* $n \to \infty$, $j = 1, 2$.

Example 2.2. Consider again the BPI given in the previous example. The estimates for μ and μ^I calculated from expressions (7) and (8) are showed in Figures 4 and 5, respectively. Note that $\widehat{\mu}_n^I$ has a very irregular behaviour, as $n \to \infty$. In fact, it can be proved that it is not a consistent estimator for μ^I.

Remark 2.2. Using the results in [8], it can be established that $[W > 0] = [Z_n \to \infty]$ almost surely. In consequence, one can replace in Theorems 2.1, 2.2 and 2.4, P_Q by $P_{[Z_n \to \infty]}$. Also, by considering Lemma 2.3 in [9], $P_{[Z_n \to \infty]}$ can be replaced by $P_{[Z_{n-1} > 0]}$. This is important from a practical viewpoint since the condition $[Z_{n-1} > 0]$ is a more verifiable condition that $[Z_n \to \infty]$.

Remark 2.3. Taking into account the previous remark and the results established in Theorems 2.1 (iii), Theorem 2.2 (iii) and Theorem 2.4, depending on the sample available, one can determine asymptotic confidence intervals for μ_i, $i = 1, 2$.

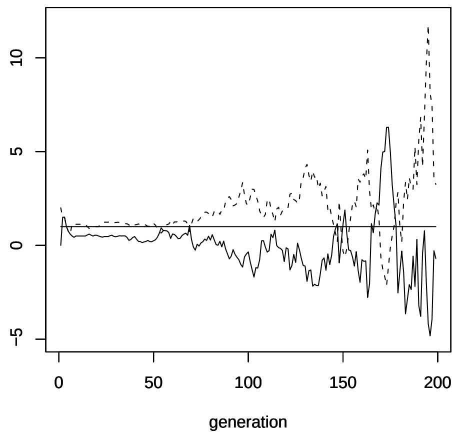

Figure 5: Estimates obtained from $\widehat{\mu}^I_{1n}$ (solid line) and $\widehat{\mu}^I_{2n}$ (dashed line)

3. Proofs

P r o o f o f T h e o r e m 2.1.

i) $E[\overline{\mu}_k \mid Z_{k-1}] = (P[Z_{k-1} > 0])^{-1} \sum_{j=1}^{\infty} j^{-1} P[Z_{k-1} = j] \sum_{l=1}^{j} E[\gamma_{0l}] = \mu$, where it has been used that $E[\gamma_{k-1l}] = E[\gamma_{0l}] = \mu$, $l = 1, \dots, j$.

ii) It is a direct consequence of (3)(ii).

In order to prove iii) and iv) we define the sets:

$$C_1 = [Z_i > 0\ ,\ i \geq 1] \bigcap Q$$

and

$$C_j = [Z_{j-1} = 0\ ,\ Z_i > 0\ ,\ i \geq j] \bigcap Q, j = 2, 3, \dots$$

Obviously the sets C_j are disjoint and $\bigcup_{j=1}^{\infty} C_j = Q$. Moreover, if $P_j[\cdot] = P[\cdot \mid C_j]$, we have that:

$$P_j[(\sigma_{ii}^{-1} Z_{k-1})^{1/2} (\overline{\mu}_{ik} - \mu_i) \leq x] = P_j[S_{ik}(x)], \quad i = 1, 2,$$

where $S_{ik}(x) = [(\sigma_{ii} Z_{k-1})^{-1/2} \sum_{l=1}^{Z_{k-1}} (\gamma^i_{k-1l} - \mu_i) \leq x]$.

Making use, for $i \in \{1,2\}$, of Proposition A.1 (see Appendix) with $a_n = r^n, \nu_n = Z_n, \nu = W$ and

$$Y_n(t,w) = (\sigma_{ii} Z_n(w))^{-1/2} \sum_{l=1}^{[Z_n(w)t]} (\gamma_{n-1l}^i(w) - \mu_i),$$

we get $\lim_{k\to\infty} P_j[S_{ik}(x)] = \phi(x)$, $j = 1,2,\ldots$, and therefore

$$\lim_{k\to\infty} P_Q[S_{ik}(x)] = \lim_{k\to\infty} (P[Q])^{-1} \sum_{j=1}^{\infty} P_j[S_{ik}(x)]P[C_j] = \phi(x),$$

where it has been used that

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} P_j[S_{ik}(x)]P[C_j] = P[Q] \lim_{k\to\infty} P_j[S_{ik}(x)] = P[Q]\phi(x).$$

Thus, iii) is proved.

On the other hand, it is clear that

$$P_j[(\sigma_{ii}^{-1} r^{k-1})^{1/2} (\overline{\mu}_{ik} - \mu_i) \leq x] = P_j[S'_{ik}(x)], \quad i = 1,2,$$

where $S'_{ik}(x) = [(r^{k-1} Z_{k-1}^{-1})^{1/2} (\sigma_{ii} Z_{k-1})^{-1/2} \sum_{l=1}^{Z_{k-1}} (\gamma_{k-1l}^i - \mu_i) \leq x]$.

Taking into account Proposition A.2 (see Appendix) it can be derived that $\lim_{k\to\infty} P_j[S'_{ik}(x)] = \phi^*(x)$. Then, using a similar argument to that one used in the proof of iii) we deduce iv).

P r o o f o f T h e o r e m 2.2. Consider the σ-algebras $\mathcal{F}_n = \sigma(Z_0, \Gamma_1, \ldots, \Gamma_n)$ and $\mathcal{F}_n^I = \sigma(\Gamma_1^I, \ldots, \Gamma_n^I)$, and denote by $X_n = \sum_{k=1}^{n} Z_k$, $n = 1,2,\ldots$

i)

$$E[\widetilde{\mu}_n] = E\left[E[X_{n-1}^{-1} \sum_{k=1}^{n} (\Gamma_k - \Gamma_k^I) \mid \mathcal{F}_{n-1} \vee \mathcal{F}_{n-1}^I] \right] =$$

$$E\left[X_{n-1}^{-1} \left(\sum_{k=1}^{n-1} (\Gamma_k - \Gamma_k^I) + E[\Gamma_n - \Gamma_n^I \mid \mathcal{F}_{n-1} \vee \mathcal{F}_{n-1}^I] \right) \right] =$$

$$E[X_{n-1}^{-1} \sum_{k=1}^{n-1} (\Gamma_k - \Gamma_k^I)] \; + \; E[(X_{n-1})^{-1} Z_{n-1}]\mu.$$

ii) It is proved taking into account Theorem 1 (ii) and Toeplitz's lemma. In fact, on $[Z_n \to \infty]$, we have that:

$$\lim_{n\to\infty} \widetilde{\mu}_n = \lim_{n\to\infty} X_{n-1}^{-1} \sum_{k=1}^{n} Z_{k-1}[Z_{k-1}^{-1}(\Gamma_k - \Gamma_k^I)] =$$
$$\lim_{n\to\infty} Z_{n-1}^{-1}(\Gamma_n - \Gamma_n^I) = \mu \quad \text{a.s.}$$

iii)

$$P_Q[(\sigma_{ii}^{-1} X_{n-1})^{1/2}(\widetilde{\mu}_{in} - \mu_i) \le x] =$$
$$P_Q[(\sigma_{ii} X_{n-1})^{-1/2} \sum_{k=1}^{n} \sum_{l=1}^{Z_{k-1}} (\gamma_{k-1l}^{i} - \mu_i) \le x] =$$
$$P_Q[(\sigma_{ii} X_{n-1})^{-1/2} \sum_{l=1}^{X_{n-1}} (\gamma_{0l}^{i} - \mu_i) \le x]. \tag{9}$$

From Toeplitz's lemma we deduce that

$$\lim_{n\to\infty} r^{-n} X_n = r(r-1)^{-1} W \qquad P_Q - a.s.,$$

and therefore

$$\lim_{n\to\infty} (r-1)(r^{n+1}-1)^{-1} X_n = W \qquad P_Q - a.s.$$

Thus, from (9) and making use of Proposition A.1 with $a_n = (r-1)^{-1}(r^{n+1}-1)$, $\nu_n = X_n$, $\nu = W$ and

$$Y_n(t,w) = (\sigma_{ii} X_n(w))^{-1/2} \sum_{l=1}^{[X_n(w)t]} (\gamma_{0l}^{i}(w) - \mu_i),$$

the result is derived.

iv) It can be deduced in a similar way to that one used in iii), applying Proposition A.2 and taking into account that

$$P_Q[(\sigma_{ii}^{-1} c_n)^{1/2}(\widetilde{\mu}_{in} - \mu_i) \le x] =$$
$$P_Q[(c_n^{-1} X_{n-1})^{-1/2}(\sigma_{ii} X_n)^{-1} \sum_{l=1}^{X_{n-1}} (\gamma_{0l}^{i} - \mu_i) \le x],$$

where $c_n = (r-1)^{-1}(r^n - 1)$.

P r o o f o f T h e o r e m 2.3. $\widehat{\mu}_n$ can be rewritten in the form:

$$\widehat{\mu}_n = \frac{r^{-2n}\sum_{k=1}^{n} Z_{k-1}\Gamma_k - n^{-1}\left(r^{-n}\sum_{k=1}^{n}\Gamma_k\right)\left(r^{-n}\sum_{k=1}^{n} Z_{k-1}\right)}{r^{-2n}\sum_{k=1}^{n} Z_{k-1}^2 - n^{-1}\left(r^{-n}\sum_{k=1}^{n} Z_{k-1}\right)^2}. \tag{10}$$

Using (3)(ii) and Toeplitz's lemma it is obtained, on $[Z_n \to \infty]$, that:

$$\lim_{n\to\infty} r^{-2n}\sum_{k=1}^{n} Z_{k-1}\Gamma_k = (r^2-1)^{-1}W^2\mu \quad \text{a.s.,}$$

$$\lim_{n\to\infty} r^{-n}\sum_{k=1}^{n} \Gamma_k = (r-1)^{-1}W\mu \quad \text{a.s.,}$$

$$\lim_{n\to\infty} r^{-n}\sum_{k=1}^{n} Z_{k-1} = (r-1)^{-1}W \quad \text{a.s.,}$$

$$\lim_{n\to\infty} r^{-2n}\sum_{k=1}^{n} (Z_{k-1})^2 = (r^2-1)^{-1}W^2 \quad \text{a.s.}$$

Consequently, from (10), the proof is concluded.

P r o o f o f T h e o r e m 2.4. Previously it will be necessary to prove, for $j = 1, 2,$ and $x \in \mathbb{R}$, that:

$$\lim_{n\to\infty} P_Q[(\sigma_{jj}\sum_{k=1}^{\infty} b_k^2)^{-1/2}\sum_{k=1}^{n} b_k(Z_{n-k}+1)^{-1/2}\eta_{nk}^j \le x] = \phi(x) \tag{11}$$

and

$$\lim_{n\to\infty} P_Q\left[\frac{(r^{2\alpha+1}-1)^{1/2}\sum_{k=1}^{n}(Z_{k-1}+1)^{\alpha}}{((r-1)^{2\alpha+1}\sigma_{jj})^{1/2}\left(\sum_{k=1}^{n} Z_{k-1}+n\right)^{\alpha+1/2}}\eta_{nk}^j \le x\right] = \phi(x), \tag{12}$$

where $\eta_{nk}^j = \Gamma_{jn-k+1} - Z_{n-k}\mu_j - \mu_j^I$, $\{b_n\}_{n\ge 1}$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} b_k^2 < \infty$ and $\alpha > -1/2$.

To prove (11) we consider the random variables:

$$U^j_{kn} = \sum_{i=1}^{k} b_i \xi^j_{ni} \quad , \quad U^j_k = \sum_{i=1}^{k} b_i \xi^j_i \quad , \quad U^j = \sum_{i=1}^{\infty} b_i \xi^j_i ,$$

where

$$\xi^j_{ni} = (Z_{n-i}+1)^{-1/2} \eta^j_{ni} \quad \text{if} \quad i = 1, \ldots, n \quad \text{or} \quad 0 \quad \text{if} \quad i \geq n+1,$$

and $\{\xi^j_i\}$ is a sequence of i.i.d. random variables with normal distribution of mean 0 and variance σ_{jj}.

It is clear that:

$$\xi^j_{ni} = (Z_{n-i}+1)^{-1/2} \sum_{i=1}^{Z_{n-i}} \gamma^j_{ni} \quad \text{if} \quad i = 1, \ldots, n \quad \text{or} \quad 0 \quad \text{if} \quad i \geq n+1.$$

Then, by a similar argument to that one used in Theorem 3.1 of [17], it can be proved that, for $j = 1, 2$, and any positive integer m

(13) $\quad (\xi^j_{n1}, \ldots, \xi^j_{nm})$ converges in distribution to $(\xi^j_1, \ldots, \xi^j_m)$ as $n \to \infty$.

From (13) we obtain that $\{U^j_{kn}\}$ converges in distribution to U^j_k, as $n \to \infty$, and clearly $\{U^j_k\}$ converges in distribution to U^j, as $k \to \infty$.

Obviously, for $k \geq n$, $U^j_{kn} = U^j_{nn}$. For $k = 1, \ldots, n$, taking into account that $E[\Gamma_{jk} - Z_{k-1}\mu_j - \Gamma^I_{jk} \mid \mathcal{F}_{k-1}] = 0$ almost surely and Chebyshev's inequality, we deduce that:

$$P_Q[|U^j_{kn} - U^j_{nn}| > \epsilon] \leq \epsilon^{-2} E[(U^j_{kn} - U^j_{nn})^2] \leq \sigma_{jj} \epsilon^{-2} \sum_{i=k+1}^{\infty} b_i^2 .$$

Thus, $P_Q[|U^j_{kn} - U^j_{nn}| > \epsilon]$ converges to 0 as $k \to \infty$, and applying Proposition A.3 (see Appendix) we derive that $\{U^j_{nn}\}$ converges in distribution to U^j as $n \to \infty$.

Now

$$\sum_{i=1}^{n} b_i (Z_{n-i}+1)^{-1/2} \eta^j_{ni} = U^j_{nn} + \sum_{i=1}^{n} b_i \frac{\Gamma^I_{jn-i+1} - \mu^I_j}{(Z_{n-i}+1)^{1/2}} \tag{14}$$

and since

$$P_Q \left[\left| \sum_{i=1}^{n} b_i \frac{\Gamma^I_{jn-i+1} - \mu^I_j}{(Z_{n-i}+1)^{1/2}} \right| > \epsilon \right] \leq 2\epsilon^{-1} \mu^I_j E \left[\sum_{i=1}^{n} |b_i| (Z_{n-i}+1)^{-1/2} \right],$$

the second term of the sum in (14) converges in probability to 0, as $n \to \infty$, and (11) holds.

On the other hand, on Q and for $j=1,2$, it may be written:

$$\frac{\sum_{k=1}^{n}(Z_{k-1}+1)^{\alpha}}{(\sum_{k=1}^{n} Z_{k-1}+n)^{\alpha+1/2}}\eta_{nk}^{j} =$$

$$\left(W^{-1}(\sum_{k=1}^{n} Z_{k-1}+n)\right)^{-(\alpha+1/2)} \sum_{k=1}^{n} \frac{r^{(k-1)(\alpha+1/2)}}{(Z_{k-1}+1)^{1/2}}\eta_{nk}^{j}+$$

$$\frac{(\sum_{k=1}^{n} Z_{k-1}+n)^{-(\alpha+1/2)}}{(Z_{k-1}+1)^{1/2}} \sum_{k=1}^{n}\left((Z_{k-1}+1)^{\alpha+1/2}-(r^{k-1}W)^{\alpha+1/2}\right)\eta_{nk}^{j} \tag{15}$$

Now $(Wr^n)^{-1}(r-1)(\sum_{k=1}^{n} Z_{k-1}+n)$ converges almost surely to 1, as $n\to\infty$, and from (11),

$$\frac{(r-1)^{\alpha+1/2}}{(Z_{k-1}+1)^{1/2}}\sum_{k=1}^{n} r^{-(n-k+1)(\alpha+1/2)}\eta_{nk}^{j}$$

converges in distribution to a normal of mean 0 and variance σ_{jj}^{*} where

$$\sigma_{jj}^{*}=\sigma_{jj}(r^{2\alpha+1}-1)^{-1}(r-1)^{2\alpha+1},$$

so that, it is deduced that the first summand in (15) converges in distribution to a normal of mean 0 and variance σ_{jj}^{*}.

It is not difficult to verify that:

$$\frac{\left(\sum_{k=1}^{n} Z_{k-1}+n\right)^{-(\alpha+1/2)}}{(Z_{k-1}+1)^{1/2}} \sum_{k=1}^{n}\left((Z_{k-1}+1)^{\alpha+1/2}-(r^{k-1}W)^{\alpha+1/2}\right)\eta_{nk}^{j} \leq$$

$$\left(r^{-n}\sum_{k=1}^{n} Z_{k-1}+n\right)^{-(\alpha+1/2)} A_n^{1/2}(B_n^j)^{1/2}$$

where

$$A_n = r^{-n(\alpha+1/2)}\sum_{k=1}^{n} r^{(k-1)(\alpha+1/2)}\left[(r^{-(k-1)}(Z_{k-1}+1))^{\alpha+1/2}-W^{\alpha+1/2}\right]^2$$

and

$$B_n^j = r^{-n(\alpha+1/2)} \sum_{k=1}^{n} r^{(k-1)(\alpha+1/2)} \left(\frac{\Gamma_{jk} - Z_{k-1}\mu_j - \mu_j^I}{Z_{k-1}+1} \right)^2 .$$

We know that $r^{-n}\left(\sum_{k=1}^{n} Z_{k-1} + n\right)$ converges almost surely to $W(r-1)^{-1}$ as $n \to \infty$. Then, by Toeplitz's lemma, it is obtained that A_n converges almost surely to 0, as $n \to \infty$.

It is matter of some calculations to verify, for $j = 1, 2$, that:

$$E[B_n^j] = r^{-n(\alpha+1/2)} \sum_{k=1}^{n} r^{(k-1)(\alpha+1/2)} E\left[\frac{\sigma_{jj} Z_{k-1} + \sigma_{jj}^I}{Z_{k-1}+1} \right].$$

Consequently, on Q , it is derived that $\sup_n E[B_n^j] < \infty$, $j = 1, 2$, and (12) holds.

We now prove the Theorem. It is readily obtained that:

$$\left(\sum_{k=1}^{n} Z_{k-1} + n \right)^{1/2} (\widehat{\mu}_{jn} - \mu_j) = \frac{\overline{A}_n^j - n^{-1}\overline{B}_n^j}{\overline{C}_n - n^{-1}},$$

where:

$$\overline{A}_n^j = \left(\sum_{k=1}^{n} Z_{k-1} + n \right)^{-3/2} \left(\sum_{k=1}^{n} Z_{k-1} + n \right) (\Gamma_{jk} - Z_{k-1}\mu_j - \mu_j^I),$$

$$\overline{B}_n^j = \left(\sum_{k=1}^{n} Z_{k-1} + n \right)^{1/2} \sum_{k=1}^{n} (\Gamma_{jk} - Z_{k-1}\mu_j - \mu_j^I),$$

and

$$\overline{C}_n = \left(\sum_{k=1}^{n} Z_{k-1} + n \right)^{-2} \sum_{k=1}^{n} (Z_{k-1}+1)^2.$$

From (12), with $\alpha = 1$, we obtain that $\overline{A}_n^j$ converges in distribution to a normal of mean 0 and variance $(r^2+r+1)^{-1}(r-1)^2\sigma_{jj}$ as $n \to \infty$. Using again (12), with $\alpha = 0$, we have that $\overline{B}_n^j$ converges in distribution to a normal of mean 0 and variance σ_{jj} and therefore $n^{-1}\overline{B}_n^j$ converges in distribution to 0, as $n \to \infty$. Then, taking into account that $\overline{C}_n$ converges almost surely to $(r+1)^{-1}(r-1)$ as $n \to \infty$, the proof is concluded.

Appendix

Consider a sequence of i.i.d. random variables $\{\xi_n\}_{n\geq 1}$ on the probability space $(\Omega, \mathcal{A}, P)$ such that $E[\xi_1] = 0$ and $E[\xi_1^2] = \sigma^2 < \infty$. Denote by $S_n = \sum_{i=1}^{n} \xi_i$, $n = 1, 2, \ldots$ and, for $t \in [0, 1]$, we shall define, for $n = 1, 2, \ldots$,

$$X_n(t, w) = \sigma^{-1} n^{-1/2} S_{[nt]}(w) \ , \ w \in \Omega,$$

and

$$Y_n(t, w) = X_{\nu_n}(t, w) \quad \text{if} \quad \nu_n(w) > 0 \quad \text{or} \quad 0 \quad \text{otherwise},$$

where $\{\nu_n\}_{n\geq 1}$ is a sequence of non negative integer-valued random variables on $(\Omega, \mathcal{A}, P)$

Proposition A.1. *If there exists a sequence of real numbers $\{a_n\}_{n\geq 1}$ such that:*

i) $\{a_n\}_{n\geq 1}$ *converges to* ∞, *as* $n \to \infty$.

ii) $\{a_n^{-1}\nu_n\}_{n\geq 1}$ *converges in probability to a non negative random variable* ν *such that* $P[\nu > 0] > 0$.

Then, for every probability $P \ll P_D$, where $P_D[\cdot] = P[\cdot \mid D]$, being $D = [\nu > 0]$, we have, for $t \in [0, 1]$ and $x \in \mathbb{R}$, that:

$$\lim_{n\to\infty} P^*[w : Y_n(t, w) \leq x] = P^*[w : V^*(t, w) \leq x],$$

where $\{V^*(t, \cdot), 0 \leq t \leq 1)\}$ is a Wiener process.

The proof can be read in [3].

Proposition A.2. *Under the hypotheses of Proposition* A.1 *if, for $t \in [0, 1]$, we define, for $n = 1, 2, \ldots$:*

$$Y_n'(t, w) = \nu_n^{-1/2}(w) a_n^{1/2} Y_n(t, w) \ , \ w \in D.$$

Then, for every probability $P^* << P_D$, $t \in [0, 1]$, $x \in \mathbb{R}$, we have that

$$\lim_{n\to\infty} P^*[w : Y_n'(t, w) \leq x] = P^*[w : V(t, w)\psi^{-1/2} \leq x],$$

where ψ is P^*-independent of the Wiener process $\{V^*(t, \cdot), 0 \leq t \leq 1)\}$ and such that $P^*[w : \psi(w) \leq x] = P^*[w : \nu(w) \leq x]$.

The proof can be read in [3].

Proposition A.3. *Consider the sequences of random variables $\{V_n\}_{n\geq 1}$ and $\{U_{kn}\}_{n,k\geq 1}$ on $(\Omega, \mathcal{A}, P)$ such that:*

i) For each $k = 1, 2, \ldots,$ $\{U_{kn}\}_{n\geq 1}$ *converges in distribution to* U_k*, as* $n \to \infty$.

ii) $\{U_k\}_{k\geq 1}$ *converges in distribution to* U *as* $k \to \infty$.

iii) For every $\varepsilon > 0$, $\lim\limits_{k\to\infty} \lim\limits_{n\to\infty} P[|U_{kn} - V_n| \geq \varepsilon] = 0$.

Then $\{V_n\}_{n\geq 1}$ *converges in distribution to* U*, as* $n \to \infty$.

The proof can be read in [1], p. 28.

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