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# HYPERBOLIC DOUBLE-COMPLEX LAPLACE OPERATOR

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*Dedicated to the 65 years Anniversary of Professor Petar Popivanov*

Abstract. In this paper is introduced the hyperbolic double-complex Laplace operator. The hyperbolic decomplexification of the hyperbolic double-complex Laplace operator and its characteristic set is found. The exponential eigenfunctions of the zero eigenvalue of the hyperbolic double-complex Laplace operator are found as well.

**1. Hyperbolic complex and hyperbolic double-complex numbers.** Let us recall two basic definitions.

**Definition 1.** *The elements of the commutative, associative algebra with zero divisors*

$$\tilde{\mathbf{C}} := \{x + \mathbf{j}y = (x, y) : \ \mathbf{j}^2 = 1, \ x, y, \in \mathbf{R}\}.$$

*are called* hyperbolic complex numbers.

These numbers are used in geometry, mechanics, physics, etc. For more details one may consult, for instance, the book of I. Yaglom [3], where hyperbolic complex numbers are called double numbers.

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The addition in $\tilde{\mathbf{C}}$ is componentwise, and the multiplication is given by

$$(x_1, y_1).(x_2, y_2) := (x_1x_2 + y_1y_2, x_1y_2 + y_1x_2).$$

The multiplication by a scalar, that is a real number, is defined by $\lambda(x, y) := (\lambda x, \lambda y)$.

The numbers $(x, x)$ and $(x, -x)$ are zero divisors, since $(x^2 - y^2, 0) = (x, y).(x, -y)$ and therefore $(x, x)(x, -x) = (x^2 - x^2, 0) = (0, 0)$. *Conjugate number* of the hyperbolic complex number $x + \mathbf{j}y$ is called the hyperbolic complex number $x - \mathbf{j}y$.

**Definition 2.** *We call* hyperbolic double-complex numbers *the elements of the fourth dimensional commutative, associative hyperbolic double-complex algebra*

$$D\tilde{\mathbf{C}}_2 := \{X = x_0 + jx_1 + j^2x_2 + j^3x_3 = Z + jW,\ Z = x_0 + j^2x_2,\ W = x_1 + j^2x_3\}$$

*where $j$ is a symbol such that $j^4 = +1$ and $j^2 = \mathbf{j}$; $x_k \in \mathbf{R}$ for $k = 0, 1, 2, 3$.*

The numbers $Z$ and $W$ are hyperbolic complex numbers. The algebra $D\tilde{\mathbf{C}}_2$ inherites from $\tilde{\mathbf{C}}$ the componentwise addition and the multiplication with a real scalar $\lambda \in \mathbf{R}$. The multiplication of two hyperbolic double-complex numbers is defined in an obvious way using the identities for the degrees of $j$ and the distributive rule.

Algebra $D\tilde{\mathbf{C}}_2$ has zero divisors as well. Indeed, for example, the numbers $X(1 - j^2)$ and $Y(1 + j^2)$ satisfy $X(1 - j^2)Y(1 + j^2) = XY(1 - j^4) = 0$.

**2. Holomorphic hyperbolic double-complex functions.** Let us consider a function $f : U \to D\tilde{\mathbf{C}}_\mathbf{2}$, where $U$ is a open subset of the algebra $D\tilde{\mathbf{C}}_\mathbf{2}$. Then $f(X) = f_0(Z, W) + jf_1(Z, W)$, $f_0, f_1 : U \to \tilde{\mathbf{C}}$. The function $f_0$ is called an even part, and the function $f_1$ is called an odd part of the hyperbolic double-complex function $f$.

For the introduction of *formal hyperbolic complex derivatives* we use the partial derivatives with respect to the real variables $x_0, x_1, x_2, x_3$. We denote as follows:

$$\frac{\partial}{\partial Z} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} + j^2\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial W} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + j^2\frac{\partial}{\partial x_3}\right). \tag{1}$$

The following examples are important for the formal calculus using so defined formal hyperbolic complex derivatives 1.)$\dfrac{\partial Z}{\partial Z} = 1$, 2.)$\dfrac{\partial Z}{\partial W} = 0$, 3.)$\dfrac{\partial W}{\partial W} = 1$ and 4.)$\dfrac{\partial W}{\partial Z} = 0$.

We consider also the following formal derivatives

$$\frac{\partial}{\partial X} = \frac{1}{2}\left(\frac{\partial}{\partial Z} + \frac{1}{j}\frac{\partial}{\partial W}\right), \quad \frac{\partial}{\partial X^*} = \frac{1}{2}\left(\frac{\partial}{\partial Z} - \frac{1}{j}\frac{\partial}{\partial W}\right). \tag{2}$$

The formal derivatives defined in (2) are called *formal hyperbolic double-complex derivatives.*

**Definition 3.** *We say that the hyperbolic double-complex function* $f$ *is holomorphic hyperbolic double-complex functions if and only if*

$$\frac{\partial f}{\partial X^*} = \frac{1}{2}\left(\frac{\partial f}{\partial Z} - \frac{1}{j}\frac{\partial f}{\partial W}\right) = 0, \tag{3}$$

**Example 1.** The hyperbolic double-complex function $X = Z + jW$ is a holomorphic hyperbolic double-complex function, because

$$\frac{\partial(Z + jW)}{\partial Z} - \frac{1}{j}\frac{\partial(Z + jW)}{\partial W} = 1 - 1 = 0.$$

*Exponential hyperbolic double-complex function* $e^X$ is defined by the power series $e^X = \sum_{k=0}^{\infty} \dfrac{X^k}{k!}$. The following identity for the exponential function is fulfilled

$$e^X = e^{x_0}(\cosh x_2 + j^2 \sinh x_2)\times$$

$$\times((1 + j^2)\cosh x_1 + (j + j^3)\sinh x_1 + (1 - j^2)\cos x_1 + (j - j^3)\sin x_1)\times$$

$$\times((1 + j^2)\cosh x_3 + (j^3 + j)\sinh x_3 + (1 - j^2)\cos x_3 + (j^3 - j)\sin x_3) =$$

$$= e^{x_0}(\cosh x_2 + j^2 \sinh x_2)\left[(1 + j^2)(\cosh x_1 + j\sinh x_1)(\cosh x_3 + j\sinh x_3) +\right.$$

$$\left. + (1 - j^2)(\cos x_1 - j\sin x_1)(\cos x_3 - j\sin x_3)\right].$$

**Example 2.** The exponential hyperbolic double-complex function $e^X$ and the following composite functions

$$e^{p(1+j+j^2+j^3)(Z+W)}, \quad e^{p(1+j^2)(jZ+W)} \quad \text{and} \quad e^{q(Z+jW)},$$

where $p$, $q$ are hyperbolic double-complex numbers and $Z$, $W$ are hyperbolic complex variables, are holomorphic hyperbolic double-complex functions.

On the other hand, it is easy to check that the composite exponential hyperbolic double complex function

$$e^{p(1-j+j^2-j^3)(Z+W)}, \quad p \neq 0$$

is not a holomorphic hyperbolic double-complex function. This follows from the definition, applying the rules for computation of the formal hyperbolic complex derivatives, which are similar to those for the partial derivatives.

**Theorem 1 (see [1], [2]).** *The function $f = f_0 + jf_1$ is holomorphic hyperbolic double-complex function if and only if the Cauchy-Riemann type system*

$$\frac{\partial f_0}{\partial Z} = \frac{\partial f_1}{\partial W}, \quad \frac{\partial f_0}{\partial W} = j^2 \frac{\partial f_1}{\partial Z} \tag{4}$$

*is satisfied by the hyperbolic complex functions $f_0$ and $f_1$.*

**3. Hyperbolic double-complex Laplace operator.** The even part $f_0$ and the odd part $f_1$ of the holomorphic hyperbolic double-complex function $f$ satisfy two second order partial differential equations with constant hyperbolic complex coefficients. By the Cauchy-Riemann type system (4) we derive the following system of equations for $f_0$ and $f_1$:

$$\frac{\partial^2 f_0}{\partial Z \partial W} = \frac{\partial^2 f_1}{\partial W^2}, \quad \frac{\partial^2 f_0}{\partial W \partial Z} = j^2 \frac{\partial^2 f_1}{\partial Z^2}, \quad \frac{\partial^2 f_1}{\partial Z \partial W} = \frac{\partial^2 f_0}{\partial Z^2}, \quad \frac{\partial^2 f_1}{\partial W \partial Z} = j^2 \frac{\partial^2 f_0}{\partial W^2}.$$

Therefore the functions $f_0$ and $f_1$ satisfy the equations $\frac{\partial^2 f_1}{\partial Z^2} = \mathbf{j} \frac{\partial^2 f_1}{\partial W^2}$ and, respectively, $\frac{\partial^2 f_0}{\partial Z^2} = \mathbf{j} \frac{\partial^2 f_0}{\partial W^2}$. The second order partial differential operator with hyperbolic complex coefficient $\mathbf{j}$ and hyperbolic complex variables $Z$ and $W$

$$\Delta_{hdc} = \frac{\partial^2}{\partial Z^2} - \mathbf{j} \frac{\partial^2}{\partial W^2} = \tag{5}$$

$$= \frac{1}{4}\left(\frac{\partial}{\partial x_0} + j^2\frac{\partial}{\partial x_2}\right)^2 - j^2\frac{1}{4}\left(\frac{\partial}{\partial x_1} + j^2\frac{\partial}{\partial x_3}\right)^2 =$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_2^2} - 2\frac{\partial^2}{\partial x_1 \partial x_3} - j^2\frac{\partial^2}{\partial x_1^2} - j^2\frac{\partial^2}{\partial x_3^2} + 2j^2\frac{\partial^2}{\partial x_0 \partial x_2}\right)$$

is called *a hyperbolic double-complex Laplace operator.*

The hyperbolic complex-valued function $f$, solution of the equation $\Delta_{hdc} f = 0$ is called harmonic hyperbolic complex function. Then the even and the odd parts $f_0$ and $f_1$ of a holomorphic hyperbolic double-complex function are harmonic hyperbolic complex functions.

In order to write the hyperbolic complex symbol of the hyperbolic double-complex Laplace operator, we replace in (5) the variable $\partial/\partial x_0$ by $\xi_0$, $\partial/\partial x_1$ by $\xi_1$, $\partial/\partial x_2$ by $\xi_2$ and $\partial/\partial x_3$ by $\xi_3$, respectively.

This way we obtain the quadratic form

$$B(\xi_0, \xi_1, \xi_2, \xi_3) = \frac{1}{4}(\xi_0^2 + \xi_2^2 - 2\xi_1\xi_3 - j^2\xi_1^2 - j^2\xi_3^2 + 2j^2\xi_0\xi_2),$$

with the following matrix

$$Q = \frac{1}{4}\begin{pmatrix} 1 & 0 & j^2 & 0 \\ 0 & -j^2 & 0 & -1 \\ j^2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -j^2 \end{pmatrix}.$$

The quadratic form $B(\xi_0, \xi_1, \xi_2, \xi_3)$ is called a hyperbolic complex symbol of the operator $\Delta_{hdc}$.

**4. The characteristic set of the hyperbolic double-complex Laplace operator.** In order to find the characteristic set $\Sigma$ of the hyperbolic double-complex Laplace operator we solve the following equation of second order with hyperbolic complex coefficients

$$(\xi_0 + j^2\xi_2)^2 - j^2(\xi_1 + j^2\xi_3)^2 = 0 \iff \xi_0^2 + \xi_2^2 - j^2\xi_1^2 - j^2\xi_3^2 + 2j^2\xi_0\xi_2 - 2\xi_1\xi_3 = 0,$$

or the equivalent system of two real quadratic equations

$$\begin{aligned} &\xi_0^2 + \xi_2^2 - 2\xi_1\xi_3 = 0, \\ &\xi_1^2 + \xi_3^2 - 2\xi_0\xi_2 = 0. \end{aligned} \tag{6}$$

It is true that $\Sigma$ is a subset in $T^*(\mathbf{R}^4)$, but it is remarkable that $\Sigma$ contains two hyper-planes. Indeed, the sum of the equations in the system (6) is

$$(\xi_0 - \xi_2)^2 + (\xi_1 - \xi_3)^2 = 0 \quad \Longleftrightarrow \quad \xi_0 = \xi_2 \quad \text{and} \quad \xi_1 = \xi_3, \tag{7}$$

as all variables $\xi_0, \xi_1, \xi_2, \xi_3$ are real.

The difference between the first and the second equation in (6) is

$$(\xi_0 + \xi_2)^2 - (\xi_1 + \xi_3)^2 = 0. \tag{8}$$

which gives the equations $(\xi_0 + \xi_2) = \pm(\xi_1 + \xi_3)$.

So we obtain parts $\Sigma_1$ and $\Sigma_2$ of the characteristic $\Sigma \supset (\Sigma_1 \cup \Sigma_2)$ as follows

$$\Sigma_1 = \{(\xi_0, \xi_1, \xi_2, \xi_3) \in T_0^* D\tilde{\mathbf{C}}_2 \setminus \{0\} : \xi_0 = \xi_1\}$$

$$\Sigma_2 = \{(\xi_0, \xi_1, \xi_2, \xi_3) \in T_0^* D\tilde{\mathbf{C}}_2 \setminus \{0\} : \xi_0 = -\xi_1\}$$

which are two transversal hyper-planes in the cotangent bundle $T_0^* D\tilde{\mathbf{C}}$ of the algebra of the hyperbolic double-complex numbers $D\tilde{\mathbf{C}}$.

**5. Hyperbolic decomplexification of the hyperbolic double-complex Laplace operator.** Let us present the harmonic hyperbolic complex-valued functions $f_0$ and $f_1$ by real functions $g_0, g_1, g_2$ and $g_3$ as follows: $f_0 = g_0 + j^2 g_2$, $f_1 = g_1 + j^2 g_3$. Then the functions $g_k$, $k = 0, 1, 2, 3$ satisfy the following equation:

$$4\Delta_{hdc}(f_0 + jf_1) = 4\Delta_{hdc}(g_0 + jg_1 + j^2 g_2 + j^3 g_3) =$$

$$= \frac{\partial^2(g_0 + jg_1 + j^2 g_2 + j^3 g_3)}{\partial x_0^2} + \frac{\partial^2(g_0 + jg_1 + j^2 g_2 + j^3 g_3)}{\partial x_2^2} -$$

$$-2\frac{\partial^2(g_0 + jg_1 + j^2 g_2 + j^3 g_3)}{\partial x_1 \partial x_3} - j^2 \frac{\partial^2(g_0 + jg_1 + j^2 g_2 + j^3 g_3)}{\partial x_1^2} -$$

$$-j^2 \frac{\partial^2(g_0 + jg_1 + j^2 g_2 + j^3 g_3)}{\partial x_3^2} + 2j^2 \frac{\partial^2(g_0 + jg_1 + j^2 g_2 + j^3 g_3)}{\partial x_0 \partial x_2} =$$

$$= \frac{\partial^2 g_0}{\partial x_0^2} + \frac{\partial^2 g_0}{\partial x_2^2} - 2\frac{\partial^2 g_0}{\partial x_1 \partial x_3} - \frac{\partial^2 g_2}{\partial x_1^2} - \frac{\partial^2 g_2}{\partial x_3^2} + 2\frac{\partial^2 g_2}{\partial x_0 \partial x_2} +$$

$$+j\left(\frac{\partial^2 g_1}{\partial x_0^2} + \frac{\partial^2 g_1}{\partial x_2^2} - 2\frac{\partial^2 g_1}{\partial x_1 \partial x_3} - \frac{\partial^2 g_3}{\partial x_1^2} - \frac{\partial^2 g_3}{\partial x_3^2} + 2\frac{\partial^2 g_3}{\partial x_0 \partial x_2}\right) +$$

$$+j^2\left(\frac{\partial^2 g_2}{\partial x_0^2} + \frac{\partial^2 g_2}{\partial x_2^2} - 2\frac{\partial^2 g_2}{\partial x_1 \partial x_3} - \frac{\partial^2 g_0}{\partial x_1^2} - \frac{\partial^2 g_0}{\partial x_3^2} + 2\frac{\partial^2 g_0}{\partial x_0 \partial x_2}\right) +$$

$$+j^3\left(\frac{\partial^2 g_3}{\partial x_0^2} + \frac{\partial^2 g_3}{\partial x_2^2} - 2\frac{\partial^2 g_3}{\partial x_1 \partial x_3} - \frac{\partial^2 g_1}{\partial x_1^2} - \frac{\partial^2 g_1}{\partial x_3^2} + 2\frac{\partial^2 g_1}{\partial x_0 \partial x_2}\right) = 0.$$

It is equivalent to the following system of two equations for two couples of real-valued functions $g_0, g_2$, and $g_1, g_3$, namely

$$\left|\begin{array}{l} \frac{\partial^2 g_0}{\partial x_0^2} + \frac{\partial^2 g_0}{\partial x_2^2} - 2\frac{\partial^2 g_0}{\partial x_1 \partial x_3} - \frac{\partial^2 g_2}{\partial x_1^2} - \frac{\partial^2 g_2}{\partial x_3^2} + 2\frac{\partial^2 g_2}{\partial x_0 \partial x_2} = 0 \\ \frac{\partial^2 g_0}{\partial x_1^2} + \frac{\partial^2 g_0}{\partial x_3^2} - 2\frac{\partial^2 g_0}{\partial x_0 \partial x_2} - \frac{\partial^2 g_2}{\partial x_0^2} - \frac{\partial^2 g_2}{\partial x_2^2} + 2\frac{\partial^2 g_2}{\partial x_1 \partial x_3} = 0 \end{array}\right.$$

and the same system for the couple $g_1, g_3$.

Denoting $A(g) = \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial x_2^2} - 2\frac{\partial^2 g}{\partial x_1 \partial x_3}$ and $B(g) = \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_3^2} - 2\frac{\partial^2 g}{\partial x_0 \partial x_2}$ the above system can be rewritten as

$$(A) \qquad \left|\begin{array}{l} A(g_0) - B(g_2) = 0 \\ B(g_0) - A(g_2) = 0. \end{array}\right. \qquad \left|\begin{array}{l} A(g_1) - B(g_3) = 0 \\ B(g_1) - A(g_3) = 0. \end{array}\right.$$

The symbol of this system is the following antisymmetric matrix

$$(B) \qquad \sigma = \begin{pmatrix} \sigma(A) = \xi_0^2 + \xi_2^2 - 2\xi_1\xi_3 & -\sigma(B) = -\xi_1^2 - \xi_3^2 + 2\xi_0\xi_2 \\ \sigma(B) = \xi_1^2 + \xi_3^2 - 2\xi_0\xi_2 & -\sigma(A) = -\xi_0^2 - \xi_2^2 + 2\xi_1\xi_3 \end{pmatrix},$$

where $(\xi_0, \xi_1, \xi_2, \xi_3) \in T^*(\mathbf{R}^4) \setminus \{0\}$.

We see that $(A^2-B^2)g_0 = A(A(g_0)) - B(B(g_0)) = A(B(g_2)) - B(A(g_2)) = 0$ as the operators $A$ and $B$ have constant coefficients. So the system (A) implies the following partial differential equation of fourth order with one unknown function:

$$(A^2-B^2)g_k = 0 \quad \text{for } k = 0,1,2,3$$

or in explicit form

$$\left(\frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial x_2^2} - 2\frac{\partial^2 g}{\partial x_1 \partial x_3}\right)^2 - \left(\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_3^2} - 2\frac{\partial^2 g}{\partial x_0 \partial x_2}\right)^2 = 0$$

or

$$\left(\left(\frac{\partial g}{\partial x_0} - \frac{\partial g}{\partial x_2}\right)^2 + \left(\frac{\partial g}{\partial x_1} - \frac{\partial g}{\partial x_3}\right)^2\right)\left(\left(\frac{\partial g}{\partial x_0} + \frac{\partial g}{\partial x_2}\right)^2 - \left(\frac{\partial g}{\partial x_1} + \frac{\partial g}{\partial x_3}\right)^2\right) = 0.$$

for the functions $g = g_0, g_1, g_2, g_3$.

The symbol of the operator $-A^2 + B^2$ has the following form $-(\xi_0^2 + \xi_2^2 - 2\xi_1\xi_3)^2 + (\xi_1^2 + \xi_3^2 - 2\xi_0\xi_2)^2$. The determinant of the matrix (B) is the following one:

$$\det\sigma = \det\begin{pmatrix} \xi_0^2 + \xi_2^2 - 2\xi_1\xi_3 & -\xi_1^2 - \xi_3^2 + 2\xi_0\xi_2 \\ \xi_1^2 + \xi_3^2 - 2\xi_0\xi_2 & -\xi_0^2 - \xi_2^2 + 2\xi_1\xi_3 \end{pmatrix} =$$

$$= -(\xi_0^2 + \xi_2^2 - 2\xi_1\xi_3)^2 + (\xi_1^2 + \xi_3^2 - 2\xi_0\xi_2)^2 =$$

$$= -(\xi_0^2 + \xi_2^2 - 2\xi_1\xi_3 + \xi_1^2 + \xi_3^2 - 2\xi_0\xi_2)(\xi_0^2 + \xi_2^2 - 2\xi_1\xi_3 - \xi_1^2 - \xi_3^2 + 2\xi_0\xi_2) =$$

$$= \left((\xi_0 - \xi_2)^2 + (\xi_1 - \xi_3)^2\right)\left((\xi_0 + \xi_2)^2 - (\xi_1 + \xi_3)^2\right) =$$

$$= \left((\xi_0 - \xi_2)^2 + (\xi_1 - \xi_3)^2\right)(\xi_0 + \xi_2 + \xi_1 + \xi_3)(\xi_0 + \xi_2 - \xi_1 - \xi_3).$$

Therefore the real solutions of the equation

$$(9)\ \det\sigma = \left((\xi_0 - \xi_2)^2 + (\xi_1 - \xi_3)^2\right)(\xi_0 + \xi_2 + \xi_1 + \xi_3)(\xi_0 + \xi_2 - \xi_1 - \xi_3) = 0,$$

forms the characteristic set of the hyperbolic decomplexification of the hyperbolic double-complex Laplace operator.

This way we may write the characteristic set in an explicit form $\Sigma = \Sigma_1 \cup \Sigma_3 \cup \Sigma_3$ where

$$\Sigma_1 = \{(\xi_0, \xi_1, \xi_2, \xi_3) : \xi_0 + \xi_2 + \xi_1 + \xi_3 = 0\},$$
$$\Sigma_2 = \{(\xi_0, \xi_1, \xi_2, \xi_3) : \xi_0 + \xi_2 - \xi_1 - \xi_3 = 0\}$$

and

$$\Sigma_3 = \{(\xi_0, \xi_1, \xi_2, \xi_3) : \xi_0 = \xi_2, \quad \xi_1 = \xi_3\}, \quad \text{where } (\xi_0, \xi_1, \xi_2, \xi_3) \in T^*(\mathbf{R}^4) \setminus \{0\}.$$

**6. Hyperbolic double-complex exponential eigenfuncions for the zero eigenvalue of the hyperbolic double-complex Laplace operator.** Let us consider the eigenfunction problem for the hyperbolic double-complex Laplace operator $\Delta_{hdc}$, i.e. to solve the equation

$$\Delta_{hdc} F(Z + jW) = \lambda F(Z + jW), \tag{10}$$

where $F(Z + jW) : D\tilde{\mathbf{C}}_2 \to D\tilde{\mathbf{C}}_2$ is a hyperbolic double-complex function of hyperbolic double-complex variable.

Applying the method of separation of variables for the above problem we are looking for the solutions of the form $F(Z + jW) = f(Z).g(W)$, where $f(Z) = f_1(Z) + jf_2(Z)$ and $g(W) = g_1(W) + jg_2(W)$, $f_1, f_2, g_1, g_2 : \tilde{\mathbf{C}} \to \tilde{\mathbf{C}}$ are hyperbolic complex valued functions of one hyperbolic complex variable. Equation (10) becomes

$$g(W)\frac{\partial^2 f(Z)}{\partial Z^2} - \mathbf{j} f(Z)\frac{\partial^2 g(W)}{\partial W^2} = \lambda f(Z) g(W),$$

where $\lambda \in D\tilde{C}_2$. For the zero eigenvalue $\lambda$ we have

$$f''_{ZZ}(Z) = (m_0 + jm_1 + j^2 m_2 + j^3 m_3) f(Z)$$

and

$$g''_{WW}(W) = j^2(m_0 + jm_1 + j^2 m_2 + j^3 m_3) g(W)$$

where $m_0, m_1, m_2, m_3$ are real constants. Solutions of this ODEs are $f(Z) = e^{aZ}$ and $g(W) = e^{bW}$, where $a = a_0 + ja_1 + j^2a_2 + j^3a_3$ and $b = b_0 + jb_1 + j^2b_2 + j^3b_3$, $a_k, b_k \in \mathbf{R}$ for $k = 0, 1, 2, 3$, are hyperbolic double-complex numbers such that

$$a^2 = j^2b^2 = m_0 + jm_1 + j^2m_2 + j^3m_3. \tag{11}$$

If $m_0 + m_2 \geq |m_1 + m_3|$ then

$$a_0 = \frac{\varepsilon_1}{4}\sqrt{m_0 + m_1 + m_2 + m_3} + \frac{\varepsilon_2}{4}\sqrt{m_0 - m_1 + m_2 - m_3} +$$

$$+ \frac{\varepsilon}{2\sqrt{2}}\sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} + m_0 - m_2},$$

$$a_1 = \frac{\varepsilon_1}{4}\sqrt{m_0 + m_1 + m_2 + m_3} - \frac{\varepsilon_2}{4}\sqrt{m_0 - m_1 + m_2 - m_3} +$$

$$+ \frac{\varepsilon \ \mathrm{sign}(m_1 - m_3)}{2\sqrt{2}}\sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} - (m_0 - m_2)},$$

$$a_2 = \frac{\varepsilon_1}{4}\sqrt{m_0 + m_1 + m_2 + m_3} + \frac{\varepsilon_2}{4}\sqrt{m_0 - m_1 + m_2 - m_3} -$$

$$- \frac{\varepsilon}{2\sqrt{2}}\sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} + m_0 - m_2},$$

$$a_3 = \frac{\varepsilon_1}{4}\sqrt{m_0 + m_1 + m_2 + m_3} - \frac{\varepsilon_2}{4}\sqrt{m_0 - m_1 + m_2 - m_3} -$$

$$- \frac{\varepsilon \ \mathrm{sign}(m_1 - m_3)}{2\sqrt{2}}\sqrt{\sqrt{(m_0 - m_2)^2 + (m_1 - m_3)^2} - (m_0 - m_2)},$$

where the constants $\varepsilon, \varepsilon_1$ and $\varepsilon_2$ are equal to $\pm 1$.

Analogously $(b_0 + jb_1 + j^2b_2 + j^3b_3)^2 = m_2 + jm_3 + j^2m_0 + j^3m_1$ and

$$b_0 = \frac{\varepsilon_1}{4}\sqrt{m_2 + m_3 + m_0 + m_1} + \frac{\varepsilon_2}{4}\sqrt{m_2 - m_3 + m_0 - m_1} +$$

$$+\frac{\varepsilon\ \mathrm{sign}(m_1-m_3)}{2\sqrt{2}}\sqrt{\sqrt{(m_2-m_0)^2+(m_3-m_1)^2}+m_2-m_0},$$

$$b_1=\frac{\varepsilon_1}{4}\sqrt{m_2+m_3+m_0+m_1}-\frac{\varepsilon_2}{4}\sqrt{m_2-m_3+m_0-m_1}+$$

$$+\frac{\varepsilon}{2\sqrt{2}}\sqrt{\sqrt{(m_2-m_0)^2+(m_3-m_1)^2}-(m_2-m_0)},$$

$$b_2=\frac{\varepsilon_1}{4}\sqrt{m_2+m_3+m_0+m_1}+\frac{\varepsilon_2}{4}\sqrt{m_2-m_3+m_0-m_1}-$$

$$-\frac{\varepsilon\ \mathrm{sign}(m_1-m_3)}{2\sqrt{2}}\sqrt{\sqrt{(m_2-m_0)^2+(m_3-m_1)^2}+m_2-m_0},$$

$$b_3=\frac{\varepsilon_1}{4}\sqrt{m_2+m_3+m_0+m_1}-\frac{\varepsilon_2}{4}\sqrt{m_2-m_3+m_0-m_1}-$$

$$-\frac{\varepsilon}{2\sqrt{2}}\sqrt{\sqrt{(m_2-m_0)^2+(m_3-m_1)^2}-(m_2-m_0)},$$

where the constants $\varepsilon, \varepsilon_1$ and $\varepsilon_2$ are equal to $\pm 1$.

So for the eigenfunctions, corresponding to the zero eigenvalue, we obtain that $F(Z+jW)=e^{aZ+bW}=e^{a(Z+j^2W)}$ depends on hyperbolic double-complex parameter $m=m_0+jm_1+j^2m_2+m_3$ and are the following

$$F_{\varepsilon,\varepsilon_1,\varepsilon_2,m}(Z,W)=e^{\frac{\varepsilon_1(1+j+j^2+j^3)}{4}\sqrt{m_0+m_1+m_2+m_3}(Z+W)}\times \tag{12}$$

$$\times\ e^{\frac{\varepsilon(1-j^2)}{2\sqrt{2}}\sqrt{\sqrt{(m_0-m_2)^2+(m_1-m_3)^2}+m_0-m_2}(Z+jW)}\times$$

$$\times\ e^{\frac{\varepsilon\ \mathrm{sign}(m_1-m_3)(1-j^2)}{2\sqrt{2}}\sqrt{\sqrt{(m_0-m_2)^2+(m_1-m_3)^2}-m_0+m_2}(jZ+W)}\times$$

$$\times\ e^{\frac{\varepsilon_2(1-j+j^2-j^3)}{4}\sqrt{m_0-m_1+m_2-m_3}(Z+W)}.$$

Here $Z$, $W$ are hyperbolic complex numbers, $\varepsilon, \varepsilon_1$, $\varepsilon_2$ are equal to $\pm 1$ and $m_0, m_1, m_2$ and $m_3$ are real numbers such that $m_0 + m_2 \geq |m_1 + m_3|$ holds.

Note that the numbers $1 + j + j^2 + j^3$, and $1 - j^2$ in the formulae above are zero divisors in the algebra $D\tilde{\mathbf{C}}_2$.

The first three of the composite exponential hyperbolic double-complex functions in the product (12) are holomorphic hyperbolic double-complex functions and the fourth one is not a holomorphic hyperbolic double-complex function.

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