PLISKA
STUDIA MATHEMATICA
BULGARICA

ПЛИСКА
БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. Bulgar. **21** (2012), 71–96

PLISKA
STUDIA MATHEMATICA
BULGARICA

L^p MICROLOCAL PROPERTIES FOR MULTI-QUASI-ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

Gianluca Garello, Alessandro Morando

Abstract. In the present paper microlocal properties of a class of suitable L^p bounded pseudodifferential operators are stated in the framework of weighted Sobolev spaces of L^p type. Applications to microlocal regularity of solutions to multi-quasi-elliptic partial differential equations are also given.

1. Introduction. Consider the class of pseudodifferential operators with standard quantization:

$$a(x,D)u := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, d\xi, \tag{1}$$

where $x \cdot \xi = \sum_{j=1}^{n} x_j\xi_j$, $\hat{u}$ is the Fourier transform of $u \in C_0^\infty(\Omega)$, Ω is an open subset of $\mathbb{R}^n$ and $a(x,\xi)$ belongs to the class $S^m_\lambda(\Omega)$ of smooth symbols satisfying for any compact set $K \subset\subset \Omega$ and all multi-indices $\alpha, \beta \in \mathbb{Z}^n_+$:

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{K,\alpha,\beta} \lambda(\xi)^{m-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n. \tag{2}$$

$\lambda(\xi)$ is a continuous function with polynomial growth at infinitely, which satisfies a slowly varying condition, that is $1/C \le \lambda(\xi)/\lambda(\eta) \le C$ when $|\xi - \eta| \le c\lambda(\xi)^{\frac{1}{\mu}}$, for suitable $\mu, C, c > 0$.

2010 *Mathematics Subject Classification:* Primary 35S05; Secondary 35A17.
Key words: pseudodifferential operators, weighted Sobolev spaces, microlocal properties.

Assume moreover that the symbol $a(x,\xi)$ is elliptic in the generalized sense $|a(x,\xi)| > C_K\lambda(\xi)^m$, for x in any compact set $K \subset\subset \Omega$ and $|\xi|$ large. In the L^2 framework, continuity, local solvability and regularity of solutions to pseudodifferential equations, also in microlocal sense, are standard arguments, see R. Beals [1], Rodino [10], Garello [4], and finally they may be in someway summarized in the Weyl-Hörmander pseudodifferential calculus [8, Ch. 18].
On the other hand, as it is specified in the next inclusion (25), the symbols in $S^0_\lambda(\Omega)$ are a generalization of the Hörmander class $S^0_{\rho,0}(\Omega)$, with $\rho < 1$, then as well known the respective pseudodifferential operators are not L^p bounded for $p \neq 2$.
A wide literature is devoted to the problem of L^p continuity of classical pseudodifferential operators with $\rho < 1$, we quote here only the paper of Fefferman [3].
Following the arguments in Taylor [11], the authors in [6] consider the class $M^m_\lambda(\Omega)$ of symbols satisfying $\xi^\gamma\partial^\gamma_\xi a(x,\xi) \in S^m_\lambda(\Omega)$ when the components of $\gamma \in \mathbb{Z}^n_+$ are equal to zero or one. They state the L^p continuity for pseudodifferential operators of zero order and, in the generalized elliptic case, they show the regularity of solutions to pseudodifferential equations, in the frame of λ weighted Sobolev spaces of L^p type.

For introducing at this point the study of microlocal properties, the main problem arises from the lack of any homogeneity of the weight $\lambda(\xi)$ and the presence of the multiplicative factor ξ^γ, which do not allow us to use in a suitable way conic neighborhoods in $\mathbb{R}^n_\xi$, as done in the classical definition of Hörmander wave front set, see [8].
The focus point in the present paper is then to find suitable neighborhoods of a set $X \subset \mathbb{R}^n_\xi$, which allow us to construct useful microlocal properties. To this aim the slowly varying condition is relaxed in the form expressed in the next Definition 1.

In §2 the weight functions and the respective weighted symbols are introduced and their main properties stated. Then in §3 the attention is focused on the introduction of microlocal Sobolev regularity of weighted L^p type for a distribution $u \in \mathcal{D}'(\Omega)$. Here the construction of suitable neighborhoods of a set X in the phase space $\mathbb{R}^n_\xi$ is carefully described.
In §4 the authors prove that the microlocal regularity is preserved under the action of pseudodifferential operators in $M^m_\lambda(\Omega)$ and the solutions of the equations $a(x,D)u = f$ keep the same microlocal Sobolev regularity of the data f, clearly with different order, when $a(x,D)$ is λ-elliptic in microlocal sense.
In §5 applications to multi-quasi-elliptic equations are given.

2. Weight Functions and Symbol Classes.

Definition 1 (Weight Functions). *A continuous real valued map* $\lambda(\xi)$, $\xi \in \mathbb{R}^n$, *is a weight function if there exist suitable constants* $\mu \geq \mu_1 \geq \mu_0 > 0$, $C \geq 1 \geq c > 0$ *such that, for any* $\xi, \eta \in \mathbb{R}^n$

$$\frac{1}{C}\,(1+|\xi|)^{\mu_0} \leq \lambda(\xi) \leq C\,(1+|\xi|)^{\mu_1}\,; \tag{3}$$

$$\frac{1}{C} \leq \frac{\lambda(\eta)}{\lambda(\xi)} \leq C \quad \textit{when} \quad \sum_{j=1}^{n} |\xi_j - \eta_j| \left(\lambda(\eta)^{\frac{1}{\mu}} + |\eta_j|\right)^{-1} \leq c. \tag{4}$$

We say that $\tilde{\lambda}(\xi)$ is equivalent to $\lambda(\xi)$, write $\lambda \asymp \tilde{\lambda}$, if $\dfrac{1}{C} \leq \dfrac{\lambda(\xi)}{\tilde{\lambda}(\xi)} \leq C$, for some $C \geq 1$. It is trivial that $\tilde{\lambda}(\xi)$ is again a weight function.
As the reader can easily verify, the elliptic weight of order $m \in \mathbb{N}$, $\lambda_m(\xi) := \sqrt{1+\sum_{j=1}^n \xi_j^{2m}}$, the quasi-elliptic weight of anisotropic order $M = (m_1, \dots, m_n)$, $m_j \in \mathbb{N}$, $\inf_j m_j \geq 1$, $\lambda_M(\xi) := \sqrt{1+\sum_{j=1}^n \xi_j^{2m_j}}$, are weight functions. Other examples will be given in the last Section.

Consider $\xi, \eta \in \mathbb{R}^n$ such that $|\xi_j - \eta_j| \leq \varepsilon \left(\lambda(\eta)^{\frac{1}{\mu}} + |\eta_j|\right)$, for any $j = 1, \dots, n$. At least one among (i) $|\xi_j - \eta_j| \leq 2\varepsilon\lambda(\eta)^{\frac{1}{\mu}}$ or (ii) $|\xi_j - \eta_j| \leq 2\varepsilon|\eta_j|$ is surely verified. In the case (i) with $\varepsilon \leq \dfrac{c}{n}$ we obtain from (4), $|\xi_j - \eta_j| \leq 2\varepsilon C^{\frac{1}{\mu}} \lambda(\xi)^{\frac{1}{\mu}} \leq 2\varepsilon C^{\frac{1}{\mu}} \left(\lambda(\xi)^{\frac{1}{\mu}} + |\xi_j|\right)$. When $\varepsilon < \dfrac{1}{2}$, we obtain from (ii) that $(1-2\varepsilon)|\eta_j| \leq |\xi_j| \leq (1+2\varepsilon)|\eta_j|$. It then follows $|\xi_j - \eta_j| \leq \dfrac{2\varepsilon}{1-2\varepsilon}|\xi_j| \leq \dfrac{2\varepsilon}{1-2\varepsilon}\left(\lambda(\xi)^{\frac{1}{\mu}} + |\xi_j|\right)$.
For any $0 < \varepsilon < \min\left\{\dfrac{1}{2}, \dfrac{c}{n}\right\}$ we can say that $|\xi_j - \eta_j| \leq H\left(\lambda(\xi)^{\frac{1}{\mu}} + |\xi_j|\right)$ when for any $i = 1, \dots, n$ $|\xi_i - \eta_i| \leq \varepsilon\left(\lambda(\eta)^{\frac{1}{\mu}} + |\eta_i|\right)$, $H = \max\left\{\dfrac{2\varepsilon}{1-2\varepsilon}, 2\varepsilon C^{\frac{1}{\mu}}\right\}$.
Moreover, taking ξ, η as above, we have

$$\lambda(\eta)^{\frac{1}{\mu}} + |\eta_j| \leq \lambda(\eta)^{\frac{1}{\mu}} + |\xi_j - \eta_j| + |\xi_j| \leq (C^{\frac{1}{\mu}} + H)\lambda(\xi)^{\frac{1}{\mu}} + (H+1)|\xi_j|,$$

$$\lambda(\xi)^{\frac{1}{\mu}} + |\xi_j| \leq \lambda(\xi)^{\frac{1}{\mu}} + |\xi_j - \eta_j| + |\eta_j| \leq (C^{\frac{1}{\mu}} + \varepsilon)\lambda(\eta)^{\frac{1}{\mu}} + (\varepsilon+1)|\eta_j|.$$

Taking now $\varepsilon < \dfrac{c}{2n}$, $H = \max\left\{\dfrac{2\varepsilon}{1-2\varepsilon}, 2\varepsilon C^{\frac{1}{\mu}}\right\}$, $K = H + C^{\frac{1}{\mu}}$ we can conclude:

$$\begin{aligned}&\frac{1}{K} \le \frac{\lambda(\xi)^{\frac{1}{\mu}} + |\xi_j|}{\lambda(\eta)^{\frac{1}{\mu}} + |\eta_j|} \le K \quad \text{and} \quad |\xi_j - \eta_j| \le H\left(\lambda(\xi)^{\frac{1}{\mu}} + |\xi_j|\right),\\ &\text{when}\\ &\sum_{i=1}^{n} |\xi_i - \eta_i| \left(\lambda(\eta)^{\frac{1}{\mu}} + |\eta_i|\right)^{-1} \le \varepsilon.\end{aligned} \tag{5}$$

The following proposition is then immediately obtained.

Proposition 1. *Set*

$$\lambda_j(\xi) := \lambda(\xi)^{\frac{1}{\mu}} + |\xi_j|, \quad j = 1, \ldots, n. \tag{6}$$

Then the vector-valued function $\Lambda(\xi) = (\lambda_1(\xi), \ldots \lambda_n(\xi))$ is a weight vector in the sense that there exist suitable constants $\nu_1 \ge \nu_0 > 0$, $K \ge 1 \ge k > 0$ such that, for any $\xi \in \mathbb{R}^n$ and $j = 1, \ldots, n$

$$\frac{1}{K}\left(1 + |\xi|\right)^{\nu_0} \le \lambda_j(\xi) \le K\left(1 + |\xi|\right)^{\nu_1}; \tag{7}$$

$$\frac{1}{K} \le \frac{\lambda_j(\eta)}{\lambda_j(\xi)} \le K \quad \textit{when} \quad \sum_{i=1}^{n} |\xi_i - \eta_i| \lambda_i(\eta)^{-1} \le k. \tag{8}$$

For details on the weight vectors notation see Beals [1] and Rodino [10].
In the following we write $\lambda(\xi) \approx \lambda(\eta)$ in $D \subset \mathbb{R}^n$ if, for some $C > 1$, $\dfrac{1}{C} \le \dfrac{\lambda(\eta)}{\lambda(\xi)} \le C$, when $\xi, \eta \in D$.

Definition 2 (Symbol Classes). *We say that $a(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ is a symbol in $S^m_\lambda(\Omega)$, Ω open subset of $\mathbb{R}^n$, $m \in \mathbb{R}$, if for any compact $K \subset\subset \Omega$, α, $\beta \in \mathbb{Z}^n_+$ and some positive constants $c_{K,\alpha,\beta}$:*

$$\sup_{x \in K} \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le c_{K,\alpha,\beta} \lambda(\xi)^{m - \frac{|\alpha|}{\mu}}, \quad \xi \in \mathbb{R}^n. \tag{9}$$

Moreover $a(x,\xi)$ belongs to $M^m_\lambda(\Omega)$ if:

$$\xi^\gamma \partial_\xi^\gamma a(x,\xi) \in S^m_\lambda(\Omega), \textit{ for every } \gamma \in \mathbb{K}. \tag{10}$$

Here the set of multi-indices $\mathbb{K} := \{0,1\}^n \subset \mathbb{Z}^n_+$, is considered in such a way that ∂_ξ^γ are the derivatives made at most one time with respect to any components.

Proposition 2. *For any $a(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$, $m \in \mathbb{R}$, the following properties are equivalent:*

$$\xi^\gamma \partial_\xi^\gamma a(x,\xi) \in S_\lambda^m(\Omega), \quad \textit{for any} \quad \gamma \in \mathbb{Z}_+^n; \tag{11}$$

$$\sup_{x\in K} |\xi^\gamma \partial_\xi^{\alpha+\gamma} \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta,\gamma,K} \lambda(\xi)^{m-\frac{1}{\mu}|\alpha|}, \quad \textit{for any} \quad \alpha,\beta,\gamma \in \mathbb{Z}_+^n; \tag{12}$$

$$\sup_{x\in K} |\partial_\xi^\nu \partial_x^\beta a(x,\xi)| \le C_{\nu,\beta,K} \lambda(\xi)^m \Lambda(\xi)^{-\nu}, \quad \textit{for any} \quad \nu,\beta \in \mathbb{Z}_+^n. \tag{13}$$

Here K is a generic compact subset of Ω, $C_{\alpha,\beta,\gamma,K}$, $C_{\nu,\beta,K}$ are suitable positive constants and, with usual multi-index notation, $\Lambda(\xi)^{-\nu} := \prod_{j=1}^n \lambda_j(\xi)^{-\nu_j}$.

The equivalence of conditions (11) and (12) is still true when $\gamma \in \mathbb{K}$.

P r o o f. The equivalence of conditions (11), (12) is proved in [6, Proposition 3.4].
(12) $\Rightarrow$ (13): For an arbitrarily fixed vector $\xi \in \mathbb{R}^n$, we set

$$\begin{aligned} J_1 &= J_1(\xi) := \{j \in \{1,\dots,n\} : |\xi_j| > \lambda(\xi)^{\frac{1}{\mu}}\}, \\ J_2 &= J_2(\xi) := \{1,\dots,n\} \setminus J_1(\xi). \end{aligned} \tag{14}$$

Moreover, we can split any multi-index $\nu \in \mathbb{Z}_+^n$ in $\nu = \alpha + \gamma$, with $\alpha = \alpha(\xi) \in \mathbb{Z}_+^n$ and $\gamma = \gamma(\xi) \in \mathbb{Z}_+^n$ defined by

$$\alpha_j := \begin{cases} \nu_j, & j \in J_2 \\ 0, & \text{otherwise} \end{cases} \qquad \gamma_j := \begin{cases} 0, & j \in J_2 \\ \nu_j, & \text{otherwise}. \end{cases} \tag{15}$$

By considering now $\alpha, \gamma \in \mathbb{Z}_+^n$, defined as in (15), $\beta \in \mathbb{Z}_+^n$ and $K \subset\subset \Omega$, the estimate (12) reads:

$$\prod_{j\in J_1} |\xi_j|^{\nu_j} |\partial_\xi^\nu \partial_x^\beta a(x,\xi)| \le \tilde{C}_{\alpha,\beta,\gamma,K} \lambda(\xi)^m \prod_{j\in J_2} \lambda(\xi)^{-\frac{\nu_j}{\mu}}, \ \forall x \in K, \tag{16}$$

where $\tilde{C}_{\alpha,\beta,\gamma,K}$ is a suitable positive constant.
For any $j \in J_1$ we get $2|\xi_j| > \lambda_j(\xi)$, hence

$$\prod_{j\in J_1} |\xi_j|^{\nu_j} \ge \prod_{j\in J_1} \frac{1}{2^{\nu_j}} \lambda_j(\xi)^{\nu_j}. \tag{17}$$

Similarly, for $j \in J_2$ we have $\lambda_j(\xi) \le 2\lambda(\xi)^{\frac{1}{\mu}}$, hence

$$\prod_{j\in J_2} \lambda(\xi)^{-\frac{\nu_j}{\mu}} \le \prod_{j\in J_2} 2^{\nu_j}\lambda_j(\xi)^{-\nu_j}\,. \tag{18}$$

Since $\alpha = \alpha(\xi)$ and $\gamma = \gamma(\xi)$, at a first glance the constant $\tilde{C}_{\alpha,\beta,\gamma,K}$ involved in (16) seems to depend on ξ. However for any fixed ν, β and K, the trivial estimate $\tilde{C}_{\alpha,\beta,\gamma,K} \le \max\limits_{\alpha+\gamma=\nu}\{C_{\alpha,\beta,\gamma,K}\}$, where $C_{\alpha,\beta,\gamma,K}$ are the constants involved in (12), shows that $\tilde{C}_{\alpha,\beta,\gamma,K}$ is independent of ξ.

Then (13) follows at once, collecting (16), (17) and (18).

(13) $\Rightarrow$ (12): for given α, β, γ and K, the estimate (13) with $\nu = \alpha+\gamma$ gives

$$|\partial_\xi^{\alpha+\gamma}\partial_x^\beta a(x,\xi)| \le C_{\alpha+\gamma,\beta,K}\lambda(\xi)^m\Lambda(\xi)^{-\alpha-\gamma}\,. \tag{19}$$

Then (12) follows from the trivial inequality: $|\xi^\gamma|\Lambda(\xi)^{-\alpha-\gamma} \le \lambda(\xi)^{-\frac{|\alpha|}{\mu}}$. $\square$

Remark. Consider for $m \in \mathbb{R}$ the class $S^m_\Lambda(\Omega)$ of smooth symbols $a(x,\xi)$ such that for any $K \subset\subset \Omega$

$$\sup_{x\in K}|\partial_\xi^\nu\partial_x^\beta a(x,\xi)| \le c_{\nu,\beta,K}\lambda(\xi)^m\Lambda(\xi)^{-\nu} \quad \nu,\beta\in\mathbb{Z}^n_+. \tag{20}$$

Then Proposition 2 easily shows that

$$S^m_\Lambda(\Omega) \subset M^m_\lambda(\Omega). \tag{21}$$

$S^m_\Lambda(\Omega)$ are particular cases of the symbol classes studied in [1], [10], [4] [9].

Moreover by means of Proposition 2, $M^m_\lambda(\Omega)$ may be identified with the symbol class $M^m_{\rho,\lambda}(\Omega)$, with $\rho = \frac{1}{\mu}$, introduced in [6]. Again $M^m_{\rho,\langle\xi\rangle}(\Omega)$, $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, $0 < \rho \le 1$, are the Taylor symbol classes $M^m_\rho(\Omega)$, see [11, Ch. XI, §4]. Notice at the end that $M^m_{\langle\xi\rangle}(\Omega)$ is exactly the standard Hörmander symbol class $S^m_{1,0}(\Omega)$.

Proposition 3. *Any weight function $\lambda(\xi)$ admits an equivalent smooth weight function $\tilde{\lambda}(\xi) \in M^1_\lambda(\mathbb{R}^n)$.*

P r o o f. For fixed $\varepsilon > 0$, in the set of smooth compactly supported functions $C^\infty_0(\mathbb{R}^n)$, consider a non negative $\varphi(\zeta)$ such that $|\zeta_j| \le \varepsilon$ in supp $\varphi(\zeta)$ and $\varphi(\zeta) = 1$ when $|\zeta_j| \le \dfrac{\varepsilon}{2}$, $j = 1,\dots,n$. Assuming $\lambda_j(\xi)$ as in (6) set:

$$\Phi(\xi,\eta) := \varphi\left(\frac{\xi_1-\eta_1}{\lambda_1(\eta)},\dots,\frac{\xi_n-\eta_n}{\lambda_n(\eta)}\right),$$

Notice now that $\Phi(\xi,\eta)$ is supported in the set where, for any $j=1,\dots,n$, $|\xi_j-\eta_j|\le\varepsilon\lambda_j(\eta)$ and it is identically equal to 1 when $|\xi_j-\eta_j|\le\frac{\varepsilon}{2}\lambda_j(\eta)$. Then, assuming $\varepsilon<\dfrac{k}{2n}$ and ξ,η in supp $\Phi(\xi,\eta)$, (5) assures that, for some $K,H>0$,

$$\frac{1}{K}\le\frac{\lambda_j(\eta)}{\lambda_j(\xi)}\le K\quad\text{and}\quad|\xi_j-\eta_j|\le H\lambda_j(\xi),\quad j=1,\dots,n\,.$$

The same is true when $\Phi(\xi,\eta)=1$ by changing the constant H with a suitable smaller one $\tilde H$. Then

$$\int\Phi(\xi,\eta)\,d\eta\le\|\varphi\|_\infty\int\chi_{B(\xi)}(\xi-\eta)\,d\eta=(2H)^n\|\varphi\|_\infty\prod_{j=1}^n\lambda_j(\xi),$$

$$\int\Phi(\xi,\eta)\,d\eta\ge\int\chi_{\tilde B(\xi)}(\xi-\eta)\,d\eta=(2\tilde H)^n\prod_{j=1}^n\lambda_j(\xi).$$

$\chi_{B(\xi)}$ is the characteristic function of the cube $B(\xi)=\prod_{j=1}^n\left[-H\lambda_j(\xi),H\lambda_j(\xi)\right]$ and $\chi_{\tilde B(\xi)}$ is the same for the cube $\tilde B(\xi)$ obtained by changing H with $\tilde H$. It then follows that $\int\Phi(\xi,\eta)\,d\eta\asymp\prod\limits_{j=1}^n\lambda_j(\xi)$. Set now:

$$\tilde\lambda(\xi)=\int\lambda(\eta)\Phi(\xi,\eta)\prod_{j=1}^n\lambda_j(\eta)^{-1}\,d\eta. \tag{22}$$

Since for $\varepsilon<\dfrac{c}{2n}$ and any $j=1,\dots,n$, $|\xi_j-\eta_j|\le\varepsilon\lambda_j(\eta)$ in supp $\Phi(\xi,\eta)$, it follows from (4) and (8), $\lambda(\eta)\approx\lambda(\xi)$ and $\lambda_j(\eta)\approx\lambda_j(\xi)$, for any $j=1,\dots,n$, then $\tilde\lambda(\xi)\asymp\lambda(\xi)$. Moreover $\tilde\lambda(\xi)$ is obviously smooth and for any $\nu\in\mathbb{Z}_+^n$:

$$\partial^\nu\tilde\lambda(\xi)=\int\lambda(\eta)\partial_\zeta^\nu\varphi\left(\frac{\xi_1-\eta_1}{\lambda_1(\eta)},\dots,\frac{\xi_n-\eta_n}{\lambda_n(\eta)}\right)\prod_{j=1}^n\lambda_j(\eta)^{-\nu_j-1}\,d\eta. \tag{23}$$

Since supp $\partial_\zeta^\nu\varphi\subset$ supp φ, we obtain, for some positive constant M:

$$|\partial^\nu\tilde\lambda(\xi)|\le M\tilde\lambda(\xi)\Lambda(\xi)^{-\nu}, \tag{24}$$

which concludes the proof in view of Proposition 2, see also (21). $\square$

For any $m \in \mathbb{R}$ we have the following relations with the usual Hörmander symbol classes $S^m_{\rho,\delta}(\Omega)$ [8]:

$$S^h_{\frac{\mu_1}{\mu},0}(\Omega) \subset S^m_\lambda(\Omega) \subset S^k_{\frac{\mu_0}{\mu},0}(\Omega), \quad h(k) = \min(\max)\{m\mu_0, m\mu_1\}; \tag{25}$$

$$S^{m-N_0}_\lambda(\Omega) \subset M^m_\lambda(\Omega) \subset S^m_\lambda(\Omega), \quad N_0 = n\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right); \tag{26}$$

$$\bigcap_{m\in\mathbb{R}} M^m_\lambda(\Omega) = \bigcap_{m\in\mathbb{R}} S^m_\lambda(\Omega) = \bigcap_{m\in\mathbb{R}} S^m_{0,0}(\Omega) := S^{-\infty}(\Omega). \tag{27}$$

Generally speaking we say that two symbols $a(x,\xi), b(x,\xi)$ in some of the previous classes are equivalent if $a(x,\xi) - b(x,\xi) \in S^{-\infty}(\Omega)$ (we write $a(x,\xi) \sim b(x,\xi)$).
Let now $\{a_j\}_{j=1}^\infty$ be a sequence of symbols $a_j(x,\xi) \in S^{m_j}_\lambda(\Omega)(M^{m_j}_\lambda(\Omega))$ such that $m_j > m_{j+1}$, $m_j \overset{j\to\infty}{\longrightarrow} -\infty$. Then there exists a symbol $a(x,\xi) \in S^{m_1}_\lambda(\Omega)(M^{m_1}_\lambda(\Omega))$ such that

$$a(x,\xi) \sim \sum_{j=1}^\infty a_j(x,\xi), \tag{28}$$

where (28) means in particular $a(x,\xi) - \sum\limits_{j<N} a_j(x,\xi) \in S^{m_N}_\lambda(\Omega)(M^{m_N}_\lambda(\Omega))$.

Here and in the following we can refer to [11] and [6] for omitted proofs and details.

We denote with $\mathcal{S}^m_\lambda(\Omega)$ $(\mathcal{M}^m_\lambda(\Omega))$ the class of pseudodifferential operators introduced in (1) with symbols in $S^m_\lambda(\Omega)$ $(M^m_\lambda(\Omega))$.
Recall that a pseudodifferential operator is said *properly supported* provided that it maps $C^\infty_0(\Omega)$ to $\mathcal{E}'(\Omega)$ and the same happens for its transposed, hence $a(x,D) : C^\infty(\Omega) \mapsto \mathcal{D}'(\Omega)$. For any $a(x,\xi) \in S^m_\lambda(\Omega)\,(M^m_\lambda(\Omega))$ there exists $a'(x,\xi) \in S^m_\lambda(\Omega)\,(M^m_\lambda(\Omega))$ such that $a'(x,D)$ is properly supported and $a'(x,\xi) \sim a(x,\xi)$. For the classes of properly supported pseudodifferential operators with symbols respectively in $S^m_\lambda(\Omega)\,(M^m_\lambda(\Omega))$, we use the notation $\tilde{\mathcal{S}}^m_\lambda(\Omega)\left(\tilde{\mathcal{M}}^m_\lambda(\Omega)\right)$.

Proposition 4 (Symbolic Calculus). *For any $m, m' \in \mathbb{R}$, consider the operators $a(x,D) \in \mathcal{M}^m_\lambda(\Omega)$, $b(x,D) \in \tilde{\mathcal{M}}^{m'}_\lambda(\Omega)$. Then $c(x,D) := b(x,D)a(x,D) \in \mathcal{M}^{m+m'}_\lambda(\Omega)$ and moreover:*

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi b(x,\xi) D^\alpha_x a(x,\xi), \quad D^\alpha = (-i)^\alpha \partial^\alpha. \tag{29}$$

Definition 3 (Sobolev Spaces). *For $s \in \mathbb{R}$ and $1 < p < \infty$, we define the spaces:*

(30) $$H^{s,p}_\lambda := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \quad ; \quad \lambda(D)^s u \in L^p(\mathbb{R}^n) \right\} ;$$

(31) $$H^{s,p}_{\lambda,\mathrm{loc}}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) \quad ; \quad \varphi u \in H^{s,p}_\lambda \text{ for any } \varphi \in C_0^\infty(\Omega) \right\} ;$$

(32) $$H^{s,p}_{\lambda,\mathrm{comp}}(\Omega) := H^{s,p}_\lambda \cap \mathcal{E}'(\Omega).$$

Proposition 5. *Consider $a(x,\xi) \in M^m_\lambda(\Omega)$, $m \in \mathbb{R}$. Then for any $1 < p < \infty$, $s \in \mathbb{R}$*

(33) $$a(x,D) : H^{s+m,p}_{\lambda,comp}(\Omega) \mapsto H^{s,p}_{\lambda,loc}(\Omega), \quad \textit{continuously.}$$

If moreover $a(x,D)$ is properly supported,

(34) $$a(x,D) : H^{s+m,p}_{\lambda,\mathrm{loc}}(\Omega) \mapsto H^{s,p}_{\lambda,\mathrm{loc}}(\Omega), \quad \textit{continuously};$$

(35) $$a(x,D) : H^{s+m,p}_{\lambda,\mathrm{comp}}(\Omega) \mapsto H^{s,p}_{\lambda,\mathrm{comp}}(\Omega), \quad \textit{continuously.}$$

We say that a symbol $a(x,\xi) \in S^m_\lambda(\Omega)$ is λ-elliptic if for every compact set $K \subset\subset \Omega$ there exist two positive constants C_K, R_K such that:

(36) $$|a(x,\xi)| \geq C_K \lambda(\xi)^m, \quad \text{when } x \in K, \ |\xi| \geq R_K.$$

Proposition 6. *Consider $a(x,\xi) \in M^m_\lambda(\Omega)$ λ-elliptic symbol. Then there exists a properly supported operator $b(x,D) \in \tilde{\mathcal{M}}^{-m}_\lambda(\Omega)$ such that $b(x,D)a(x,D) = identity + r(x,D)$, where $r(x,\xi) \in S^{-\infty}(\Omega)$.*

We conclude the Section with the following

Proposition 7. *Let $a(x,\xi) \in M^m_\lambda(\Omega)$ be a λ-elliptic symbol. For $1 < p < \infty$ and $s \in \mathbb{R}$ assume that $a(x,D)u \in H^{s,p}_{\lambda,\mathrm{loc}}(\Omega)$ and $u \in \mathcal{E}'(\Omega)$, then $u \in H^{s+m,p}_{\lambda,\mathrm{comp}}(\Omega)$. If moreover $a(x,D)$ is assumed properly supported and $u \in \mathcal{D}'(\Omega)$, we obtain that $u \in H^{s+m,p}_{\lambda,\mathrm{loc}}(\Omega)$.*

3. Microlocal Properties in M^m_λ.

Definition 4. *We say that $a(x,\xi) \in M^m_\lambda(\Omega)$ is λ-microelliptic in $X \subset \mathbb{R}^n_\xi$ at the point $x_0 \in \Omega$ if for some positive constants M, R:*

(37) $$|a(x_0,\xi)| \geq M\lambda^m(\xi), \quad \textit{when} \quad \xi \in X, \quad |\xi| > R.$$

Define now the λ-neighborhood of $X \subset \mathbb{R}^n$ with length $\varepsilon > 0$ as:

$$X_{\varepsilon\lambda} := \bigcup_{\xi^0 \in X} \left\{ |\xi_j - \xi_j^0| < \varepsilon\lambda_j(\xi^0), \quad \text{for}\, j = 1, \ldots, n \right\}. \tag{38}$$

Moreover for $x_0 \in \Omega$ we introduce:

$$X(x_0) := \{x_0\} \times X, \quad X_{\varepsilon\lambda}(x_0) := B_\varepsilon(x_0) \times X_{\varepsilon\lambda}, \tag{39}$$

where $B_\varepsilon(x_0)$ is the open ball centered at x_0 with radius equal to ε.

Remark. Consider $0 < \varepsilon^* < \dfrac{k}{2n}$ and $\xi \in (X_{\varepsilon^*\lambda})_{\varepsilon^*\lambda}$. Then for some $\xi^1 \in X_{\varepsilon^*\lambda}$ and $\xi^0 \in X$ we have $|\xi_j - \xi_j^0| \leq |\xi_j - \xi_j^1| + |\xi_j^1 - \xi_j^0| < \varepsilon^*\lambda_j(\xi^1) + \varepsilon^*\lambda_j(\xi^0) \leq 2K\varepsilon^*\lambda_j(\xi^0)$, for any $j = 1, \ldots n$. Then for any fixed $X \subset \mathbb{R}^n$ and $\varepsilon > 0$:

$$(X_{\varepsilon^*\lambda})_{\varepsilon^*\lambda} \subset X_{\varepsilon\lambda}, \quad \text{for} \quad \varepsilon^* < \min\left\{ \frac{k}{2n}, \frac{\varepsilon}{2K} \right\}. \tag{40}$$

Fix now $\varepsilon > 0$, let ε^* satisfy (40) and consider, for $\varepsilon^\circ < \dfrac{k}{2n}$, $\xi \in (\mathbb{R}^n \setminus X_{\varepsilon\lambda})_{\varepsilon^\circ\lambda} \subset (\mathbb{R}^n \setminus (X_{\varepsilon^*\lambda})_{\varepsilon^*\lambda})_{\varepsilon^\circ\lambda}$. So for some $\xi^1 \in (\mathbb{R}^n \setminus (X_{\varepsilon^*\lambda})_{\varepsilon^*\lambda})$ and any $j = 1, \ldots, n$: $|\xi_j - \xi_j^1| < \varepsilon^\circ\lambda_j(\xi^1) \leq K\varepsilon^\circ\lambda_j(\xi)$. Assuming now $K\varepsilon^\circ \leq \varepsilon^*$ we can verify that ξ cannot belong to $X_{\varepsilon^*\lambda}$, since for any $\zeta \in X_{\varepsilon^*\lambda}$ and some $j = 1, \ldots, n$ $|\xi_j^1 - \zeta_j| \geq \varepsilon^*\lambda_j(\zeta)$. Since $\mathbb{R}^n \setminus X_{\varepsilon^*\lambda} \subset \mathbb{R}^n \setminus X_{\varepsilon^\circ\lambda}$ we conclude that for any fixed $\varepsilon > 0$:

$$(\mathbb{R}^n \setminus X_{\varepsilon\lambda})_{\varepsilon^\circ\lambda} \subset \mathbb{R}^n \setminus X_{\varepsilon^\circ\lambda}, \quad \text{for} \quad \varepsilon^\circ < \min\left\{ \frac{k}{2Kn}, \frac{\varepsilon}{2K^2} \right\}. \tag{41}$$

Proposition 8. *Any symbol $a(x,\xi) \in \mathcal{M}_\lambda^m(\Omega)$, λ-microelliptic in $X \subset \mathbb{R}_\xi^n$ at the point $x_0 \in \Omega$, satisfies the same property in $X_{\varepsilon\lambda}(x_0)$, for suitable $\varepsilon > 0$, that is for some $M, R > 0$:*

$$|a(x,\xi)| \geq M\lambda(\xi)^m, \quad \textit{for}\, (x,\xi) \in X_{\varepsilon\lambda}(x_0), \quad |\xi| > R. \tag{42}$$

Proof. Consider $a(x,\xi) \in M_\lambda^m(\Omega)$ λ-microelliptic in $X \subset \mathbb{R}^n$ at the point $x_0 \in \Omega$ and fix an arbitrary point $\xi^0 \in X$. For $(x,\xi) \in X_{\varepsilon\lambda}(x_0)$ set $(x_t, \xi^t) := ((1-t)x_0 + tx, (1-t)\xi^0 + t\xi)$, $|t| \leq 1$. Since $|\xi_j^t - \xi_j^0| = |t||\xi_j - \xi_j^0|$, $j = 1, \ldots, n$, it follows from of (4), (5) and $\varepsilon < \dfrac{c}{2n}$: $\lambda(\xi) \approx \lambda(\xi^t) \approx \lambda(\xi^0)$, and the same for $\lambda_j(\xi)$, $j = 1, \ldots, n$.

By means of Taylor expansion we have, for any $(x,\xi) \in X_{\varepsilon\lambda}(x_0)$:

$$
\begin{aligned}
&|a(x,\xi) - a(x_0,\xi^0)| \le \textstyle\sum_{j=1}^n |x^j - x_0^j||\partial_{x^j} a(x_t,\xi^t)| + |\xi_j - \xi_j^0||\partial_{\xi_j} a(x_t,\xi^t)| \le \\
&\le nc_1\varepsilon\lambda(\xi^t)^m + \varepsilon \textstyle\sum_{j=1}^n \lambda_j(\xi^0)|\partial_{\xi_j} a(x_t,\xi^t)| \le \\
&\le nc_1\varepsilon\lambda(\xi^t)^m + K\varepsilon \textstyle\sum_{j=1}^n \lambda_j(\xi^t)|\partial_{\xi_j} a(x_t,\xi^t)| \le \\
&\le nc_1\varepsilon\lambda(\xi^t)^m + Knc_1\varepsilon\lambda(\xi^t)^{\frac{1}{\mu}}\lambda(\xi^t)^{m-\frac{1}{\mu}} + K\varepsilon \textstyle\sum_{j=1}^n |\xi_j^t \partial_{\xi_j} a(x_t,\xi^t)| \le \\
&\le 3Knc_1\varepsilon\lambda(\xi^t)^m \le 3KC^{|m|}nc_1\varepsilon\lambda(\xi^0)^m,
\end{aligned}
$$

where C, K, c_1 are respectively the positive constants in (4), (8), (9). Since, for suitable $M, R > 0$, $|a(x_0,\xi^0)| \ge M\lambda(\xi^0)^m$ when $|\xi^0| > R$, considering $0 < \varepsilon < \dfrac{c}{2n}$ such that $r = 3KC^{|m|}nc_1\varepsilon < M$ and suitable $\tilde{R} > 0$, we obtain for any $(x,\xi) \in X_{\varepsilon\lambda}(x_0)$, $|\xi| > \tilde{R}$: $|a(x,\xi)| \ge (M-r)\lambda(\xi^0)^m \ge \dfrac{M-r}{C^{|m|}}\lambda(\xi)^m$, which ends the proof. $\square$

We can notice that the class $S^0_\Lambda(\Omega)$ defined in (20) is exactly the symbol class $S^0_\Psi(\Omega)$ considered in [10], by setting $\Psi(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$. Then the following Lemma can be obtained from [10, Lemma 1.10] and (21). For sake of completeness we provide however a detailed outline of the proof.

Lemma 1. *Fix $\varepsilon > 0$ and $X \subset \mathbb{R}^n$. Then there exists $\sigma(\xi) \in M^0_\lambda(\mathbb{R}^n)$ such that $\operatorname{supp}\sigma \subset X_{\varepsilon\lambda}$ and $\sigma(\xi) = 1$ when $\xi \in X_{\varepsilon'\lambda}$, for suitable $0 < \varepsilon' < \varepsilon$.*

Proof. Fix $\varepsilon > 0$ and $X \subset \mathbb{R}^n$, then by means of (40), (41) we can find $0 < \varepsilon' < \varepsilon/2$ such that

$$
(\mathbb{R}^n \setminus X_{\varepsilon/2\lambda})_{\varepsilon'\lambda} \cap (X_{\varepsilon'\lambda})_{\varepsilon'\lambda} = \emptyset\,. \tag{43}
$$

Let u be the characteristic function of the set $(X_{\varepsilon'\lambda})_{\varepsilon'\lambda}$ and take $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\displaystyle\int \varphi(t)dt = 1$ and $\operatorname{supp}\varphi \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Then we set

$$
\sigma(\xi) := \frac{1}{\varepsilon'^{\,n}\prod_{j=1}^n c_j^{-1}\tilde{\lambda}_j(\xi)} \int u(\eta)\varphi\left(\frac{c_1(\xi_1-\eta_1)}{\varepsilon'\tilde{\lambda}_1(\xi)}, \dots, \frac{c_n(\xi_n-\eta_n)}{\varepsilon'\tilde{\lambda}_n(\xi)}\right) d\eta\,. \tag{44}
$$

For each integer $1 \le j \le n$ the function $\tilde{\lambda}_j(\xi)$ is defined by (22), where the weight-function $\lambda(\xi)$ is replaced by $\lambda_j(\xi)$ introduced in (6) and c_j is some positive constant such that

$$
\frac{1}{c_j}\lambda_j(\xi) \le \tilde{\lambda}_j(\xi) \le c_j\lambda_j(\xi)\,, \quad \forall\, \xi \in \mathbb{R}^n\,. \tag{45}
$$

Arguing as in the proof of Proposition 3, it tends out that for any $\alpha \in \mathbb{Z}^n_+$ and $1 \le j \le n$ there exists a constant $C_{\alpha,j} > 0$ such that

$$|\partial^\alpha_\xi \tilde\lambda_j(\xi)| \le C_\alpha \tilde\lambda_j(\xi)\tilde\Lambda(\xi)^{-\alpha}\,, \quad \forall\, \xi \in \mathbb{R}^n, \tag{46}$$

where $\tilde\Lambda(\xi) = (\tilde\lambda_1(\xi), \dots, \tilde\lambda_n(\xi))$.
Let us show, firstly, that σ belongs to the symbol class $M^0_\lambda(\mathbb{R}^n)$. Actually, we will prove a little more, namely that $\sigma \in S^0_\Lambda(\mathbb{R}^n) \subset M^0_\lambda(\mathbb{R}^n)$, see (21). For a given multi-index $\alpha \in \mathbb{Z}^n_+$, Leibniz's formula and differentiation under the integral sign give

$$\partial^\alpha_\xi \sigma(\xi) = \varepsilon'^{\,-n} \sum C \partial^{\beta^1}_\xi \tilde\lambda_1(\xi)^{-1} \dots \partial^{\beta^n}_\xi \tilde\lambda_n(\xi)^{-1} \int u(\eta) \partial^\nu_\xi \left(\varphi\left(\zeta(\xi,\eta)\right)\right) d\eta\,, \tag{47}$$

where the vector valued function $\zeta(\xi,\eta) := \left(\frac{c_1(\xi_1-\eta_1)}{\varepsilon'\tilde\lambda_1(\xi)}, \dots, \frac{c_n(\xi_n-\eta_n)}{\varepsilon'\tilde\lambda_n(\xi)}\right)$ is introduced. Moreover the sum in the right-hand side of (47) is performed over all sets $\{\beta^1, \dots, \beta^n, \nu\}$ of $n+1$ multi-indices such that $\beta^1 + \dots + \beta^n + \nu = \alpha$, and the positive constants C only depends on the multi-indices $\alpha, \nu, \beta^1, \dots, \beta^n$.
Faà di Bruno's formula and (46) easily give

$$|\partial^{\beta^j}_\xi \tilde\lambda_j(\xi)^{-1}| \le C_{\beta^j} \tilde\lambda_j(\xi)^{-1} \tilde\Lambda(\xi)^{-\beta^j}\,, \quad \forall\, \xi \in \mathbb{R}^n\,, \tag{48}$$

for each index $1 \le j \le n$ a suitable constants C_{β^j}.
On the other hand, again by Faà di Bruno's formula we compute for $\nu \ne 0$

$$\begin{aligned} \partial^\nu_\xi \left(\varphi\left(\zeta(\xi,\eta)\right)\right) &= \sum_{0<|\delta|\le|\nu|} \left(\partial^\delta \varphi\right)\left(\zeta(\xi,\eta)\right) \times \\ \sum_{\nu^1+\dots+\nu^{|\delta|}=\nu} & C_{\nu,\delta,\nu^1,\dots,\nu^{|\delta|}} \partial^{\nu^1}_\xi \left(\frac{c_1(\xi_1-\eta_1)}{\varepsilon'\tilde\lambda_1(\xi)}\right) \dots \partial^{\nu^{\delta_1}}_\xi \left(\frac{c_1(\xi_1-\eta_1)}{\varepsilon'\tilde\lambda_1(\xi)}\right) \dots \\ & \dots \partial^{\nu^{\delta_1+\dots+\delta_{n-1}+1}}_\xi \left(\frac{c_n(\xi_n-\eta_n)}{\varepsilon'\tilde\lambda_n(\xi)}\right) \dots \partial^{\nu^{|\delta|}}_\xi \left(\frac{c_n(\xi_n-\eta_n)}{\varepsilon'\tilde\lambda_n(\xi)}\right)\,, \end{aligned} \tag{49}$$

where, for each $\delta \in \mathbb{Z}^n_+$ such that $0 < |\delta| \le |\nu|$, the second sum in the right-hand side of (49) is performed over all sets $\{\nu^1, \dots, \nu^{|\delta|}\}$ of multi-indices $\nu^J \ne 0$, for $1 \le J \le |\delta|$, such that $\nu^1 + \dots + \nu^{|\delta|} = \nu$, with $C_{\nu,\delta,\nu^1,\dots,\nu^{|\delta|}} > 0$.
Once again, Leibniz's rule and (48) yield that for each $1 \le j \le n$ and $\nu \in \mathbb{Z}^n_+$ there exists a constant $C_{\nu,j} > 0$ such that

$$\left|\partial^\nu_\xi \left(\frac{\xi_j - \eta_j}{\tilde\lambda_j(\xi)}\right)\right| \le C_{\nu,j} \left(\frac{|\xi_j - \eta_j|}{\tilde\lambda_j(\xi)} + 1\right) \tilde\Lambda(\xi)^{-\nu}\,, \quad \forall\, \xi \in \mathbb{R}^n\,. \tag{50}$$

From (49), (50) we then get

$$
\begin{aligned}
&\left|\int u(\eta)\partial_\xi^\nu\left(\varphi\left(\zeta(\xi,\eta)\right)\right)\,d\eta\right| \\
&\le C_\nu\tilde\Lambda(\xi)^{-\nu}\sum_{0<|\delta|\le|\nu|}\int\left|\partial^\delta\varphi\left(\zeta(\xi,\eta)\right)\right|\prod_{j=1}^n\left(\frac{c_j|\xi_j-\eta_j|}{\varepsilon'\tilde\lambda_j(\xi)}+\frac{1}{\varepsilon'}\right)^{\delta_j}d\eta \\
&\le C_\nu{\varepsilon'}^n\prod_{j=1}^n c_j^{-1}\tilde\lambda_j(\xi)\tilde\Lambda^{-\nu}(\xi)\sum_{0<|\delta|\le|\nu|}\left(1+\frac{1}{\varepsilon'}\right)^{|\delta|}\int|\partial^\delta\varphi(\zeta)|d\zeta \\
&\le C_{\nu,\varepsilon}{\varepsilon'}^n\prod_{j=1}^n c_j^{-1}\tilde\lambda_j(\xi)\tilde\Lambda^{-\nu}(\xi)\,,\quad\forall\,\xi\in\mathbb{R}^n\,,
\end{aligned}\tag{51}
$$

where the change of variables $\zeta=\zeta(\xi,\eta)$ has been performed under the integral in the second line above. Finally, from (47), (48), (51) it follows that

$$
|\partial^\alpha\sigma(\xi)|\le C_{\alpha,\varepsilon'}\tilde\Lambda^{-\alpha}(\xi)\,,\quad\forall\,\xi\in\mathbb{R}^n\,,\tag{52}
$$

with $C_{\alpha,\varepsilon'}>0$, which in view of (45) proves that $\sigma\in S^0_\Lambda(\mathbb{R}^n)$.
Now for proving that σ vanishes identically on $\mathbb{R}^n\setminus X_{\varepsilon\lambda}$, consider $\xi\in\mathbb{R}^n$ and set for simplicity $g_\xi(\eta):=\varphi\left(\frac{c_1(\xi_1-\eta_1)}{\varepsilon'\tilde\lambda_1(\xi)},\dots,\frac{c_n(\xi_n-\eta_n)}{\varepsilon'\tilde\lambda_n(\xi)}\right)$. In view of (45) we have that

$$
\begin{aligned}
\operatorname{supp}g_\xi&\subseteq\left\{\eta\in\mathbb{R}^n:\,|\eta_j-\xi_j|<\frac{\varepsilon'}{2c_j}\tilde\lambda_j(\xi)\,,\ 1\le j\le n\right\}\\
&\subseteq\left\{\eta\in\mathbb{R}^n:\,|\eta_j-\xi_j|<\varepsilon'\lambda_j(\xi)\,,\ 1\le j\le n\right\}\,.
\end{aligned}\tag{53}
$$

For $\xi\in\mathbb{R}^n\setminus X_{\varepsilon\lambda}$ it follows that $\operatorname{supp}g_\xi\subseteq(\mathbb{R}^n\setminus X_{\varepsilon\lambda})_{\varepsilon'\lambda}$ (cf. (38)), hence we obtain from (43) that $\sigma(\xi)=0$ for any $\xi\in\mathbb{R}^n\setminus X_{\varepsilon\lambda}$.
It follows that σ is identically one on $X_{\varepsilon'\lambda}$ observing that from (53) we get $\operatorname{supp}g_\xi\subseteq(X_{\varepsilon'\lambda})_{\varepsilon'\lambda}$, as long as $\xi\in X_{\varepsilon'\lambda}$. Thus for $\xi\in X_{\varepsilon'\lambda}$, passing once again to the new integration variables $\zeta_j=\dfrac{c_j(\xi_j-\eta_j)}{\varepsilon'\tilde\lambda_j(\xi)}$ in (44) and using that $u=1$ on $(X_{\varepsilon'\lambda})_{\varepsilon'\lambda}$ we obtain at once that $\sigma(\xi)=\int\varphi(\zeta)\,d\zeta=1$. $\square$

Consider $\chi_0\in C_0^\infty(\Omega)$ such that supp $\chi_0\subset B_\varepsilon(x_0)$ and $\chi_0(x)=1$ when $x\in B_{\varepsilon'}(x_0)$. Then by means of the previous Lemma, for any $x_0\in\Omega$, $X\subset\mathbb{R}^n$ and $\varepsilon>0$ we can construct $\tau_0(x,\xi)=\chi_0(x)\sigma(\xi)$ such that:

$$
\tau_0(x,\xi)\in M^0_\lambda(\Omega),\quad\operatorname{supp}\tau_0\subset X_{\varepsilon\lambda}(x_0),\quad\tau_0(x,\xi)=1\text{ in }X_{\varepsilon'\lambda}(x_0).\tag{54}
$$

Definition 5. *A symbol $a(x,\xi) \in S^m_\lambda(\Omega)$, $m \in \mathbb{R}$ is said to be rapidly decreasing in $\Theta \subset \Omega \times \mathbb{R}^n$ if there exists $a_0(x,\xi) \in S^m_\lambda(\Omega)$ such that $a(x,\xi) \sim a_0(x,\xi)$ and $a_0(x,\xi) = 0$ in Θ*

Theorem 1. *For $m \in \mathbb{R}$, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, consider $a(x,D) \in \tilde{\mathcal{M}}^m_\lambda(\Omega)$ whose symbol is λ-microelliptic in $X(x_0)$. Then there exists $b(x,D) \in \tilde{\mathcal{M}}^{-m}_\lambda(\Omega)$ such that*

$$b(x,D)a(x,D) = \textit{identity} + c(x,D), \tag{55}$$

where $c(x,\xi) \in M^0_\lambda(\Omega)$ is rapidly decreasing in $X_{r\lambda}(x_0)$ for some some $r > 0$.

Proof. It is not restrictive to assume that $\lambda(D)^{-m} \in \mathcal{M}^{-m}(\Omega)$ is properly supported. Then multiplying $a(x,D)$ by $\lambda(D)^{-m}$ we are reduced to the case $m = 0$.
Assuming then $a(x,\xi) \in M^0_\lambda(\Omega)$ λ-microelliptic in $X(x_0)$ and using Proposition 8 we can find $\varepsilon > 0$ such that $a(x,\xi)$ is still λ-microelliptic in $X_{\varepsilon\lambda}(x_0)$. Take now $\tau_0 \in M^0_\lambda(\Omega)$ as in (54) and set

$$b_0(x,\xi) := \begin{cases} \dfrac{\tau_0(x,\xi)}{a(x,\xi)} & \text{for} \quad (x,\xi) \in X_{\varepsilon\lambda}(x_0), \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

Since $a(x,\xi)$ satisfies (42), $b_0(x,\xi)$ is well defined for large $|\xi|$ and moreover it belongs to $M^0_\lambda(\Omega)$, see [6, Lemma 6.3] .
Then arguing by recurrence, consider for $j = 1, 2, \dots$

$$b_{-j}(x,\xi) := \begin{cases} -\sum\limits_{0<|\alpha|\le j} \frac{1}{\alpha!}\partial^\alpha_\xi b_{-j+|\alpha|}(x,\xi)\dfrac{D^\alpha_x a(x,\xi)}{a(x,\xi)}, & \text{for} \quad (x,\xi) \in X_{\varepsilon\lambda}(x_0), \\ 0 & \text{otherwise.} \end{cases}$$

For large $|\xi|$, $b_{-j}(x,\xi)$ is well defined, it belongs to $M^{-j}_\lambda(\Omega)$ and it is supported in $X_{\varepsilon\lambda}(x_0)$. Using then standard arguments we can construct $b(x,\xi) \sim \sum\limits_{j\ge 0} b_{-j}(x,\xi)$ in such a way that $b(x,D) \in \tilde{\mathcal{M}}^0_\lambda(\Omega)$. Notice now that, thanks to (29), the symbol $c_1(x,\xi)$ of the product $b(x,D)a(x,D)$ realizes to be equivalent to $\tau_0(x,\xi)$, then the symbol $c(x,\xi)$ of the operator $b(x,D)a(x,D) - \textit{identity}$ belongs to $M^0_\lambda(\Omega)$ and it is rapidly decreacreasing in $X_{r\lambda}(x_0)$ for any $0 < r \le \varepsilon'$, assuming ε' as in (54). The proof is then concluded $\square$

Proposition 9. *For $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, $s \in \mathbb{R}$, $1 < p < \infty$, the following properties are equivalent:*

i) there exists $a(x, D) \in \tilde{\mathcal{M}}^0_\lambda(\Omega)$ with symbol $a(x, \xi)$ λ-microelliptic in $X(x_0)$, such that $a(x, D)u \in H^{s,p}_{\lambda,\mathrm{loc}}(\Omega)$;

ii) $\sigma(D)(\phi u) \in H^{s,p}_\lambda$ for some $\phi \in C^\infty_0(\Omega)$, $\phi(x_0) = 1$ and $\sigma \in M^0_\lambda(\mathbb{R}^n)$, such that $\operatorname{supp} \sigma \subset X_{\varepsilon\lambda}$, $\sigma(\xi) = 1$ when $\xi \in X_{\varepsilon'\lambda}$, for suitable $\varepsilon > \varepsilon' > 0$.

Proof. Assume that $a(x, D)$ and u satisfy the assumptions in i). Then there exists $b(x, D) \in \tilde{\mathcal{M}}^0_\lambda(\Omega)$ such that $b(x, D)a(x, D) = \mathit{identity} + c(x, D)$, where the symbol $c(x, \xi) \in M^0_\lambda(\Omega)$ is rapidly decreasing in $X_{\varepsilon\lambda}(x_0)$, for some $0 < \varepsilon < 1$. Then:

$$u = b(x, D)a(x, D)u - c(x, D)u. \tag{57}$$

Using Lemma 1 we can consider, for suitable $0 < \varepsilon' < \varepsilon$, $\sigma \in M^0_\lambda(\mathbb{R}^n)$ such that $\operatorname{supp} \sigma \subset X_{\varepsilon\lambda}$, $\sigma(\xi) = 1$ in $X_{\varepsilon'\lambda}$ and $\phi \in C^\infty_0(\Omega)$, with $\operatorname{supp} \phi \subset B_\varepsilon(x_0)$, $\phi(x_0) = 1$. Then

$$\sigma(D)(\phi u) = \sigma(D)\phi(x)b(x, D)a(x, D)u - \sigma(D)\phi(x)c(x, D)u. \tag{58}$$

Since $a(x, D)u \in H^{s,p}_{\lambda,\mathrm{loc}}(\Omega)$, from (34) we obtain $\sigma(D)\phi(x)b(x, D)a(x, D)u \in H^{s,p}_\lambda$. Assuming now $c(x, D)$ properly supported we can find $\tilde{\phi}(x) \in C^\infty_0(\Omega)$ such that $\phi(x)c(x, D)u = \phi(x)c(x, D)(\tilde{\phi}u)$. Moreover for some $c_0(x, \xi) \in M^0_\lambda(\Omega)$ supported in $(\Omega \times \mathbb{R}^n) \setminus X_{\varepsilon\lambda}(x_0)$ we have $c(x, \xi) = c_0(x, \xi) + \rho(x, \xi)$, with $\rho(x, \xi) \in S^{-\infty}(\Omega)$. Consider now the operator $d(x, D) = \sigma(D)\phi(x)c_0(x, D)$, whose symbol obtained by (29) is $d(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi \sigma(\xi) D^\alpha_x(\phi(x)c_0(x, \xi))$. Observe that: $\partial^\alpha_\xi \sigma(\xi) = 0$ when $\xi \notin X_{\varepsilon\lambda}$; $c_0(x, \xi) = 0$ when $\xi \in X_{\varepsilon\lambda}$ and $x \in B_\varepsilon(x_0)$; $\phi(x) = 0$ when $\xi \in X_{\varepsilon\lambda}$ and $x \notin B_\varepsilon(x_0)$. Then $\partial^\alpha_\xi \sigma(\xi) D^\alpha_x(\phi(x)c_0(x, \xi)) = 0$, thus $d(x, \xi) \in S^{-\infty}(\Omega)$. Observing that $\sigma(D)(\phi(x)c(x, D)u) = d(x, D)(\tilde{\phi}u) + \sigma(D)(\phi(x)\rho(x, D)(\tilde{\phi}u))$, we conclude that ii) follows from i).
Consider now ii) satisfied by $\sigma \in M^0_\lambda(\mathbb{R}^n)$, $\phi \in C^\infty_0(\Omega)$ and take $a(x, D) \in \tilde{\mathcal{M}}^0_\lambda(\Omega)$ equivalent to the operator $\sigma(D)\phi(x)$. Since $a(x, \xi) \sim \sum_\alpha \frac{1}{\alpha}\partial^\alpha_\xi \sigma(\xi) D^\alpha_x \phi(x)$ realizes to be λ-microelliptic in $X(x_0)$, i) easily follows. □

Definition 6. *For $X \subset \mathbb{R}^n$ we say that that $u \in \mathcal{D}'(\Omega)$ is λ microlocally regular in X at the point $x_0 \in \Omega$, we write $u \in H^{s,p}_\lambda X(x_0)$, if one of the equivalent properties in Proposition 9 is satisfied.*

4. Microlocal Sobolev Continuity and Regularity.

Theorem 2. *Let $x_0 \in \Omega$ and $X \subset \mathbb{R}^n$ be given. Then for every $m, s \in \mathbb{R}$, $p \in]1, +\infty[$, $a(x, D) \in \tilde{\mathcal{M}}^m_\lambda(\Omega)$ and $u \in H^{s+m,p}_\lambda X(x_0)$ one has $a(x, D)u \in H^{s,p}_\lambda X(x_0)$.*

Proof. From Proposition 9, we know there exists an operator $b(x, D) \in \tilde{\mathcal{M}}^0_\lambda(\Omega)$ with symbol $\lambda-$microelliptic in X at $x_0 \in \Omega$, such that $b(x, D)u \in H^{s+m,p}_{\lambda,loc}(\Omega)$. From Theorem 1 there also exists an operator $c(x, D) \in \tilde{\mathcal{M}}^0_\lambda(\Omega)$ such that

$$c(x, D)b(x, D) = \text{identity} + \rho(x, D), \tag{59}$$

where $\rho(x, \xi) \in M^0_\lambda(\Omega)$ is rapidly decreasing in $X_{r\lambda}(x_0)$ for some $0 < r < 1$. Using (41), $0 < r^* < r$ may be chosen in such a way that

$$(\mathbb{R}^n \setminus X_{r\lambda})_{r^*\lambda} \subset \mathbb{R}^n \setminus X_{r^*\lambda}\,. \tag{60}$$

By means of Lemma 1 we can consider a symbol $\sigma = \sigma(\xi) \in M^0_\lambda(\mathbb{R}^n)$, satisfying $\operatorname{supp} \sigma \subset X_{r^*\lambda}$, $\sigma = 1$ on $X_{r'\lambda}$, for a suitable $0 < r' < r^*$, and define as in (54) a symbol $\tau_0(x, \xi) = \chi_0(x)\sigma(\xi)$, with $\chi_0(\xi) \in C^\infty_0(\Omega)$ supported in $B_{r^*}(x_0)$ and identically equal to one on $B_{r'}(x_0)$. Of course, one has that

$$\operatorname{supp} \tau_0 \subset X_{r^*\lambda}(x_0) \qquad \text{and} \qquad \tau_0 = 1 \ \text{ on } X_{r'\lambda}(x_0)\,. \tag{61}$$

Let $\tau(x, D)$ be a properly supported operator in $\tilde{\mathcal{M}}^0_\lambda(\Omega)$ such that $\tau(x, \xi) \sim \tau_0(x, \xi)$ and set $\theta_0(x, \xi) := \tau(x, \xi) - \tau_0(x, \xi) \in S^{-\infty}(\Omega)$. It turns out that the symbol $\tau(x, \xi)$ is $\lambda-$microelliptic in $X(x_0)$; indeed for $(x, \xi) \in X_{r'\lambda}(x_0)$ one has:

$$|\tau(x, \xi)| \geq |\tau_0(x, \xi)| - |\theta_0(x, \xi)| = 1 - c_0\lambda^{-1}(\xi) \geq 1/2\,, \qquad \text{if } |\xi| > R_0\,, \tag{62}$$

for a suitable constant $c_0 > 0$ and $R_0 > 1$ sufficiently large. Moreover

$$\tau(x, \xi) = \theta_0(x, \xi) \in S^{-\infty}(\Omega)\,, \qquad \text{for } (x, \xi) \notin X_{r^*\lambda}(x_0)\,. \tag{63}$$

For proving $\tau(x, D)a(x, D)u \in H^{s,p}_{\lambda,loc}(\Omega)$, consider that in view of (59) we have

$$\tau(x, D)a(x, D)u = \tau(x, D)a(x, D)c(x, D)\,(b(x, D)u) - \tau(x, D)a(x, D)\rho(x, D)u\,.$$

Since $\tau(x, D)a(x, D)c(x, D) \in \tilde{\mathcal{M}}^m_\lambda(\Omega)$ and $b(x, D)u \in H^{s+m,p}_{\lambda,loc}(\Omega)$, from Proposition 5, $\tau(x, D)a(x, D)c(x, D)\,(b(x, D)u) \in H^{s,p}_{\lambda,loc}(\Omega)$ follows at once.
As to $\tau(x, D)a(x, D)\rho(x, D)$, we will see that $\tau(x, D)a(x, D)\rho(x, D)u \in H^{t,p}_{\lambda,loc}(\Omega)$

for every $t \in \mathbb{R}$; this amounts to prove that $\varphi(x)\tau(x,D)a(x,D)\rho(x,D)u$ belongs to $H^{t,p}_\lambda$, for every $\varphi \in C_0^\infty(\Omega)$. Since $\tau(x,D)a(x,D)\rho(x,D)$ is properly supported, there exists another function $\tilde\varphi \in C_0^\infty(\Omega)$ such that

$$\varphi(x)\tau(x,D)a(x,D)\rho(x,D)u = \varphi(x)\tau(x,D)a(x,D)\rho(x,D)\left(\tilde\varphi u\right) .$$

Let $\rho_0(x,\xi)$ be a symbol in $M^0_\lambda(\Omega)$ such that

$$\rho_0 = 0 \ \text{ in } X_{r\lambda}(x_0) \qquad \text{and} \qquad \tilde\sigma(x,\xi) := \rho(x,\xi) - \rho_0(x,\xi) \in S^{-\infty}(\Omega)\,. \tag{64}$$

Hence

$$\begin{aligned}&\varphi(x)\tau(x,D)a(x,D)\rho(x,D)(\tilde\varphi u)\\ &= \varphi(x)\tau(x,D)a(x,D)\rho_0(x,D)(\tilde\varphi u) + \varphi(x)\tau(x,D)a(x,D)\tilde\sigma(x,D)(\tilde\varphi u)\,.\end{aligned}$$

Since $\tilde\sigma(x,\xi) \in S^{-\infty}(\Omega)$ and $\tilde\varphi u \in \mathcal{E}'(\Omega)$, $\tau(x,D)a(x,D)\tilde\sigma(x,D)(\tilde\varphi u) \in C^\infty(\Omega)$ hence $\varphi(x)\tau(x,D)a(x,D)\tilde\sigma(x,D)(\tilde\varphi u) \in C_0^\infty(\Omega) \subset H^{t,p}_\lambda$, for any $t \in \mathbb{R}$. Concerning the operator $\eta(x,D) := \tau(x,D)a(x,D)\rho_0(x,D)$, in view of the symbolic calculus in Proposition 4, for an arbitrary positive integer N the following identities hold for any $(x,\xi) \in \Omega \times \mathbb{R}^n$

$$\begin{aligned}\eta(x,\xi) &= \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha(\tau\sharp a)(x,\xi)D_x^\alpha\rho_0(x,\xi) + R_N(x,\xi)\,,\\ \tau\sharp a(x,\xi) &= \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha\tau(x,\xi)D_x^\alpha a(x,\xi) + S_N(x,\xi),\end{aligned} \tag{65}$$

where $R_N(x,\xi), S_N(x,\xi) \in M^{m-N/\mu}_\lambda(\Omega)$ and $\tau\sharp a(x,\xi)$ denotes the symbol of $\tau(x,D)a(x,D)$.
Since the set $X_{r\lambda}(x_0)$ is open, from (65) and (64) it follows that

$$\eta(x,\xi) = R_N(x,\xi) \in M^{m-N/\mu}_\lambda(\Omega)\,, \qquad \forall\,(x,\xi) \in X_{r\lambda}(x_0)\,. \tag{66}$$

On the other hand, as a consequence of (60) it can be proved that

$$(\Omega\times\mathbb{R}^n)\setminus X_{r\lambda}(x_0) \subset int\left((\Omega\times\mathbb{R}^n)\setminus X_{r^*\lambda}(x_0)\right)\,, \tag{67}$$

where $intS$ denotes the interior of a set $S \subseteq \Omega\times\mathbb{R}^n$. Thus from (63) and (65) we have

$$\begin{aligned}\tau\sharp a(x,\xi) &= \sum_{|\alpha|<N} \tfrac{1}{\alpha!}\partial_\xi^\alpha\theta_0(x,\xi)D_x^\alpha a(x,\xi) + S_N(x,\xi)\\ &=: T_N(x,\xi) \in M^{m-N/\mu}_\lambda(\Omega),\end{aligned} \tag{68}$$

hence for any $(x,\xi)\in(\Omega\times\mathbb{R}^n)\setminus X_{r\lambda}(x_0)$:

$$\eta(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\xi^\alpha T_N(x,\xi)D_x^\alpha\rho_0(x,\xi)+R_N(x,\xi)\in M_\lambda^{m-N/\mu}(\Omega). \tag{69}$$

From (66), (69) we conclude that $\eta(x,\xi)\in M_\lambda^{m-N/\mu}(\Omega)$ for all integers $N\geq 1$, hence $\eta(x,\xi)\in S^{-\infty}(\Omega)$ (see (27)). Thus, for any $t\in\mathbb{R}$, $\varphi(x)\eta(x,D)(\tilde{\varphi}u)\in C_0^\infty(\Omega)\subset H_\lambda^{t,p}$, and the proof is concluded. □

Theorem 3. *For given $x_0\in\Omega$, $X\subset\mathbb{R}^n$, $m\in\mathbb{R}$, assume that $a(x,D)\in\tilde{\mathcal{M}}_\lambda^m(\Omega)$ has $\lambda-$microelliptic symbol in $X(x_0)$. Then for every $s\in\mathbb{R}$, $p\in]1,+\infty[$ and $u\in\mathcal{D}'(\Omega)$ such that $a(x,D)u\in H_\lambda^{s,p}X(x_0)$ one has $u\in H_\lambda^{s+m,p}X(x_0)$.*

Proof. We follow the arguments used in the proof of Theorem 2.
Let $u\in\mathcal{D}'(\Omega)$ satisfy the assumptions of Theorem 3. Using Proposition 9, we can find an operator $b(x,D)\in\tilde{\mathcal{M}}_\lambda^0(\Omega)$, with $\lambda-$microelliptic symbol, such that

$$b(x,D)a(x,D)u\in H_{\lambda,loc}^{s,p}(\Omega). \tag{70}$$

By Theorem 1 there exist some operators $c(x,D)\in\tilde{\mathcal{M}}_\lambda^0(\Omega)$, $q(x,D)\in\tilde{\mathcal{M}}_\lambda^{-m}(\Omega)$ such that

$$\begin{aligned} c(x,D)b(x,D)&=\text{identity}+\rho(x,D),\\ q(x,D)a(x,D)&=\text{identity}+\sigma(x,D)\end{aligned} \tag{71}$$

hold true, for suitable symbols $\rho(x,\xi),\sigma(x,\xi)\in M_\lambda^0(\Omega)$ rapidly decreasing in $X_{r\lambda}(x_0)$ and some constant $0<r<1$.
Choose $r^*>0$ correspondingly to r as in the proof of Theorem 2 (see (60)). In the same way consider the symbols $\tau_0(x,\xi),\tau(x,\xi)\in M_\lambda^0(\Omega)$, satisfying (61), (62), (63).
For proving that $u\in H_\lambda^{s+m,p}(X(x_0))$, we will check that $\tau(x,D)u\in H_{\lambda,loc}^{s+m,p}(\Omega)$. By the use of (71) we can write

$$\begin{aligned}\tau(x,D)u&=\tau(x,D)q(x,D)\left(a(x,D)u\right)-\tau(x,D)\sigma(x,D)u\\ &=\tau(x,D)q(x,D)c(x,D)\left(b(x,D)a(x,D)u\right)\\ &\quad-\tau(x,D)q(x,D)\rho(x,D)a(x,D)u-\tau(x,D)\sigma(x,D)u\,.\end{aligned} \tag{72}$$

Since $\tau(x,D)q(x,D)c(x,D)\in\tilde{\mathcal{M}}_\lambda^{-m}(\Omega)$, then

$$\tau(x,D)q(x,D)c(x,D)\left(b(x,D)a(x,D)u\right)\in H_{\lambda,loc}^{s+m,p}(\Omega) \tag{73}$$

follows from (70) by Proposition 5.
Concerning the other terms $\tau(x,D)q(x,D)\rho(x,D)a(x,D)u$, $\tau(x,D)\sigma(x,D)u$, appearing in the right-hand side of (72), following the argumets used in the proof of Theorem 2 we can prove that for any $t\in\mathbb{R}$

$$\tau(x,D)q(x,D)\rho(x,D)a(x,D)u, \quad \tau(x,D)\sigma(x,D)u\in H^{t,p}_{\lambda,loc}(\Omega). \tag{74}$$

Arguing explicitly on $\tau(x,D)\sigma(x,D)u$, we need that $\varphi(x)\tau(x,D)\sigma(x,D)u\in H^{t,p}_\lambda$, for every $\varphi\in C^\infty_0(\Omega)$. Since the operator $\tau(x,D)\sigma(x,D)$ is properly supported, for every $\varphi\in C^\infty_0(\Omega)$, another function $\tilde\varphi\in C^\infty_0(\Omega)$ exists such that

$$\varphi(x)\tau(x,D)\sigma(x,D)u=\varphi(x)\tau(x,D)\sigma(x,D)(\tilde\varphi u)\,.$$

Since $\sigma(x,\xi)$ is rapidly decreasing in $X_{r\lambda}(x_0)$, there exists $\sigma_0(x,\xi)\in M^0_\lambda(\Omega)$ such that, for any $(x,\xi)\in X_{r\lambda}(x_0)$, $\sigma_0(x,\xi)=0$ and $\eta_0(x,\xi):=\sigma(x,\xi)-\sigma_0(x,\xi)\in S^{-\infty}(\Omega)$. Let us decompose $\varphi(x)\tau(x,D)\sigma(x,D)(\tilde\varphi u)$ as

$$\varphi(x)\tau(x,D)\sigma(x,D)(\tilde\varphi u)=\varphi(x)\tau(x,D)\sigma_0(x,D)(\tilde\varphi u)+\varphi(x)\tau(x,D)\eta_0(x,D)(\tilde\varphi u)\,.$$

Since $\eta_0(x,\xi)\in S^{-\infty}(\Omega)$ and $\tilde\varphi u\in\mathcal{E}'(\Omega)$, we can easily obtain that, for any $t\in\mathbb{R}$, $\varphi(x)\tau(x,D)\eta_0(x,D)(\tilde\varphi u)\in C^\infty_0(\Omega)\subset H^{t,p}_\lambda$.
For the operator $\tau(x,D)\sigma_0(x,D)$, one may argue on the asymptotic expansion of its symbol $\tau\sharp\sigma_0(x,\xi)$, as it was done in the proof of Theorem 2:

$$\tau\sharp\sigma_0(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial^\alpha_\xi\tau(x,\xi)D^\alpha_x\sigma_0(x,\xi)+R_N(x,\xi)\,, \tag{75}$$

with $R_N(x,\xi)\in M^{-N/\mu}_\lambda(\Omega)$, for every integer $N\ge1$. Since $\sigma_0(x,\xi)=0$ for any (x,ξ) in the open set $X_{r\lambda}(x_0)$, (75) gives $\tau\sharp\sigma_0(x,\xi)=R_N(x,\xi)\in M^{-N/\mu}_\lambda(\Omega)$, for any $(x,\xi)\in X_{r\lambda}(x_0)$. Since r^* is chosen such that (67) is still true, from (75) and (63) one obtains again $\tau\sharp\sigma_0(x,\xi)\in M^{-N/\mu}_\lambda(\Omega)$ for any $(x,\xi)\in(\Omega\times\mathbb{R}^n)\setminus X_{r\lambda}(x_0)$. Since $N\ge1$ is an arbitrary integer, as in Theorem 2 we conclude that $\tau\sharp\sigma_0(x,\xi)\in S^{-\infty}(\Omega)$, hence $\varphi(x)\tau(x,D)\sigma_0(x,D)(\tilde\varphi u)\in C^\infty_0(\Omega)\subset H^{t,p}_\lambda$ for any $t\in\mathbb{R}$.
The same arguments above can be applied to prove the first statement in (74). Then $\tau(x,D)u\in H^{s+m,p}_{\lambda,loc}(\Omega)$ follows from (72)-(74), and the proof is complete. □

Let $a(x,D)$ be a properly supported pseudodifferential operator with symbol $a(x,\xi)\in M^m_\lambda(\Omega)$ and $x_0\in\Omega$. Following [4], [10], we can define, for any $x_0\in\Omega$,

- λ filter of Sobolev singularities of $u\in\mathcal{D}'(\Omega)$:

$$\mathcal{W}^{s,p}_{x_0,\lambda}u:=\left\{X\subset\mathbb{R}^n\,;\,u\in H^{s,p}_\lambda(\mathbb{R}^n\setminus X)(x_0)\right\}\,,\quad s\in\mathbb{R},\,1<p<\infty; \tag{76}$$

- λ characteristic filter of $a(x,D) \in \tilde{\mathcal{M}}^m_\lambda(\Omega)$, $m \in \mathbb{R}$:

$$(77)\quad \Sigma^\lambda_{x_0} a(x,D) := \{X \subset \mathbb{R}^n\,,\ a(x,\xi)\,\text{is}\,\lambda\,\text{microelliptic in}\,(\mathbb{R}^n \setminus X)(x_0)\}\,.$$

It is trivial that $\mathcal{W}^{s,p}_{x_0,\lambda}u$ and $\Sigma_{x_0,\lambda}a(x,D)$ are actually filters in the sense that they are closed with respect to the intersection of a finite number of sets and any $Y \supset X \in \mathcal{W}^{s,p}_{x_0,\lambda}u\,(\Sigma^\lambda_{x_0}a(x,D)u)$ belongs to the same set family.
It is also straightforward to show that the results of Theorem 2, 3 can be restated as follows.

Proposition 10. *Let $a(x,D) \in \tilde{\mathcal{M}}^m_\lambda(\Omega)$, $x_0 \in \Omega$, $s \in \mathbb{R}$ and $p \in]1,+\infty[$ be given. Then the following inclusions are satisfied*

$$\mathcal{W}^{s,p}_{x_0,\lambda}a(x,D)u \cap \Sigma^\lambda_{x_0}a(x,D) \subset \mathcal{W}^{s+m,p}_{x_0,\lambda}u \subset \mathcal{W}^{s,p}_{x_0,\lambda}a(x,D)u\,, \qquad \forall\, u \in \mathcal{D}'(\Omega)\,.$$

5. Complete polyhedra of $\mathbb{R}^n$ and applications to multi-quasi-elliptic equations. An useful tool in the study of hypoelliptic partial differential equations is given by the weight functions associated to a complete polyhedron, introduced by Gindikin and Volevich [7], see also [2].
Recall that a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n$ may be obtained as the convex hull of a finite subset $V(\mathcal{P}) \subset \mathbb{R}^n$ of convex-linearly independent points called *vertices of* $\mathcal{P}$ and univocally determined by $\mathcal{P}$. Moreover there exist two finite subsets $\mathcal{N}_0(\mathcal{P})$, $\mathcal{N}_1(\mathcal{P}) \subset \mathbb{R}^n$ such that:

$$(78)\quad \mathcal{P} = \{\zeta \in \mathbb{R}^n; \nu\cdot\zeta \geq 0, \forall \nu \in \mathcal{N}_0(\mathcal{P})\} \cap \{\zeta \in \mathbb{R}^n; \nu\cdot\zeta \leq 1, \forall \nu \in \mathcal{N}_1(\mathcal{P})\}.$$

Again $\mathcal{N}_0(\mathcal{P})$, $\mathcal{N}_1(\mathcal{P}) \subset \mathbb{R}^n$ are univocally determined by $\mathcal{P}$, if it has a non-empty interior. The boundary of $\mathcal{P}$ is made of the faces $\mathcal{F}_\nu$, which are the convex hull of the vertices of $\mathcal{P}$ lying on the hyperplane H_ν, orthogonal to $\nu \in \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$ of equation $\nu\cdot\zeta = 0$ if $\nu \in \mathcal{N}_0(\mathcal{P})$, $\nu\cdot\zeta = 1$ if $\nu \in \mathcal{N}_1(\mathcal{P})$.
A convex polyhedron $\mathcal{P} \subset [0,+\infty[^n$ is said *complete polyhedron* if:

- $V(\mathcal{P}) \subset \mathbb{N}^n$, $(0,\dots,0) \in V(\mathcal{P})$ and $V(\mathcal{P}) \neq \{(0,\dots,0)\}$,
- $\mathcal{N}_1(\mathcal{P}) \subset]0,+\infty[^n$, $\mathcal{N}_0(\mathcal{P}) = \{e_j\}_{j=1}^n$, where $e_j = (0,\dots,1_{j-\text{entry}},\dots,0)$.

We can now associate to any complete polyhedron $\mathcal{P}$ the positive function $\lambda_\mathcal{P}$, defined by

$$(79)\qquad \lambda_\mathcal{P}(\xi) = \left(\sum_{\gamma\in V(\mathcal{P})} \xi^{2\gamma}\right)^{1/2}.$$

It can be easily proved that $\lambda_{\mathcal{P}}(\xi)$ satisfies (3) with $\mu_0 = \min\limits_{\gamma\in\mathcal{V}(\mathcal{P})\setminus\{0\}} |\gamma|$ and $\mu_1 = \max\limits_{\gamma\in\mathcal{V}(\mathcal{P})} |\gamma|$.
For proving that $\lambda_{\mathcal{P}}(\xi)$ satisfies also (4), we introduce the following two lemmas, whose proof may be found in [5], [6].

Lemma 2. *For every $\alpha \in \mathbb{Z}^n_+$ one has that*

$$|\xi^\alpha| \le \lambda_{\mathcal{P}}(\xi)^{k(\alpha,\mathcal{P})}\,, \qquad \xi \in \mathbb{R}^n\,, \tag{80}$$

where we have set $k(\alpha,\mathcal{P}) := \inf\{t > 0 \,:\; t^{-1}\alpha \in \mathcal{P}\} = \max\{\nu\cdot\alpha \,:\; \nu \in \mathcal{N}_1(\mathcal{P})\}$.

Lemma 3. *For any $\alpha, \gamma \in \mathbb{Z}^n_+$ and $m \in \mathbb{R}$ there exists a constant $C_{m,\alpha,\gamma} > 0$ such that*

$$\left|\xi^\gamma \partial_\xi^{\alpha+\gamma}\lambda_{\mathcal{P}}(\xi)^m\right| \le C_{m,\alpha,\gamma}\lambda_{\mathcal{P}}(\xi)^{m-\frac{|\alpha|}{\mu}}\,, \qquad \xi \in \mathbb{R}^n\,, \tag{81}$$

where $\mu := \max\left\{\dfrac{1}{\nu_j} \;:\; j = 1,\dots,n\,,\; \nu \in \mathcal{N}_1(\mathcal{P})\right\}$ is called formal order of $\mathcal{P}$.

In view of Proposition 2, Lemma 3 yields that $\lambda^m_{\mathcal{P}}(\xi) \in M^m_{\lambda_{\mathcal{P}}}(\mathbb{R}^n)$, for every $m \in \mathbb{R}$.
We can prove now the following result.

Lemma 4. *There exist $C > 0$ and $0 < \varepsilon_0 < 1$ such that for $0 < \varepsilon \le \varepsilon_0$*

$$\lambda_{\mathcal{P}}(\eta) \le C\lambda_{\mathcal{P}}(\xi) \qquad \textit{holds when} \qquad \sum_{j=1}^n |\eta_j - \xi_j|(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}} + |\xi_j|)^{-1} < \varepsilon\,. \tag{82}$$

P r o o f. For an arbitrary $\xi \in \mathbb{R}^n$, we define the subsets $J_1 = J_1(\xi)$ and $J_2 = J_2(\xi)$ contained in $\{1,\dots,n\}$ as in the proof of Proposition 2 (see (14)). There exist some constants $C_* > 1$ and $\varepsilon_* < 1$, independent of η,ξ, such that for $0 < \varepsilon \le \varepsilon_*$ the following can be proved

i) If $|\eta_j - \xi_j| < \varepsilon(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}} + |\xi_j|)$ and $j \in J_1(\xi)$ then $\frac{1}{C_*}|\xi_j| \le |\eta_j| \le C_*|\xi_j|$.

ii) If $|\eta_j - \xi_j| < \varepsilon(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}} + |\xi_j|)$ and $j \in J_2(\xi)$ then $|\eta_j| \le C_*\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}$.

We assume, for a while, that statements *i*, *ii* are true, and we go on to prove (82). Let us take $\eta, \xi \in \mathbb{R}^n$ satisfying

$$\sum_{j=1}^n |\eta_j - \xi_j|(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}} + |\xi_j|)^{-1} < \varepsilon\,, \tag{83}$$

with $0 < \varepsilon \le \varepsilon_*$. The arbitrary vertex $\gamma \in V(\mathcal{P})$ of $\mathcal{P}$ is decomposed as $\gamma = \tilde{\gamma} + \hat{\gamma}$, where the multi-indices $\tilde{\gamma} = \tilde{\gamma}(\xi)$ and $\hat{\gamma} = \hat{\gamma}(\xi)$ are defined by setting

$$\tilde{\gamma}_j := \begin{cases} \gamma_j\,, & j \in J_1(\xi)\,, \\ 0 & \text{otherwise} \end{cases} \tag{84}$$

and $\hat{\gamma} := \gamma - \tilde{\gamma}$.
Using (83), the statement i and Lemma 2, we get

$$\eta^{2\tilde{\gamma}} = \prod_{j \in J_1} |\eta_j|^{2\gamma_j} \le C_*^{2|\tilde{\gamma}|} \prod_{j \in J_1} |\xi_j|^{2\gamma_j} = C_*^{2|\tilde{\gamma}|} \xi^{2\tilde{\gamma}} \le C_*^{2|\tilde{\gamma}|} \lambda_{\mathcal{P}}(\xi)^{2\tilde{\nu}\cdot\tilde{\gamma}}\,, \tag{85}$$

where $\tilde{\nu} \in \mathcal{N}_1(\mathcal{P})$ is chosen such that $\tilde{\nu} \cdot \tilde{\gamma} = k(\tilde{\gamma}, \mathcal{P})$.
On the other hand, from (83) and the statement ii it follows

$$\eta^{2\hat{\gamma}} = \prod_{j \in J_2} |\eta_j|^{2\gamma_j} \le C_*^{2|\hat{\gamma}|} \lambda_{\mathcal{P}}(\xi)^{\frac{2|\hat{\gamma}|}{\mu}} \le C_*^{2|\hat{\gamma}|} \lambda_{\mathcal{P}}(\xi)^{2\tilde{\nu}\cdot\hat{\gamma}}\,, \tag{86}$$

where we have also used that $\tilde{\nu}_j \ge \dfrac{1}{\mu}$ is true for every $j = 1, \dots, n$.
From (85), (86), considering that $\gamma = \tilde{\gamma} + \hat{\gamma}$ and $\tilde{\nu} \cdot \gamma \le 1$, we get

$$\eta^{2\gamma} = \eta^{2\tilde{\gamma}} \eta^{2\hat{\gamma}} \le C_*^{2|\gamma|} \lambda_{\mathcal{P}}(\xi)^{2\tilde{\nu}\cdot\gamma} \le C_*^{2|\gamma|} \lambda_{\mathcal{P}}(\xi)^2\,. \tag{87}$$

Finally, summing (87) over all vertices $\gamma \in V(\mathcal{P})$ yields

$$\lambda_{\mathcal{P}}(\eta)^2 = \sum_{\gamma \in V(\mathcal{P})} \eta^{2\gamma} \le C^2 \lambda_{\mathcal{P}}(\xi)^2\,, \tag{88}$$

where we have set $C^2 := \sum_{\gamma \in V(\mathcal{P})} C_*^{2|\gamma|}$.
To complete the proof of the Lemma, it remains to show the statements i and ii.
Statement i. Assume that $j \in J_1$, that is $\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}} < |\xi_j|$, and

$$|\eta_j - \xi_j| < \varepsilon(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}} + |\xi_j|)\,. \tag{89}$$

Using now the triangle inequality we get $||\eta_j| - |\xi_j|| < 2\varepsilon|\xi_j|$, hence $(1-2\varepsilon)|\xi_j| < |\eta_j| < (1+2\varepsilon)|\xi_j|$. Then i follows for $0 < \varepsilon \le \varepsilon_*$, by choosing for instance $\varepsilon_* \le \dfrac{1}{4}$ and $C_* \ge 2$.
Statement ii. We assume now that $j \in J_2$, that is $|\xi_j| \le \lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}$, and (89) are

satisfied. Again, in view of the triangle inequality, we get $||\eta_j| - |\xi_j|| < 2\varepsilon\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}$, hence $|\eta_j| \le (1+2\varepsilon)\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}$. Then *ii* follows for $0<\varepsilon\le\varepsilon_*$, if ε_* and C_* are chosen as before.
The proof of the Lemma is then completed. □

As a consequence of the previous Lemmas 2-4, we prove now the following result.

Proposition 11. *There exist positive constants* $C>1$ *and* $\varepsilon_0<1$ *such that for any* $0<\varepsilon\le\varepsilon_0$ *we have*

$$\frac{1}{C}\lambda_{\mathcal{P}}(\xi) \le \lambda_{\mathcal{P}}(\eta) \le C\lambda_{\mathcal{P}}(\xi)\,, \qquad \textit{when } \sum_{j=1}^{n}|\eta_j-\xi_j|(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}+|\xi_j|)^{-1}<\varepsilon\,. \tag{90}$$

Proof. Let $\eta,\xi\in\mathbb{R}^n$ satisfy (83). By means of Taylor expansion we get the estimate

$$\begin{aligned}\left|\lambda_{\mathcal{P}}(\eta)^{\frac{1}{\mu}}-\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}\right| &\le \sum_{j=1}^{n}|\eta_j-\xi_j|\int_0^1|\partial_j\lambda_{\mathcal{P}}^{\frac{1}{\mu}}(\xi^t)|\,dt\\ &=\sum_{j\in J_1(\xi)}|\eta_j-\xi_j|\int_0^1|\partial_j\lambda_{\mathcal{P}}^{\frac{1}{\mu}}(\xi^t)|\,dt+\sum_{j\in J_2(\xi)}|\eta_j-\xi_j|\int_0^1|\partial_j\lambda_{\mathcal{P}}^{\frac{1}{\mu}}(\xi^t)|\,dt\,,\end{aligned} \tag{91}$$

where $\xi^t:=(1-t)\xi+t\eta$, and $J_1(\xi)$, $J_2(\xi)$ are defined as in the proof of Lemma 4. Considering the definition of $J_2(\xi)$ and applying estimates (81), we have

$$\begin{aligned}\sum_{j\in J_2(\xi)}|\eta_j-\xi_j|\int_0^1|\partial_j\lambda_{\mathcal{P}}^{\frac{1}{\mu}}(\xi^t)|\,dt &\le C_2\varepsilon\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}\int_0^1\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}-\frac{1}{\mu}}\,dt\\ &\le C_2\varepsilon\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}\,,\end{aligned} \tag{92}$$

for $0<\varepsilon\le\varepsilon_*$ and a suitable positive constant C_2, independent of ε.
On the other hand, observing that

$$|\xi_j^t-\xi_j|=|(1-t)\xi_j+t\eta_j-\xi_j|=t|\eta_j-\xi_j|<\varepsilon(\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}+|\xi_j|) \tag{93}$$

using the definition of $J_1(\xi)$ and applying the statement i with ξ^t instead of η (see Lemma 4), we also have

$$\begin{aligned}&\sum_{j\in J_1(\xi)} |\eta_j-\xi_j|\int_0^1 |\partial_j\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}}|\,dt \le 2\varepsilon\sum_{j\in J_1(\xi)}|\xi_j|\int_0^1 |\partial_j\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}}|\,dt\\ &\le 2C_*\varepsilon\sum_{j\in J_1(\xi)}\int_0^1 |\xi_j^t||\partial_j\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}}|\,dt\,,\end{aligned}\tag{94}$$

for ε as above and the constant C_* involved in the statements i and ii.
Again applying estimates (81), (82) and in view of (93) we obtain

$$|\xi_j^t||\partial_j\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}}| \le C_j\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}} \le \hat{C}_j\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}\,,\tag{95}$$

where $\hat{C}_j$ is a suitable positive constant, independent of ε. Combining (94), (95) yields

$$\sum_{j\in J_1(\xi)} |\eta_j-\xi_j|\int_0^1 |\partial_j\lambda_{\mathcal{P}}(\xi^t)^{\frac{1}{\mu}}|\,dt \le C_1\varepsilon\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}\,,\tag{96}$$

where again C_1 is a positive constant independent of ε. By adding (92), (96), from (91) we deduce that $\left|\lambda_{\mathcal{P}}(\eta)^{\frac{1}{\mu}}-\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}\right| \le \tilde{C}\varepsilon\lambda_{\mathcal{P}}(\xi)^{\frac{1}{\mu}}$, hence

$$(1-\tilde{C}\varepsilon)^{\mu}\lambda_{\mathcal{P}}(\xi) \le \lambda_{\mathcal{P}}(\eta) \le (1+\tilde{C}\varepsilon)^{\mu}\lambda_{\mathcal{P}}(\xi)\,,\tag{97}$$

with a suitable $\varepsilon-$independent positive constant $\tilde{C}$. Estimates (97) imply at once (90) by taking any $\varepsilon>0$ sufficiently small. □

Directly from [6] we have the following

Lemma 5. *For $\mathcal{P}$ complete polyhedron let $Q=\sum_{\alpha\in\mathcal{P}} a_\alpha(x)D^\alpha$ be a linear partial differential operator with coefficients $a_\alpha(x)\in C^\infty(\Omega)$. Then $Q\in\tilde{\mathcal{M}}^1_{\lambda_{\mathcal{P}}}(\Omega)$.*

We say moreover that the operator Q is *multi-quasi-elliptic* if for any compact $K\subset\subset\Omega$ two positive constants C_K, R_k exist such that:

$$\left|\sum_{\alpha\in\mathcal{F}(\mathcal{P})} a_\alpha(x)\xi^\alpha\right| > C_K\lambda_{\mathcal{P}}(\xi),\ x\in K,\ |\xi|>R_K,\tag{98}$$

where $\mathcal{F}(\mathcal{P}) = \bigcup_{\nu \in \mathcal{N}_1(\mathcal{P})} \mathcal{F}_\nu$.

In the same way, as in Definition 4, we can consider an operator microlocally multi-quasi-elliptic in $X \subset \mathbb{R}^n_\xi$ at the point $x_0 \in \Omega$. Since [2, §1.8] shows that a multi-quasi-elliptic operator is λ_p-elliptic in the usual sense considered in (36), using Theorem 3 we obtain the following

Proposition 12. *For $\mathcal{P}$ complete polyhedron, consider the partial differential operator $Q = \sum_{\alpha \in \mathcal{P}} a_\alpha(x) D^\alpha$, with smooth coefficients. Assume moreover that Q is microlocally multi-quasi-elliptic in $X \subset \mathbb{R}^n_\xi$ at the point x_0. If $u \in \mathcal{D}'(\Omega)$ and $Qu \in H^{s,p}_{\lambda_\mathcal{P}} X(x_0)$ then $u \in H^{s+1,p}_{\lambda_\mathcal{P}} X(x_0)$, for every $1 < p < \infty$ and $s \in \mathbb{R}$.*

REFERENCES

[1] R. Beals. A general calculus of pseudodifferential operators. *Duke Math. J.* **42** (1978), 1–42.

[2] P. Boggiatto, E. Buzano, L. Rodino. Global Hypoellipticity and Spectral Theory. Berlin, Akademie-Verlag, 1996.

[3] C. Fefferman. L^p bounds for pseudodifferential operators. *Israel J.Math.* **14** (1973), 413–417.

[4] G. Garello. Inhomogeneous microlocal analysis for C^∞ and H^ν_ψ singularities. *Rend. Sem. Mat. Univ. Polit. Torino* **2** (1992), 165–181.

[5] G. Garello. Generalized Sobolev algebras and regularity of solutions of multi-quasi-elliptic semilinear equations. *Comm. in Appl. Analysis* **3**, *4* (1999), 199–209.

[6] G. Garello, A. Morando. L^p-bounded pseudodifferential operators and regularity for multi-quasi-elliptic equations. *Integral Equations Oper. Theory* **51**, *4* (2005), 501–517.

[7] S. G. Gindikin, L. R. Volevich. The Method of Newton's Polyhedron in the Theory of Partial Differential Equations. Mathematics and its Applications. Soviet Series, Vol. **86**, Dordrecht, Kluwer Academic Publishers, 1992.

[8] L. Hörmander. The Analysis of Linear Partial Differential Operators I, III. Berlin, Heilderberg, Springer Verlag, 1985.

[9] F. Nicola, L. Rodino. Global Pseudo-Differential Operators on Euclidean Spaces. Theory and Applications, vol. **4**. Basel, Birkhäuser, 2010.

[10] L. Rodino. Microlocal analysis for spatially inhomogeneous pseudodifferential operators. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* **9** (1982), 221–253.

[11] M. E. Taylor. Pseudodifferential Operators. Princeton Mathematical Series, vol. **34**. Princeton, New Jersey, Princeton Univ. Press, 1981.

Gianluca Garello
Dipartimento di Matematica Università di Torino
Via Carlo Alberto 10
I-10123, Torino, Italy
e-mail: gianluca.garello@unito.it

Alessandro Morando
Dipartimento di Matematica
Università di Brescia
Via Valotti 9
I-25133, Brescia, Italy
e-mail: alessandro.morando@ing.unibs.it