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# CANONICALLY CONJUGATE VARIABLES FOR THE $\mu$CH EQUATION

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Abstract. We consider the $\mu$CH equation which arises as an asymptotic rotator equation in a liquid crystal with a preferred direction if one takes into account the reciprocal action of dipoles on themselves. This equation is closely related to the periodic Camassa–Holm and the Hunter-Saxton equations. The $\mu$CH equation is also integrable and bi-Hamiltonian, that is, it is Hamiltonian with respect to two compatible Poisson brackets. We give a set of conjugated variables for both brackets.

**1. Introduction.** The $\mu$CH equation was derived recently in [1, 2] as

$$-u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \tag{1.1}$$

where $u(t,x)$ is a spatially periodic real-valued function of time variable $t$ and space variable $x \in S^1 = [0,1)$, $\mu(u) = \int_0^1 u dx$. This equation is closely related to the Camassa-Holm equation (CH)

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \tag{1.2}$$

and the Hunter-Saxton equation (HS)

$$-u_{txx} = 2u_xu_{xx} + uu_{xxx}. \tag{1.3}$$

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In order to keep certain symmetry and analogy with CH, one can write the equation (1.1) in the form (see also [3])

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx}. \tag{1.4}$$

Note that $\mu(u_t) = 0$ in the periodic case. By introducing $m = \mathcal{A}u = \mu(u) - u_{xx}$, the equation (1.4) becomes

$$m_t + u m_x + 2 m u_x = 0, \quad m = \mu(u) - u_{xx}. \tag{1.5}$$

The $\mu$CH equation can be interpreted as an equation in a liquid crystal with a preferred direction if one takes into account the reciprocal action of dipoles on themselves [1, 2].

Similar to its relatives (1.2), (1.3) the $\mu$CH equation is integrable. The bi-Hamiltonian form of (1.5) is

$$m_t = -\mathcal{B}^1 \frac{\delta H_2[m]}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1[m]}{\delta m}, \tag{1.6}$$

where $\mathcal{B}^1 = \frac{1}{2}\partial\mathcal{A} = -\frac{1}{2}\partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators and the corresponding Hamiltonians are

$$H_1[m] = \frac{1}{2}\int m u dx, \quad H_2[m] = \int \left(2\mu(u)u^2 + u u_x^2\right) dx. \tag{1.7}$$

There exists an infinite sequence of conservation laws $H_n[m]$ $n = 0, \pm 1, \pm 2, \ldots$ such that (see [1] for some representatives)

$$\mathcal{B}^1 \frac{\delta H_n[m]}{\delta m} = \mathcal{B}^2 \frac{\delta H_{n-1}[m]}{\delta m}. \tag{1.8}$$

The $\mu$CH equation can be written as

$$m_t = -\{m, H_2\}_1 = -\{m, H_1\}_2, \tag{1.9}$$

where the two compatible Poisson brackets are

$$\{f,g\}_1 = \int_0^1 \frac{\delta f}{\delta m}\mathcal{B}^1 \frac{\delta g}{\delta m} dx, \quad \{f,g\}_2 = \int_0^1 \frac{\delta f}{\delta m}\mathcal{B}^2 \frac{\delta g}{\delta m} dx. \tag{1.10}$$

Note that $H_0 = \int m dx = \int \mu(u) dx = \mu(u)$ is a Casimir for the first bracket and $H_{-1} = \int \sqrt{m} dx$ is a Casimir for the second bracket.

The equation (1.1) can be expressed as a condition of compatibility between

$$\psi_{xx} = -\lambda m \psi \tag{1.11}$$

and

$$\psi_t = -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}u_x\psi, \tag{1.12}$$

that is, $(\psi_{xx})_t = (\psi_t)_{xx}$, where $\lambda$ is a spectral parameter.

The $\mu$CH equation is an Euler equation on the diffeomorphism group of the circle corresponding to a natural right invariant Sobolev metric. It is also well-posed (see [1]). This equation enjoys other geometric descriptions [3], for example it is geometrically integrable. Moreover, its Kuperschidt deformation is also geometrically integrable [4]. Let us give also another important ingredient for the integrable PDEs, the so called zero curvature representation

$$X_t - T_x + [X, T] = 0,$$

where the matrices $X$ and $T$ for the $\mu$CH equation are the following

$$X = -\begin{pmatrix} \lambda & -1 \\ \lambda m + \lambda^2 & -\lambda \end{pmatrix},$$

$$T = \frac{1}{2}\begin{pmatrix} 1 + 2\lambda u + u_x & -2u - \dfrac{1}{\lambda} \\ 2\lambda\left(u_x + um + \lambda u + \dfrac{1}{2}\right) + \mu(u) & -(1 + 2\lambda u + u_x) \end{pmatrix}.$$

In this paper, canonically conjugated variables with respect to the both brackets (1.10) are constructed. We follow Penskoi [5], where the conjugated variables are obtained for the periodic CH equation (1.2), although many essential facts can be found in Flaschka, McLaughlin [6], where the conjugated variables for the periodic KdV equation are constructed.

The paper is organized as follows. In section 2 we summarize some results for the spectral problem and formulate the main result. Then in section 3 we prove it. In section 4 we discuss the motion of the auxiliary spectrum.

**2. Spectral problem.** In what follows we assume that $m(0) > 0$. It is shown in [1] that then $m(x) > 0$ as long as $u(x,t)$ exists.

The disposition of the spectra is similar to the cases of the Korteweg de Vries equation and the CH equation. For instance, if $m \in C^2[0,1]$ with the above assumption, the Liouville transformation $\psi \to m^{-1/4}\Phi(\bar{x}), \bar{x} = \int_0^x \sqrt{m}d\tau$ brings (1.11) to a Hill's equation

$$-\frac{d^2\Phi}{d\bar{x}^2} + \left(\frac{m_{xx}}{4m^2} - \frac{5m_x^2}{16m^3}\right)\Phi = \lambda\Phi.$$

However, we shall proceed as in [9, 10] or [11].

Consider the spectral problem (1.11). Recall that $u(x+1) = u(x)$ and $m(x+1) = m(x)$.

Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be a fundamental system of solutions of (1.11) subjected to the normalization

$$y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0,$$
$$y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1.$$

Every solution $\psi$ of (1.11) can be expressed as a linear combination of $y_{1,2}$:

$$\psi(x,\lambda) = \psi(0,\lambda)y_1(x,\lambda) + \psi'(0,\lambda)y_2(x,\lambda). \tag{2.1}$$

This produces

$$\begin{pmatrix} \psi(x,\lambda) \\ \psi'(x,\lambda) \end{pmatrix} = \begin{pmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ y_1'(x,\lambda) & y_2'(x,\lambda) \end{pmatrix} \begin{pmatrix} \psi(0,\lambda) \\ \psi'(0,\lambda) \end{pmatrix}. \tag{2.2}$$

Denote the matrix in the last formula by $U(x,\lambda)$. From the definition of $y_{1,2}$ we have that $\det U(x,\lambda) = Wr(y_1,y_2) = Wr(0) = 1$. Let us define also the discriminant

$$\Delta(\lambda) = \frac{1}{2}\operatorname{tr} U(1,\lambda) = \frac{1}{2}\left(y_1(1,\lambda) + y_2'(1,\lambda)\right). \tag{2.3}$$

We first consider (1.11) conditioned by the periodic boundary conditions

$$\psi(0) = \psi(1), \quad \psi'(0) = \psi'(1).$$

There exists an infinite sequence of eigenvalues

$$\lambda_0^+ < \lambda_1^+ \le \lambda_2^+ < \lambda_3^+ \ldots, \quad \lambda_n^+ \to \infty \text{ as } n \to \infty.$$

Next we consider the antiperiodic eigenvalue problem, that is, the boundary conditions for (1.11) are of the form

$$\psi(1) = -\psi(0), \quad \psi'(1) = -\psi'(0).$$

The corresponding sequence of eigenvalues is

$$\lambda_1^- \le \lambda_2^- < \lambda_3^- \le \lambda_4^- \ldots, \quad \lambda_n^- \to \infty \text{ as } n \to \infty.$$

The quantities $\lambda_m^\pm$ are the roots of $\Delta(\lambda) = \pm 1$, $\lambda_0^+$ is always simple. It is known that

$$\lambda_0^+ < \lambda_1^- \le \lambda_2^- < \lambda_1^+ \le \lambda_2^+ < \lambda_3^- \le \lambda_4^- < \lambda_3^+ \ldots.$$

The intervals

$$(\lambda_0^+,\lambda_1^-),(\lambda_2^-,\lambda_1^+),(\lambda_2^+,\lambda_3^-),\ldots$$

are called intervals of stability. Similarly we can name the other intervals – the intervals of instability or gaps. Some of intervals of instability may disappear – $(-\infty,\lambda_0)$ always is present. Trivial arguments show that in our case $\lambda_0^+=0$ and for $\lambda\in(-\infty,0)$ the solutions of (1.11) are unbounded.

Recall that a solution of (1.11) is said to be a Floquet solution if there exists a number $\rho$ called a Floquet multiplier satisfying

$$\psi(x+1,\lambda)=\rho\psi(x,\lambda).$$

It is straightforward from (2.2) that a Floquet solution and the corresponding $\rho$ are an eigenvector and an eigenvalue of $U(x,\lambda)$.

Hence, $\rho$ is obtained from

$$\rho^2-2\Delta(\lambda)\rho+1=0. \tag{2.4}$$

Now let us consider the auxiliary eigenvalues $\mu_j$ defined as solutions of the equation $y_2(1,\mu_j)=0$. Since $m(x)$ is periodic, $y_2(x+1,\mu_j)$ is a solution of (1.11) for $\lambda=\mu_j$. Due to (2.1) we have

$$y_2(x+1,\mu_j)=y_2'(1,\mu_j)y_2(x,\mu_j),$$

that is, $y_2(x,\mu_j)$ is a Floquet solution with $\rho_j=y_2'(1,\mu_j)$. So, we have a root of (2.4) for $\lambda=\mu_j$ namely $\rho_j$. The other root is $\tilde\rho_j=\dfrac{1}{\rho_j}$. Denote by $y(x,\mu_j)$ the corresponding to $\tilde\rho_j$ Floquet solution

$$y(x+1,\mu_j)=\tilde\rho_j y(x,\mu_j). \tag{2.5}$$

Since, $y$ and $y_2$ are linearly independent, we normalize $y$ by $y(0,\mu_j)=1$.

The points of the "auxiliary spectrum" $\mu_j$ must lie in the gaps (see Fig. 1). Indeed, since $Wr(y_1,y_2)=y_1y_2'-y_1'y_2=1$, then at $x=1$ $Wr(y_1,y_2)(\mu_j)=y_1y_2'=1$ so,

$$\Delta(\mu_j)=\frac{1}{2}(y_1(1\mu_j)+y_2'(1,\mu_j))=\frac{1}{2}\left(y_1(1,\mu_j)+\frac{1}{y_1(1,\mu_j)}\right)\geq 1.$$

**Remark.** If $m$ changes the sign, there are infinite sequences of positive and negative eigenvalues for both periodic and antiperiodic spectra. This result goes back to Lyapunov.

The following lemma is more or less known.

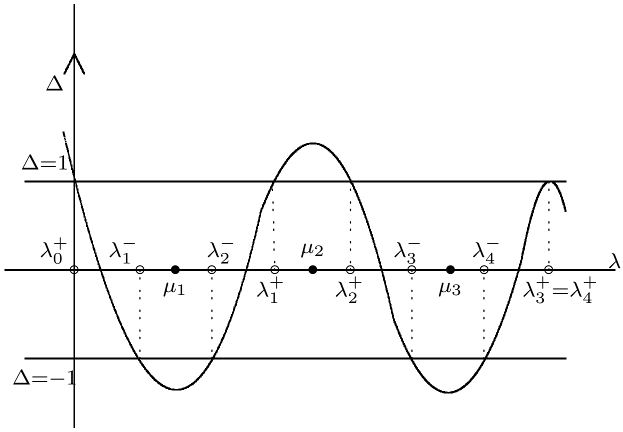


Fig. 1. Disposition of Spectra

**Lemma 1.** *Let $\psi, \varphi$ be solutions (not necessarily different) of the spectral problem* (1.11) *for the same $\lambda$. Then the following identity holds*

$$\lambda \mathcal{B}^2 \psi\varphi = \mathcal{B}^1 \psi\varphi. \tag{2.6}$$

The proof is straightforward.

Let us also give an additional formula which will be used in the next section. Suppose the functions $p$ and $q$ are such that $p, p', q, q'$ are zero at $0, 1$. Then we have

$$\int_0^1 p\mathcal{B}^s q dx = -\int_0^1 q\mathcal{B}^s p dx, \quad s = 1, 2. \tag{2.7}$$

Since $\mu_j \neq 0$ we can define the following variables $f_j = -\dfrac{\ln|\rho_j|}{\mu_j^2}$ and $g_j = -\dfrac{\ln|\rho_j|}{\mu_j^3}$. Our main result is the following

**Theorem 1.** *a) The variables $\mu_i$ and $f_j = -\dfrac{\ln|\rho_j|}{\mu_j^2}$ are conjugate with respect to the bracket $\{\,,\}_2$;*

*b) The variables $\mu_i$ and $g_j = -\dfrac{\ln|\rho_j|}{\mu_j^3}$ are conjugate with respect to the bracket* $\{\,,\}_1$.

**3. Conjugate variables.** We will prove only part a) of the Theorem 1. The part b) goes in the similar fashion. Since we follow Penskoi, only the key points will be given. We need to show that

$$\{\mu_i, \mu_j\}_2 = \int_0^1 \frac{\delta \mu_i}{\delta m} \mathcal{B}^2 \frac{\delta \mu_j}{\delta m} dx = 0, \tag{3.1}$$

$$\{\mu_i, f_j\}_2 = \int_0^1 \frac{\delta \mu_i}{\delta m} \mathcal{B}^2 \frac{\delta f_j}{\delta m} dx = \delta_{ij}, \tag{3.2}$$

$$\{f_i, f_j\}_2 = \int_0^1 \frac{\delta f_i}{\delta m} \mathcal{B}^2 \frac{\delta f_j}{\delta m} dx = 0. \tag{3.3}$$

Let us first calculate $\frac{\delta \mu_i}{\delta m}$. We have

$$y_2''(x, \mu_i) = -\mu_i m y_2(x, \mu_i), \tag{3.4}$$

which we write $y_2'' = -\mu m y_2$ for short. The variation of (3.4) reads

$$\delta y_2'' = -\delta\mu m y_2 - \mu \delta m y_2 - \mu m \delta y_2. \tag{3.5}$$

We multiply this identity by $y_2$ and integrate. Then the l.h.s. is transformed by integrating by parts to obtain

$$0 = -\delta\mu \int_0^1 m y_2^2 dx - \int_0^1 \mu \delta m y_2^2 dx. \tag{3.6}$$

Since, $\int_0^1 m y_2^2 dx \neq 0$, we get

$$\frac{\delta \mu_i}{\delta m} = -A_i \mu_i y_2^2(x, \mu_i), \tag{3.7}$$

where $A_i = \left[\int_0^1 m y_2^2(x, \mu_i) dx\right]^{-1}$.

To calculate $\frac{\delta \rho_i}{\delta m}$ we first multiply (3.5) by $y(x, \mu_i)$, defined in (2.5). Next, we multiply $y'' = -\mu m y$ by $\delta y_2$, subtract so obtained identities and, finally integrate to obtain

$$\int_0^1 (y \delta y_2'' - y'' \delta y_2) dx = -\int_0^1 \delta\mu m y y_2 dx - \int_0^1 \mu \delta m y y_2 dx.$$

The l.h.s. gives

$$\int_0^1 (y\delta y_2'' - y''\delta y_2)dx = \int_0^1 (y\delta y_2' - y'\delta y_2)'dx =$$

$$(y\delta y_2' - y'\delta y_2)|_0^1 = \delta y_2'(1,\mu_i)y(1,\mu_i) = \frac{\delta\rho_i}{\rho_i} = \delta \ln|\rho_i|.$$

Then

$$\delta \ln|\rho_i| = -\delta\mu_i B_i - \mu_i \int_0^1 \delta m y_2 y dx,$$

where $B_i = \int_0^1 m y_2(x,\mu_i)y(x,\mu_i)dx$. Using (3.7) we get

$$\frac{\delta\rho_i}{\delta m} = A_i B_i \mu_i y_2^2(x,\mu_i) - \mu_i y_2(x,\mu_i)y(x,\mu_i). \tag{3.8}$$

Now we are ready to calculate the brackets (3.1)–(3.3).

$$\{\mu_i,\mu_j\}_2 = \int_0^1 \frac{\delta\mu_i}{\delta m}\mathcal{B}^2\frac{\delta\mu_j}{\delta m}dx = A_i A_j \mu_i\mu_j \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^2 y_2^2(x,\mu_j)dx.$$

The last integral is zero. This can be seen from

$$\mu_i\mu_j \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^2 y_2^2(x,\mu_j)dx \stackrel{L1}{=} \mu_i \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^1 y_2^2(x,\mu_j)dx \stackrel{(2.7)}{=}$$

$$-\mu_i \int_0^1 y_2^2(x,\mu_j)\mathcal{B}^1 y_2^2(x,\mu_i)dx \stackrel{L1}{=} -\mu_i^2 \int_0^1 y_2^2(x,\mu_j)\mathcal{B}^2 y_2^2(x,\mu_i)dx =$$

$$\mu_i^2 \int_0^1 y_2^2(x,\mu_i)\mathcal{B}^2 y_2^2(x,\mu_j)dx.$$

Hence, $\{\mu_i,\mu_j\}_2 = 0$.

Next, we will show that $\{\mu_i, \ln|\rho_j|\}_2 = \mu_i^2\delta_{ij}$ from where (3.2) follows. The case $i \neq j$ is treated similarly as above. Let us consider the case $i = j$.

$$\{\mu_i, \ln|\rho_i|\}_2 = \int_0^1 \frac{\delta\mu_i}{\delta m}\mathcal{B}^2\frac{\delta \ln|\rho_i|}{\delta m}dx =$$

$$-\int_0^1 A_i\mu_i y_2^2(\mu_i)\mathcal{B}^2\left(A_i B_i \mu_i y_2^2(\mu_i) - \mu_i y_2 \mu_i)y(\mu_i)\right)dx =$$

$$-A_i^2 B_i \mu_i^2 \int_0^1 y_2^2(\mu_i)\mathcal{B}^2 y_2^2(\mu_i)dx + A_i\mu_i^2 \int_0^1 y_2^2(\mu_i)\mathcal{B}^2 y_2(\mu_i)y(\mu_i)dx =$$

$$A_i\mu_i^2\int_0^1 y_2^2(\mu_i)(m\partial+\partial m)y_2(\mu_i)y(\mu_i)dx =$$

$$A_i\mu_i^2\int_0^1 my_2^2(\mu_i)\left(y'(\mu_i)y_2(\mu_i)-y_2'(\mu_i)y(\mu_i)\right)dx.$$

The expression $y'(\mu_i)y_2(\mu_i)-y_2'(\mu_i)y(\mu_i)$ is the Wronskian $Wr(y,y_2)$ which is a constant. Then

$$Wr(y,y_2)=y'(1,\mu_i)y_2(1,\mu_i)-y_2'(1,\mu_i)y(1,\mu_i)=-\frac{y(0,\mu_i)}{\rho_i}y_2'(1,\mu_i)=-1.$$

So, $\{\mu_i,\ln|\rho_i|\}_2=-A_i\mu_i^2\int_0^1 my_2^2(\mu_i)dx=-\mu_i^2$ and hence

$$\{\mu_i,\ln|\rho_j|\}_2=-\mu_i^2\delta_{ij}\quad\text{and}\quad\{\mu_i,f_j\}_2=\delta_{ij}.$$

It remains to verify that $\{\ln|\rho_i|,\ln|\rho_j|\}_2=0$. These calculations are similar to those for the bracket $\{\mu_i,\mu_j\}_2=0$. Therefore, $\{f_i,f_j\}_2=0$. This finishes the proof of the part a) of the Theorem 1. The part b) follows in an analogous way.

**4. Evolution of the auxiliary spectrum.** It is natural to express the Hamiltonians $H_n$ via the variables $\mu_i, f_j$, for example. It turns out that this is a difficult task. That is why we shall study the motion of the auxiliary spectrum. To do this we assume first that $y_1, y_2$ are the Floquet solutions of (1.11)

$$y_1(0,\lambda)=1,\quad y_1'(0,\lambda)=0,$$
$$y_2(0,\lambda)=0,\quad y_2'(0,\lambda)=1,$$

in particular

$$y_1(x+1,\mu_n)=y_1(1,\mu_n)y_1(x,\mu_n),\quad y_2(x+1,\mu_n)=y_2'(1,\mu_n)y_2(x,\mu_n)$$

and according to the Wronskian relation

$$y_1(1,\mu_n)y_2'(1,\mu_n)=1. \tag{4.1}$$

Moreover, we also assume that

$$y_1(1,\mu_n)=\Delta-\sqrt{\Delta^2-1},\quad y_2'(1,\mu_n)=\Delta+\sqrt{\Delta^2-1}. \tag{4.2}$$

If we denote $y_2^\bullet$ to be the derivative with respect to $\lambda$, an easy calculation gives that

$$\int_0^1 my_2^2(x,\mu_n)dx=y_2^\bullet y_2'(1,\mu_n).$$

We may write the formula (3.7) as

$$\frac{\delta\mu_n}{\delta m} = -\mu_n \frac{y_2^2(x,\mu_n)}{y_2^\bullet y_2'(1,\mu_n)}. \tag{4.3}$$

Next, we compute (see [7])

$$\frac{\delta\Delta}{\delta m} = -\frac{\lambda}{2}\left(y_2(x+1,\lambda)y_1(x,\lambda) - y_2(x,\lambda)y_1(x+1,\lambda)\right) = -\frac{\lambda}{2} y_2^{+x}(1,\lambda), \tag{4.4}$$

where the superscript $+x$ means that $y_2(1,\lambda)$ is computed for $m$ translated in amount $0 \le x < 1$. Since $\frac{\delta\Delta}{\delta m}$ is a linear combination of products of solutions of (1.11), it satisfies Lemma 1.

$$\lambda \mathcal{B}^2 \frac{\delta\Delta}{\delta m} = \mathcal{B}^1 \frac{\delta\Delta}{\delta m}. \tag{4.5}$$

Now, with $h = \frac{\delta\mu_n}{\delta m}$ we have

$$\lambda\{\mu_n, \Delta(\lambda)\}_2 = \int_0^1 h\lambda\mathcal{B}^2 \frac{\delta\Delta}{\delta m} dx = \int_0^1 h\mathcal{B}^1 \frac{\delta\Delta}{\delta m} dx =$$

$$-\frac{1}{2}\left[h\left(\frac{\delta\Delta}{\delta m}\right)'' - h'\left(\frac{\delta\Delta}{\delta m}\right)' + h''\frac{\delta\Delta}{\delta m}\right]_0^1 - \int_0^1 \frac{\delta\Delta}{\delta m}\mathcal{B}^1 h dx.$$

Note that $h(0) = h(1) = h'(0) = h'(1) = 0$, so

$$\lambda\{\mu_n, \Delta(\lambda)\}_2 = -\frac{1}{2}\left[h''\frac{\delta\Delta}{\delta m}\right]_0^1 - \mu_n \int_0^1 \frac{\delta\Delta}{\delta m}\mathcal{B}^2 h dx$$

or

$$\lambda\{\mu_n, \Delta(\lambda)\}_2 = -\frac{1}{2}\left[h''\frac{\delta\Delta}{\delta m}\right]_0^1 + \mu_n \int_0^1 h\mathcal{B}^2 \frac{\delta\Delta}{\delta m} dx.$$

Now, it is easy to obtain from here

$$\{\mu_n, \Delta(\lambda)\}_2 = -\mu_n \frac{\lambda}{2} \frac{(y_2'(1,\mu_n))^2 - 1}{y_2^\bullet y_2'(1,\mu_n)} \frac{y_2(1,\lambda)}{\lambda - \mu_n}$$

and with the help of (4.1) and (4.2)

$$\{\mu_n, \Delta(\lambda)\}_2 = -\mu_n \lambda \frac{\sqrt{\Delta^2 - 1}}{y_2^\bullet(1,\mu_n)} \frac{y_2(1,\lambda)}{\lambda - \mu_n}. \tag{4.6}$$

It is known that the Hamiltonians $H_n, n = 1, 2, \ldots$ are coefficients in an expansion of $\Delta(\lambda) : \Delta(\lambda) = 1 - \sum_{n=1}^\infty H_n \lambda^n$. Since $\Delta^\bullet(0) = -H_1$, from (4.6) we can obtain the motion of the auxiliary spectrum under the flow of the $\mu$CH equation

$$\dot{\mu}_n = \{\mu_n, H_1\}_2 = \mu_n \frac{\sqrt{\Delta^2 - 1}}{y_2^{\bullet}(1,\mu_n)} \frac{y_2(1,0)}{-\mu_n} = -\frac{\sqrt{\Delta^2 - 1}}{y_2^{\bullet}(1,\mu_n)}, \quad n \geq 1. \tag{4.7}$$

Similarly we can obtain the motion $\mu_n$ under the flows of the higher Hamiltonians from (4.6).

It is seen that (4.7) is a system of infinitely many nonlinear differential equations in infinitely many variables. Only in the case of so called finite-gap potentials (4.7) becomes a finite system whose solutions are usually expressed via theta functions. This will be reported elsewhere.

**5. Discussion.** It turns out that the conjugate variables obtained here for the $\mu$CH equation are practically the same as for the periodic CH equation. Perhaps, the reason is that these equations, although different, have similar bi-hamiltonian structures.

Let us return to the $\mu$CH equation (1.1). Formally we may think of $\mu$CH equation in a following way. We take the HS equation and add a nonlocal term

$$-2\mu(u)u_x = -2H_0[m]u_x,$$

where $H_0$ was defined in the Introduction.

One can consider an equation obtained in this way, but the other conserved quantity is taken instead $H_0$. For example, we may take $H_1$ and obtain

$$-u_{txx} = -2H_1[m]u_x + 2u_x u_{xx} + u u_{xxx}.$$

Of course, the physical interpretation is missing, but the question is: Whether this equation, obtained in that formal way, is integrable?

## REFERENCES

[1] B. Khesin, J. Lenells, G. Misiołek. Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms. *Math. Ann.* **342** (2008), 617–656.

[2] J. Lenells, G. Misiołek, F. Tiğlay. Integrable Evolution Equations on Spaces of Tensor Densities and Their Peakon Solutions. *Commun. Math. Phys.* **299** (2010), 129–161.

[3] Ying Fu, Yue Liu, Chougzheng Ou. On the blow-up structure for the generalized periodic Camassa-Holm and Degasperis-Procesi equations. arXiv: 1009.2466v2.

[4] O. Christov. Geometric integrability of some generalizations of the Camassa-Holm equation. *International Journal of Differential Equations* 2011, doi: 10.1155/2011/738509.

[5] A. Penskoi. Canonically conjugate variables for the periodic Camassa-Holm equation. *Nonlinearity* **18** (2005), 415–421.

[6] H. Flaschka, D. McLaughlin. Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions. *Progr. Theoret. Phys.* **55**, *2* (1976), 438–456.

[7] A. Constantin, H. McKean. A shallow water equation on the circle. *Comm. Pure Appl. Math.* **51**, *5* (1998), 475–504.

[8] H. McKean, P. van Moerbeke. The spectrum of Hill's equation. *Invvent. Math.* **30** (1975), 217–274.

[9] W. Magnus, W. Winkler. Hill's Equation. New York, Interscience, Willey, 1966.

[10] M. Eastham. The Spectral Theory of Periodic Differential Equations. Scottish Academic Press, Edinburgh, 1973.

[11] A. Constantin. A General-Weighted Sturm-Liouville Problem. *Ann. della Sc. Norm. Sup. di Pisa* **24** (1997), 767–782.

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