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# ON THE HOLOMORPHIC EXTENSION OF SOLUTIONS OF ELLIPTIC PSEUDODIFFERENTIAL EQUATIONS

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Abstract. We derive analytic estimates and holomorphic extensions for the solutions of a class of elliptic pseudodifferential equations on $\mathbb{R}^d$.

**1. Introduction.** In this paper we prove analytic estimates and holomorphic extensions on conical sectors of $\mathbb{C}^d$ for weak solutions of a linear elliptic equation on $\mathbb{R}^d$ of the form

$$Pu = f. \tag{1.1}$$

We assume that $P$ is a pseudodifferential operator with symbol $p(x,\xi)$ satisfying, for some $m \geq 0$, $n \geq 0$, the following estimates

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C^{|\alpha|+|\beta|+1}\alpha!\beta!\langle\xi\rangle^{m-|\alpha|}\langle x\rangle^{n-|\beta|} \tag{1.2}$$

for every $(x,\xi) \in \mathbb{R}^{2d}, \alpha, \beta \in \mathbb{N}^d$ and for some positive constant $C$ independent of $\alpha, \beta$ (we denote as usual $\langle x\rangle = (1+|x|^2)^{1/2}$). Estimates (1.2) yield that $p(x,\xi)$ belongs to the symbol class $G^{m,n}(\mathbb{R}^d)$ introduced with different notation by Parenti [21] and Cordes [8] and considered, under different notations, by a considerable number of authors in various functional and geometric settings, see

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for instance [7, 9, 10, 11, 12, 18, 19, 20, 22, 23]. We suppose that $P$ is $G$-elliptic, in the sense that

$$|p(x,\xi)| \geq c\langle\xi\rangle^m\langle x\rangle^n, \qquad |x|+|\xi| \geq C, \tag{1.3}$$

for some positive constants $c$, $C$. As a simple example, the reader may consider the equation

$$-\Delta u + \lambda u = f, \qquad \lambda \in \mathbb{C}, \tag{1.4}$$

hence $p(x,\xi) = |\xi|^2 + \lambda$, which verifies the estimates in (1.2) and (1.3) with $m = 2$, $n = 0$, if $\lambda \notin \mathbb{R}_- \cup \{0\}$. Under the assumption (1.3) various results of global regularity have been proved for the equation (1.1). By construction of parametrices and without needing to assume analyticity of the symbol $p(x,\xi)$, Parenti [21] and Cordes [8] proved that every solution $u \in \mathcal{S}'(\mathbb{R}^d)$ of (1.1) belongs to $\mathcal{S}(\mathbb{R}^d)$ provided $f \in \mathcal{S}(\mathbb{R}^d)$, cf. also [3] for more general results. More recently, in [4, 7], the first author et al. studied the same equation in the frame of Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^d), \mu \geq 1, \nu \geq 1$, of all functions $f \in C^\infty(\mathbb{R}^d)$ satisfying the following estimate

$$|\partial^\alpha f(x)| \leq C^{|\alpha|+1}\alpha!^\mu e^{-c|x|^{1/\nu}}, \qquad x \in \mathbb{R}^d \tag{1.5}$$

for all $\alpha \in \mathbb{N}^d$ and for some positive constants $C, c$ independent of $\alpha$. In particular, in [4], the result has been proved using an inductive argument in a suitable scale of Banach spaces defining $S^\mu_\nu(\mathbb{R}^d)$. We recall that for $\mu = 1$, the condition (1.5) implies that $f$ is analytic on $\mathbb{R}^d$ and admits a holomorphic extension in a strip of the form $\{z \in \mathbb{C}^d : |\Im z| < T\}, T > 0$. Actually, using the same argument of [4] we can prove more refined estimates and derive larger holomorphic extensions for the solutions of (1.1) under suitable assumptions on the forcing term $f$. Namely, we assume that $f$ belongs to the space $\mathcal{BH}_{sect}(\mathbb{R}^d)$ of all functions satisfying the following estimates on $\mathbb{R}^d$:

$$||x^\beta \partial^\alpha f||_{L^2} \leq C^{|\alpha|+1}|\alpha|!, \quad \text{for } |\beta| \leq |\alpha|. \tag{1.6}$$

As we will show in Section 3, this implies that $f$ extends to a holomorphic bounded function $f(x+iy)$ in the sector of $\mathbb{C}^d$

$$\mathcal{C}_c = \{z = x+iy \in \mathbb{C}^d : \ |y| < c(1+|x|)\} \tag{1.7}$$

for some $c > 0$, satisfying

$$\int \sup_{|y|<c(1+|x|)} |f(x+iy)|^2\, dx < \infty. \tag{1.8}$$

Hence, we can think of $f$ as a function whose holomorphic extension in a sector is in $L^2$ uniformly with respect to its imaginary part.

The main result of the paper is the following.

**Theorem 1.** *Let $P$ be a pseudodifferential operator with symbol $p$ satisfying* (1.2) *for some $m \geq 0$, $n \geq 0$, and assume that $p$ is $G$-elliptic, that is* (1.3) *is satisfied. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ be a solution of $Pu = f \in \mathcal{BH}_{sect}(\mathbb{R}^d)$. Then $u \in \mathcal{BH}_{sect}(\mathbb{R}^d)$; in particular, it extends to a bounded holomorphic function $u(x+iy)$ in the sector of $\mathbb{C}^d$ of the type* (1.7)*, satisfying*

$$\int \sup_{|y|<c(1+|x|)} |u(x+iy)|^2 \, dx < \infty. \tag{1.9}$$

It would be interesting to extend the result above to semilinear perturbations of the equation (1.1) in the spirit of other results concerning holomorphic extensions of solutions of nonlinear elliptic equations, see [17, 2, 5, 6], and of the corresponding evolution counterparts, cf. [15]. This extension seems to be quite easy if $m \geq 1, n \geq 1$ by replacing $L^2$-norms by Sobolev $H^s$-norms with $s$ large enough, but it reveals some non trivial difficulties in the general case $m \geq 0, n \geq 0$. We postpone the study of this problem to a future paper.

## 2. Notation and preliminary results.

**2.1. Factorial and binomial coefficients.** We recall here some well known formulas involving factorials and binomial coefficients which will be often used in the sequel. We shall use the usual multi-index notation for factorial and binomial coefficients. Hence, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ we set $\alpha! = \alpha_1! \ldots \alpha_d!$ and for $\beta, \alpha \in \mathbb{N}^d$, $\beta \leq \alpha$, we set $\binom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$.

The following inequality is standard:

$$\binom{\alpha}{\beta} \leq 2^{|\alpha|}. \tag{2.1}$$

Also, we recall the identity

$$\sum_{\substack{|\alpha'|=j \\ \alpha' \leq \alpha}} \binom{\alpha}{\alpha'} = \binom{|\alpha|}{j}, \qquad j = 0, 1, \ldots, |\alpha|,$$

which follows from $\prod_{i=1}^d (1+t)^{\alpha_i} = (1+t)^{|\alpha|}$, and gives in particular

$$\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}, \quad \alpha, \beta \in \mathbb{N}^d, \ \beta \leq \alpha. \tag{2.2}$$

The estimate (2.2) implies by induction,

$$\frac{\alpha!}{\delta_1!\dots\delta_j!}\leq\frac{|\alpha|!}{|\delta_1|!\dots|\delta_j|!},\quad \alpha=\delta_1+\dots+\delta_j, \tag{2.3}$$

as well as

$$\frac{\alpha!}{(\alpha-\beta)!}\leq\frac{|\alpha|!}{|\alpha-\beta|!},\quad \beta\leq\alpha. \tag{2.4}$$

Finally we recall the so-called inverse Leibniz' formula:

$$x^\beta\partial^\alpha u(x)=\sum_{\gamma\leq\beta,\,\gamma\leq\alpha}\frac{(-1)^{|\gamma|}\beta!}{(\beta-\gamma)!}\binom{\alpha}{\gamma}\partial^{\alpha-\gamma}(x^{\beta-\gamma}u(x)). \tag{2.5}$$

**2.2. *G*-pseudo-differential operators.** $G$-pseudo-differential operators (also known as $SG$ or scattering pseudodifferential operators) are defined as standard by

$$p(x,D)u(x)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{ix\xi}p(x,\xi)\widehat{u}(\xi)\,d\xi, \tag{2.6}$$

where

$$\widehat{u}(\xi)=\int_{\mathbb{R}^d}e^{-ix\xi}u(x)\,dx$$

denotes the Fourier transform of $u$ and the symbol $p(x,\xi)$ satisfies, for some $m,n\in\mathbb{R}$, the following estimates: for every $\alpha,\beta\in\mathbb{N}^d$ there exists a constant $C_{\alpha,\beta}>0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\leq C_{\alpha,\beta}\langle x\rangle^{n-|\beta|}\langle\xi\rangle^{m-|\alpha|} \tag{2.7}$$

for every $x,\xi\in\mathbb{R}^d$. The space of functions satisfying these estimates is denoted by $G^{m,n}(\mathbb{R}^d)$ and can be endowed with the topology defined by the seminorms

$$\|p\|_N^{(G)}=\sup_{|\alpha|+|\beta|\leq N}\ \sup_{(x,\xi)\in\mathbb{R}^{2d}}\left\{|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\langle x\rangle^{-n+|\beta|}\langle\xi\rangle^{-m+|\alpha|}\right\},\quad N\in\mathbb{N}.$$

We set $\mathrm{OP}G^{m,n}(\mathbb{R}^d)$ for the space of the corresponding operators. $G$-operators represent a natural generalization of differential operators with polynomial coefficients and turn out to be very convenient for a series of problems involving global aspects of partial differential equations in $\mathbb{R}^d$. A specific calculus for these operators is presented in [8] and [21]. In fact, they are a particular case of the the general Hörmander's classes, see [16, Chapter XVIII].

Our results actually concern the subclass of $G^{m,n}(\mathbb{R}^d)$ defined by the more particular estimate (1.2) which implies in addition the analyticity of the symbols but in the proofs we will use also general properties of $G$-symbols. We recall some of them in the following referring to [8], [21] and [20, Chapter 3] for proofs and details.

First, if $p \in G^{m,n}(\mathbb{R}^d)$ then $p(x, D)$ defines a continuous map $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ which extends to a continuous map $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. The composition of two such operators is therefore well defined in $\mathcal{S}(\mathbb{R}^d)$ and in $\mathcal{S}'(\mathbb{R}^d)$; more precisely, if $p_1 \in G^{m_1,n_1}(\mathbb{R}^d)$ and $p_2 \in G^{m_2,n_2}(\mathbb{R}^d)$, then $p_1(x, D)p_2(x, D) = p_3(x, D)$ with $p_3 \in G^{m_1+m_2,n_1+n_2}(\mathbb{R}^d)$ and the map $(p_1, p_2) \mapsto p_3$ is continuous $G^{m_1,n_1}(\mathbb{R}^d) \times G^{m_2,n_2}(\mathbb{R}^d) \to G^{m_1+m_2,n_1+n_2}(\mathbb{R}^d)$.

If $p \in G^{m,n}(\mathbb{R}^d)$, with $m \leq 0$, $n \leq 0$ then

$$p(x, D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \tag{2.8}$$

continuously, and

$$\|p(x, D)\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq C\|p\|_N^{(G)}$$

for suitable $C > 0$, $N \in \mathbb{N}$ depending only on the dimension $d$ (see [20, Theorem 3.1.5]). We also recall that $\bigcap\limits_{m,n\in\mathbb{R}} G^{m,n}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d})$. In particular, operators with Schwartz symbols are (globally) regularizing, i.e. they map continuously $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$.

A symbol $p \in G^{m,n}(\mathbb{R}^d)$ (and the corresponding operator) is called *$G$-elliptic* if it satisfies (1.3) for some constants $C, c > 0$. The importance of $G$-ellipticity in the subsequent arguments relies in the fact that this condition guaranties the existence of a parametrix $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$ of $P = p(x, D)$. Namely we have the following result.

**Proposition 2.** *Let $p \in G^{m,n}(\mathbb{R}^d)$ be $G$-elliptic. Then there exists an operator $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$ such that $EP = I + R$ and $PE = I + R'$, where $R, R'$ are (globally) regularizing pseudodifferential operators, i.e. with Schwartz symbols. Hence $R$ and $R'$ are continuous maps $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. The operator $E$ is said to be a parametrix for $P$.*

Finally we recall for further reference the following formulas, which are valid for general pseudodifferential operators and can be verified by a direct computation:

for $\alpha, \beta \in \mathbb{N}^d$,

$$x^\beta Pu = \sum_{\gamma\leq\beta}(-1)^{|\gamma|}\binom{\beta}{\gamma}(D_\xi^\gamma p)(x,D)(x^{\beta-\gamma}u), \tag{2.9}$$

$$\partial^\alpha Pu = \sum_{\delta\leq\alpha}\binom{\alpha}{\delta}(\partial_x^\delta p)(x,D)\partial^{\alpha-\delta}u. \tag{2.10}$$

**2.3. Function spaces** We prove here some important properties of the space $\mathcal{BH}_{sect}(\mathbb{R}^d)$ defined in the Introduction. Namely we prove the following result.

**Theorem 3.** *Let $f \in \mathcal{BH}_{sect}(\mathbb{R}^d)$. Then $f$ extends to a holomorphic bounded function $f(x+iy)$ in a sector of $\mathbb{C}^d$ of the form* (1.7) *and satisfies there the estimate* (1.8).

Proof. If $M$ is an integer, $M > d/2$, we have

$$\|x^\beta\partial^\alpha f\|_{L^\infty} \leq C\sum_{|\gamma|\leq M}\|\partial^\gamma(x^\beta\partial^\alpha f)\|_{L^2}.$$

Hence an application of Leibniz' formula shows that (1.6) holds with the $L^2$ norm replaced by the $L^\infty$ norm.

Using $|\alpha|! \leq d^{|\alpha|}\alpha!$ we also get the estimate

$$||\langle x\rangle^{|\alpha|}\partial^\alpha f(x)||_{L^2} \leq C^{|\alpha|+1}\alpha!, \tag{2.11}$$

for a new constant $C > 0$, and similarly with the $L^2$ norm replaced by the $L^\infty$ norm.

This shows that the power series

$$\sum_\alpha \frac{\partial^\alpha f(x)}{\alpha!}(z-x)^\alpha, \tag{2.12}$$

for any fixed $x \in \mathbb{R}^d$ converges in a polydisc in $\mathbb{C}^d$ defined by $|z_k - x_k| < c'\langle x\rangle$, $1 \leq k \leq d$, for some $c' > 0$, and gives the desired bounded extension in a sector of the type (1.7), which is covered by those polydiscs. Upon setting $z = x + iy$ in (2.12), and using the estimate $|iy|^{|\alpha|} \leq (2c)^{|\alpha|}\langle x\rangle^{|\alpha|}$ in $\mathcal{C}_c$ and (2.11) we get (1.8) as well. □

In the sequel we will use the following obvious characterization of the space $\mathcal{BH}_{sect}(\mathbb{R}^d)$: for $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$f \in \mathcal{BH}_{sect}(\mathbb{R}^d) \Longleftrightarrow S_\infty^\varepsilon[f] := \sum_{|\beta|\le|\alpha|} \frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|x^\beta\partial^\alpha f\|_{L^2} < \infty, \tag{2.13}$$

for some $\varepsilon > 0$.

**3. Proof of Theorem 1.** We will prove Theorem 1 by showing that $S_\infty^\varepsilon[u] < \infty$ for some $\varepsilon > 0$. This will be achieved by an iteration argument involving the partial sum of the series in (2.13), that is

$$S_N^\varepsilon[f] = \sum_{\substack{|\alpha|+|\beta|\le N\\ |\beta|\le|\alpha|}} \frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|x^\beta\partial^\alpha f\|_{L^2}. \tag{3.1}$$

As first step in the inductive argument, observe that $S_0^\varepsilon[u] = \|u\|_{L^2} < \infty$. Indeed, $f \in L^2$ and by the equation $Pu = f$ and Proposition 2 we have $u = Ef - Ru$, where $E$ is bounded on $L^2(\mathbb{R}^d)$ and $R$ is continuous $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Let us prove some preliminary estimates.

**Proposition 4.** *Let $R \in \mathrm{OP}G^{-1,-1}(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that, for every $\varepsilon > 0, N \ge 1$ and $u \in \mathcal{S}'(\mathbb{R}^d)$, we have*

$$\sum_{\substack{0<|\alpha|+|\beta|\le N\\ |\beta|\le|\alpha|}} \frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|R(x^\beta\partial^\alpha u)\|_{L^2} \le C\varepsilon S_{N-1}^\varepsilon[u].$$

P r o o f. We first estimate the terms with $\beta = 0$, hence $\alpha \ne 0$. Let $k \in \{1,\dots,d\}$ such that $\alpha_k \ne 0$. Since $R \circ \partial_k \in \mathrm{OP}G^{0,-1}(\mathbb{R}^d)$ is bounded on $L^2(\mathbb{R}^d)$ we have[1]

$$\frac{\varepsilon^{|\alpha|}}{|\alpha|!}\|R(\partial^\alpha u)\|_{L^2} \le C\varepsilon\frac{\varepsilon^{|\alpha|-1}}{|\alpha|!}\|\partial^{\alpha-e_k}u\|_{L^2}.$$

On the other hand, when $\beta \ne 0$, hence $\beta_j \ne 0$ for some $j \in \{1,\dots,d\}$, we use the fact that $R \circ x_j \in \mathrm{OP}G^{-1,0}(\mathbb{R}^d)$ is bounded on $L^2(\mathbb{R}^d)$. We get in that case

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|R(x^\beta\partial^\alpha u)\|_{L^2} \le C\varepsilon\frac{\varepsilon^{|\alpha|+|\beta|-1}}{|\alpha|!}\|x^{\beta-e_j}\partial^\alpha u\|_{L^2}. \qquad \square$$

[1] We denote by $e_k$ the $k$th vector of the standard basis of $\mathbb{R}^d$.

**Proposition 5.** *Let $P = p(x, D)$ be a pseudodifferential operator with symbol $p(x,\xi)$ satisfying the estimates* (1.2)*, with $m \geq 0$, $n \geq 0$. Let $E \in \mathrm{OP}G^{-m,-n}(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that, for every $\varepsilon$ small enough, $N \geq 1$ and $u \in \mathcal{S}'(\mathbb{R}^d)$, we have*

$$\sum_{\substack{0<|\alpha|+|\beta|\leq N\\ |\beta|\leq|\alpha|}} \frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|E[P, x^\beta\partial^\alpha]u\|_{L^2} \leq C\varepsilon S^{\varepsilon}_{N-1}[u]. \tag{3.2}$$

P r o o f. We have

$$[P, x^\beta\partial^\alpha] = [P, x^\beta]\partial^\alpha + x^\beta[P, \partial^\alpha].$$

Hence, using (2.9), (2.10), we get

$$[P, x^\beta\partial^\alpha]u = \sum_{0\neq\gamma_0\leq\beta}(-1)^{|\gamma_0|+1}\binom{\beta}{\gamma_0}(D^{\gamma_0}_\xi p)(x,D)(x^{\beta-\gamma_0}\partial^\alpha u) - \sum_{0\neq\delta\leq\alpha}\binom{\alpha}{\delta}x^\beta\partial^\delta_x p(x,D)\partial^{\alpha-\delta}u. \tag{3.3}$$

Given $\beta$, $\delta$, let $\tilde{\delta}$ be a multi-index of maximal length among those satisfying $|\tilde{\delta}| \leq |\delta|$ and $\tilde{\delta} \leq \beta$ (hence if $|\tilde{\delta}| < |\delta|$ then $\beta - \tilde{\delta} = 0$). Writing $x^\beta = x^{\tilde{\delta}}x^{\beta-\tilde{\delta}}$ in the last term of (3.3) and using again (2.9) we get

$$[P, x^\beta\partial^\alpha]u = \sum_{\delta\leq\alpha}\sum_{\substack{\gamma_0\leq\beta-\tilde{\delta}\\ (\delta,\gamma_0)\neq(0,0)}}(-1)^{|\gamma_0|+1}\binom{\beta-\tilde{\delta}}{\gamma_0}\binom{\alpha}{\delta}x^{\tilde{\delta}}(D^{\gamma_0}_\xi\partial^\delta_x p)(x,D)(x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta}u). \tag{3.4}$$

We now consider the operator $x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta}$. Given $\gamma_0$, $\alpha$, $\delta$, let $\tilde{\gamma}_0$ be a multi-index of length $|\tilde{\gamma}_0| = |\gamma_0|$ and satisfying $\tilde{\gamma}_0 \leq \alpha - \delta$ (such a multi-index exists because $|\beta| \leq |\alpha|$). By the inverse Leibniz formula (2.5) we have

$$x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta} = x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\tilde{\gamma}_0}\partial^{\alpha-\delta-\tilde{\gamma}_0} = \partial^{\tilde{\gamma}_0}\circ x^{\beta-\tilde{\delta}-\gamma_0}\partial^{\alpha-\delta-\tilde{\gamma}_0} + \sum_{\substack{0\neq\gamma_1\leq\beta-\tilde{\delta}-\gamma_0\\ \gamma_1\leq\tilde{\gamma}_0}}\frac{(-1)^{|\gamma_1|}(\beta-\tilde{\delta}-\gamma_0)!}{(\beta-\tilde{\delta}-\gamma_0-\gamma_1)!}\binom{\tilde{\gamma}_0}{\gamma_1}\partial^{\tilde{\gamma}_0-\gamma_1}\circ x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0}. \tag{3.5}$$

We now look at the operator $x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0}$. Let $\tilde{\gamma}_1$ be a multi-index of length $|\tilde{\gamma}_1| = |\gamma_1|$ and satisfying $\tilde{\gamma}_1 \le \alpha - \delta - \tilde{\gamma}_0$. Again by (2.5) we have

$$
\begin{aligned}
(3.6)\quad x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0} &= x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\tilde{\gamma}_1}\partial^{\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1} \\
&= \partial^{\tilde{\gamma}_1} \circ x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1}\partial^{\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1}
\end{aligned}
$$

$$
+ \sum_{\substack{0\neq\gamma_2\le\beta-\tilde{\delta}-\gamma_0-\gamma_1\\ \gamma_2\le\tilde{\gamma}_1}} \frac{(-1)^{|\gamma_2|}(\beta-\tilde{\delta}-\gamma_0-\gamma_1)!}{(\beta-\tilde{\delta}-\gamma_0-\gamma_1-\gamma_2)!}\binom{\tilde{\gamma}_1}{\gamma_2}\partial^{\tilde{\gamma}_1-\gamma_2}\circ x^{\beta-\tilde{\delta}-\gamma_0-\gamma_1-\gamma_2}\partial^{\alpha-\delta-\tilde{\gamma}_0-\tilde{\gamma}_1}.
$$

We can iterate this argument and replace all in (3.4). Then we get

$$
\begin{aligned}
[P, x^\beta\partial^\alpha]u = \sum_{\delta\le\alpha}\sum_{j=0}^{h}\sum_{\substack{\gamma_0\le\beta-\tilde{\delta}\\ (\delta,\gamma_0)\neq(0,0)}}\sum_{\substack{0\neq\gamma_1\le\beta-\tilde{\delta}-\gamma_0\\ \gamma_1\le\tilde{\gamma}_0}}\cdots\sum_{\substack{0\neq\gamma_j\le\beta-\tilde{\delta}-\gamma_0-\ldots-\gamma_{j-1}\\ \gamma_j\le\tilde{\gamma}_{j-1}}} C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j} \\
\times\, p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,D)\big(x^{\beta-\tilde{\delta}-\gamma_0-\ldots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\ldots-\tilde{\gamma}_j}u\big),
\end{aligned}
$$

where $\tilde{\gamma}_j$ is defined inductively as a multi-index of length $|\tilde{\gamma}_j| \le |\gamma_j|$ and satisfying $\tilde{\gamma}_j \le \alpha - \delta - \tilde{\gamma}_0 - \ldots - \tilde{\gamma}_{j-1}$,

$$
\begin{aligned}
|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}| &= \frac{\alpha!(\beta-\tilde{\delta})!}{(\alpha-\delta)!\delta!\gamma_0!(\beta-\tilde{\delta}-\gamma_0-\ldots-\gamma_j)!}\prod_{k=1}^{j}\binom{\tilde{\gamma}_{k-1}}{\gamma_k} \\
(3.7)\qquad &\le \frac{|\alpha|!|\beta-\tilde{\delta}|!}{|\alpha-\delta|!\delta!\gamma_0!|\beta-\tilde{\delta}-\gamma_0-\ldots-\gamma_j|!}2^{|\tilde{\gamma}_0+\ldots+\tilde{\gamma}_{j-1}|},
\end{aligned}
$$

cf. (2.4) and (2.1), and

$$
(3.8)\qquad p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,\xi) = x^{\tilde{\delta}}\big(D_\xi^{\gamma_0}\partial_x^{\delta}p\big)(x,\xi)\xi^{\tilde{\gamma}_0-\gamma_1+\tilde{\gamma}_1-\ldots-\gamma_j+\tilde{\gamma}_j}, \quad j\ge 0,
$$

(if $j = 0$ we mean that there are not the binomial factors, nor the power of 2 in (3.7)). Observe that, since $\gamma_j \neq 0$ for every $j \ge 1$, this procedure in fact stops after a finite number of steps.

By (1.2), (2.1), and Leibniz' formula, for every $\theta, \sigma \in \mathbb{N}^d$ the following estimate holds:

$$
(3.9)\qquad |\partial_\xi^\theta\partial_x^\sigma p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\ldots,\gamma_j}(x,\xi)| \le C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!\langle x\rangle^{n-|\sigma|}\langle\xi\rangle^{m-|\theta|},
$$

for some constant $C$ depending only on $\theta$ and $\sigma$. In fact $|\tilde{\delta}| \le |\delta|$, $|\tilde{\gamma}_0 - \gamma_1 + \tilde{\gamma}_1 - \cdots - \gamma_j + \tilde{\gamma}_j| \le |\tilde{\gamma}_0| \le |\gamma_0|$, and the powers of $|\delta|$ and $|\gamma_0|$ which arise can be estimated by $C^{|\gamma_0|+|\delta|+1}$ for some $C > 0$.

We now use (3.9) to estimate $E \circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)$. To this end, observe that this operator belongs to $\mathrm{OP}G^{0,0}(\mathbb{R}^d)$, and therefore its norm as a bounded operator on $L^2(\mathbb{R}^d)$ is estimated by a seminorm of its symbol in $G^{0,0}(\mathbb{R}^d)$, depending only on $d$. Such a seminorm is in turn estimated by the product of a seminorm of the symbol of $E$ in $G^{-m,-n}(\mathbb{R}^d)$ and a seminorm of $p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}$ in $G^{m,n}(\mathbb{R}^d)$, again depending only on the dimension $d$. Hence by (3.9) we get

$$\|E \circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \le C^{|\gamma_0|+|\delta|+1}\gamma_0!\delta!. \tag{3.10}$$

Now, since $|\alpha| \ge |\beta|$ and $|\tilde{\gamma}_k| = |\gamma_k|$, $0 \le k \le j$, we have

$$\frac{|\beta-\tilde{\delta}|!|\alpha-\delta-\tilde{\gamma}_0-\ldots-\tilde{\gamma}_j|!}{|\alpha-\delta|!|\beta-\tilde{\delta}-\gamma_0-\ldots-\gamma_j|!} \le 1. \tag{3.11}$$

This is obvious when $|\tilde{\delta}| = |\delta|$, whereas it holds as well if $|\tilde{\delta}| < |\delta|$ since in this case $\beta - \tilde{\delta} = \gamma_0 = \cdots = \gamma_j = \tilde{\gamma}_0 = \cdots = \tilde{\gamma}_j = 0$.

By (3.7), (3.10) (3.11), we get in this case

(3.12)
$$\begin{aligned}
&\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}|C_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}|\|E\circ p_{\alpha,\beta,\delta,\gamma_0,\gamma_1,\dots,\gamma_j}(x,D)(x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u)\|_{L^2}\\
&\qquad \le C(C\varepsilon)^{|\delta|+|\gamma_0+\cdots+\gamma_j|+|\tilde{\gamma}_0+\cdots+\tilde{\gamma}_{j-1}|}\frac{\varepsilon^{|\alpha|+|\beta|-|\delta|-|\gamma_0+\cdots+\gamma_j|-|\tilde{\gamma}_0+\cdots+\tilde{\gamma}_{j-1}|}}{|\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|!}\\
&\qquad\qquad \times \|x^{\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j}\partial^{\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j}u\|_{L^2}.
\end{aligned}$$

Observe that $|\beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j| \le |\alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j|$.

We now perform the change of variables $\tilde{\alpha} = \alpha-\delta-\tilde{\gamma}_0-\cdots-\tilde{\gamma}_j$, $\tilde{\beta} = \beta-\tilde{\delta}-\gamma_0-\cdots-\gamma_j$. In fact, the map $(\alpha,\beta,\delta,j,\gamma_0,\gamma_1,\dots,\gamma_j) \to (\tilde{\alpha},\tilde{\beta},\delta,j,\gamma_0,\gamma_1,\dots,\gamma_j)$ defined in this way is not injective, because of the presence of $\tilde{\delta},\tilde{\gamma}_0,\dots,\tilde{\gamma}_j$ (of course, one should think of $\tilde{\delta}$ as a function of $\alpha,\beta,\delta$, and to every $\tilde{\gamma}_j$, $j \ge 0$, as a function of $\alpha,\beta,\delta,\gamma_k$, $k \le j$). Anyhow, since $|\tilde{\delta}| \le |\delta|$ and $|\tilde{\gamma}_j| = |\gamma_j|$, the number of pre-images of any given point is at most $2^{|\delta|+|\gamma_0|+\ldots+|\gamma_j|+d(j+2)}$. Hence we deduce from (3.12) that, if $\varepsilon$ is small enough, then

$$(3.13)\quad \sum_{\substack{|\alpha|+|\beta|\le N\\ |\beta|\le|\alpha|}} \frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|E[P,x^\beta\partial^\alpha]u\|_{L^2} \le$$

$$2^dC\sum_{\substack{|\tilde\alpha|+|\tilde\beta|\le N-1\\ |\tilde\beta|\le|\tilde\alpha|}} \frac{\varepsilon^{|\tilde\alpha|+|\tilde\beta|}}{|\tilde\alpha|!}\|x^{\tilde\beta}\partial^{\tilde\alpha}u\|_{L^2}\sum_{j=0}^r 2^{d(j+1)}\sum_\delta \sum_{\substack{\gamma_1\ne0,\dots,\gamma_j\ne0\\ \gamma_0\colon(\delta,\gamma_0)\ne(0,0)}}(2C\varepsilon)^{|\delta|+|\gamma_0+\gamma_1+\dots+\gamma_j|}$$

$$\le S^\varepsilon_{N-1}[u]\sum_{j=0}^r(C'\varepsilon)^{j+1}\le C''\varepsilon S^\varepsilon_{N-1}[u].$$

The proposition is proved. □

End of the proof of Theorem 1. It follows from the equation $Pu=f$ that, for $\alpha,\beta\in\mathbb{N}^d$, $\varepsilon>0$,

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}x^\beta\partial^\alpha Pu=\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}x^\beta\partial^\alpha f.$$

Introducing commutators we get

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}P(x^\beta\partial^\alpha u)=\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}[P,x^\beta\partial^\alpha]u+\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}x^\beta\partial^\alpha f.$$

We now apply the parametrix $E$ of $P$ to both sides. With $R=EP-I\in \mathrm{OP}G^{-1,-1}(\mathbb{R}^d)$ we obtain

$$\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}x^\beta\partial^\alpha u=-\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}R(x^\beta\partial^\alpha u)+\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}E[P,x^\beta\partial^\alpha]u+\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}E(x^\beta\partial^\alpha f),$$

Taking the $L^2$ norms and summing over $|\alpha|+|\beta|\le N$, $|\beta|\le|\alpha|$, give

$$(3.14)\quad S^{s,\varepsilon}_N[u]\ \le\ \|u\|_{L^2}+\sum_{\substack{0<|\alpha|+|\beta|\le N\\ |\beta|\le|\alpha|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|R(x^\beta\partial^\alpha u)\|_{L^2}$$

$$+\sum_{\substack{0<|\alpha|+|\beta|\le N\\ |\beta|\le|\alpha|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|E[P,x^\beta\partial^\alpha]u\|_{L^2}$$

$$+\sum_{\substack{0<|\alpha|+|\beta|\le N\\ |\beta|\le|\alpha|}}\frac{\varepsilon^{|\alpha|+|\beta|}}{|\alpha|!}\|E(x^\beta\partial^\alpha f)\|_{L^2}$$

The second and the third term in the right-hand side of (3.14) can be estimated using Propositions 4 and 5 while the term containing $f$ is clearly dominated by $S^{\varepsilon}_{\infty}[f]$. In conclusion, for $\varepsilon$ small enough, we obtain for some $C > 0$ independent of $N$:

$$S^{s,\varepsilon}_N[u] \leq \|u\|_{L^2} + CS^{\varepsilon}_{\infty}[f] + C\varepsilon S^{\varepsilon}_{N-1}[u]$$

which implies that $S^{\varepsilon}_{\infty}[u] < \infty$ for $\varepsilon$ small enough. Then $u \in \mathcal{BH}_{sect}(\mathbb{R}^d)$ by (2.13). □

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