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BEST APPROXIMATION AND MODULI OF SMOOTHNESS

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Dedicated to the 65 years Aniversary of Professor Petar Popivanov

ABSTRACT. The aim of this note is to present moduli of smoothness which are introduced by different schools of approximation for characterization of the best algebraic approximation. We observe that Potapov's generalized moduli are equivalent to the error in approximation by the algebraic version of the trigonometric Jackson integrals in uniform norm and in weighted integral metric.

Moduli of smoothness play a basic role in approximation theory. For a given function f, they measure the structure of smoothness of f via the $r-$th finite difference

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh), & \text{if } x, x+rh \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

For functions f belonging to the space $L_p[a,b], 1 \le p < \infty$, or the space $C[a,b]$ $(p=\infty)$ of continuous functions the classical r-th modulus

$$\omega_r(f,t)_p := \sup_{0<h\le t} \|\Delta_h^r f(\cdot)\|_p \tag{1}$$

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is a good measure for determining the rate of convergence of the best approximation. For 2π periodic functions f, D. Jackson (1911) and S. N. Bernstein [3] showed that the error of best approximation $E_n^*(f)_p$ by trigonometric polynomials of degree at most n has the same rate of convergence as the r-th modulus in the sense that, for $0 < \alpha < r$

$$(2) \qquad \omega_r(f,t)_p = O(t^\alpha)(t \to 0) \qquad \Longleftrightarrow \qquad E_n^*(f)_p = O(n^{-\alpha})(n \to \infty).$$

In the case of algebraic approximation, that is by algebraic polynomials $P_n \in \Pi_n$ of degree at most n this result is no longer true. Though one has here the direct estimate [3]

$$E_n(f)_p := \inf_{P_n \in \Pi_n} \|f - P_n\|_p \le c_r \omega_r \left(f, \frac{1}{n}\right)_p$$

given in the doctoral thesis of D. Jackson this inequality has no converse. It was observed by S. M. Nikolskii (1946) [8] that for functions f satisfying

$$(3) \qquad \omega_r(f,t)_p = O(t^\alpha) \qquad (0 < \alpha < r)$$

the polynomial $P_n \in \Pi_n$ of best approximation has a faster rate of convergence near the boundary of the interval [-1,1] than in the interior.

The Russian school in approximation, in particular A. F. Timan (1951) [9] and V. K. Dzjadyk (1956) [5] succeeded in characterizing (3) in terms of algebraic polynomials for $p = \infty$. They showed that (3) is equivalent to the existence of a sequence of polynomials $P_n \in \Pi_n$, such that

$$(4) \qquad |f(x) - P_n(x)| = O\left(\delta_n^\alpha(x)\right), \qquad \delta_n(x) := \left(\varphi(x) + \frac{1}{n}\right)\frac{1}{n}$$

uniformly in $x \in [-1,1]$, where $\varphi(x) := \sqrt{1-x^2}$.

However, for only integrable functions $f \in L_p[-1,1], 1 \le p < \infty$, (4) has no longer sense. Even

$$\inf_{P_n \in \Pi_n} \left\|\delta_n^{-\alpha}(f - P_n)\right\| = O(1)$$

is not equivalent to (1) as it was shown by V. P. Motornii (1971) [7] and R. DeVore (1977) [2]. The reason may be due to the fact that one should not consider (4) but

$$(5) \qquad E_n(f)_p = O(n^{-\alpha}).$$

The algebraic analogue of (2) requires a new kind of modulus of smoothness.

This long-outstanding problem was solved by different schools of approximation. Thus K. G. Ivanov (announced in 1980, proofs followed only in 1983) modified the so called τ-modulus, developed and studied intensively by the Bulgarian school (B. Sendov, A. Andreev, V. Popov). Ivanov [6] introduced the following moduli of smoothness

$$\tau_r(f, \psi(t))_{q,p} := \|\omega_r(f, .; \psi(t, .)_q\|_p ,$$

where the local moduli are given by

$$\omega_r(f, x; \psi(t,x))_q = \left((2\psi(t,x))^{-1} \int\limits_{-\psi(t,x)}^{\psi(t,x)} |\Delta_h^r f(x)|^q \, dx \right)^{1/q} , \qquad 1 \le q < \infty.$$

$$\omega_r(f, x; \psi(t,x))_\infty = \sup \{ |\Delta_h^r f(x)| : |h| \le \psi(t,x) \} ,$$

$$\psi(t,x) = t\varphi(x) + t^2, \varphi(x) = \sqrt{1-x^2}$$

and proved that (5) is equivalent to

$$\tau_r(f, \psi(t))_{q,p} = O(n^{-\alpha}).$$

In the alternative approach by Butzer, Stens, Wehrens (1976/1980) [1, 11] the classical translation $f(x+h)$ is replaced by the so-called Legendre translation

$$\left(\tau_h^L f\right)(x) := \frac{1}{\pi} \int\limits_{-1}^{1} f\left(xh + u\sqrt{(1-x^2)(1-h^2)}\right) \frac{du}{\sqrt{1-u^2}} \qquad (x, h \in [-1,1]).$$

They defined the r-th modulus of continuity by

$$\omega_r^L(f,t)_p := \sup_{1-t \le h_j \le 1} \left\| \Delta_{h_1}^L \Delta_{h_2}^L \cdots \Delta_{h_r}^L f \right\|_p$$

with difference $\Delta_h^L f := \tau_h^L f - f$ and established the complete analogue of Jackson-Bernstein result (2) for $E_n(f)_p$. In (2) one just has to replace ω_r by ω_r^L, α by 2α and $t \to 0+$ by $t \to 1-$.

Potapov [9] also solved the problem of finding an algebraic analogue of (2). He proved Jackson's theorem and its converse using generalized moduli

$$\widetilde{\omega}(f, \delta, p, \nu, \mu) = \sup_{|t| \le \delta} \|f - f(x, t, \nu, \mu)\|_{L_p}^{\nu,\mu}$$

where $\|f\|_{L_p}^{\nu,\mu} = \|f\|_{L_p,\alpha,\beta} = \left\|f(x)(1-x)^{\alpha}(1+x)^{\beta}\right\|_{L_p}$ and

$$\begin{aligned}
&\alpha = \beta = -\frac{1}{2p}, p > 1, && \text{if} \quad \nu = \mu = -\tfrac{1}{2},\\
&\alpha = \beta, \nu + \frac{1}{2} > \alpha + \frac{1}{2p} > 0, && \text{if} \quad \nu = \mu > -\tfrac{1}{2},\\
&\nu + \frac{1}{2} > \alpha + \frac{1}{2p} > 0,\ \beta = -\frac{1}{2p}, p > 1, && \text{if} \quad \nu > \mu = -\tfrac{1}{2}.
\end{aligned}$$

In some of the cases he considered generalized translation is defined by:

1) for $\nu = \mu = -\dfrac{1}{2}$

$$f(x,t,\nu,\mu) = \frac{1}{2}\left[f(x\cos t + \sqrt{1-x^2}\sin t) + f(x\cos t - \sqrt{1-x^2}\sin t)\right];$$

2) for $\nu = \mu > -\dfrac{1}{2}$

$$f(x,t,\nu,\mu) = \frac{1}{\gamma(\nu)}\int_{-1}^{1} f(x\cos t + y\sin t\sqrt{1-x^2})(1-y^2)^{\nu-\frac{1}{2}}dy;$$

3) for $\nu > \mu = -\dfrac{1}{2}$

$$f(x,t,\nu,\mu) = \frac{1}{\gamma(\nu)}\int_{-1}^{1} f(x\cos t + y\sin t\sqrt{1-x^2} - (1-y^2)(1-x)\sin^2\frac{t}{2})(1-y^2)^{\nu-\frac{1}{2}}dy,$$

where $\gamma(\nu) = \int\limits_{-1}^{1}(1-y^2)^{\nu-\frac{1}{2}}dy$.

Potapov proved that for $f(x) \in L_p^{\nu,\mu}, 1 \le p \le \infty, \nu \ge \mu \ge -\dfrac{1}{2}$

$$E_n(f, L_p^{\nu,\mu}) = \inf_{P_n}\|f - P_n\|_{L_p}^{\nu,\mu} \le c_1\widetilde{\omega}(f, \frac{1}{n}, p, \nu, \mu), \tag{6}$$

$$\widetilde{\omega}(f, \frac{1}{n}, p, \nu, \mu) \le \frac{c_2}{n^2}\sum_{k=0}^{n}(k+1)E_k(f, L_p^{\nu,\mu}), \tag{7}$$

where constants c_1 and c_2 are independent of f and n.

Ditzian and Totik [4] also introduced a new kind of modulus of smoothness for characterization of the best algebraic approximation. They defined the weighted moduli

$$\omega_\varphi^r(f,t)_p = \sup_{0<h\le t} \left\|\overline{\Delta}_{h\varphi}^r f\right\|_p, \qquad \varphi(x)=\sqrt{1-x^2}, \tag{8}$$

with $r-$th symmetric difference of function f given by

$$\overline{\Delta}_h^r f = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(x+r\frac{h}{2}-kh).$$

Using these moduli they proved an algebraic analogue of (2). Ditzian and Totik [4] as well established equivalence ($\sim$) between their moduli and the $K-$functionals

$$K_{r,\varphi}(f,t^r)_p = \inf_g \left\{\|f-g\|_p + t^r \left\|\varphi^r g^{(r)}\right\|_p \ : \ g^{(r-1)} \in AC_{loc}\right\}.$$

The expression $A(n)\sim B(n)$ means that there exists a positive constant c independent of n such that $\frac{1}{c}B(n)\le A(n)\le cB(n)$.

Apart from the theory of best approximation another field in approximation theory is the study of linear approximation processes together with the characterizations of their rate of convergence. For example, for the algebraic version of the trigonometric Jackson integrals $G_{s,n}$ defined by

$$G_{s,n}(f,x) = \int_{-\pi}^{\pi} f(\cos(\arccos x + v))K_{s,n}(v)dv,$$

where

$$K_{s,n}(v) = \mu_{s,n}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s}, \qquad \int_{-\pi}^{\pi} K_{s,n}(v)dv = 1,$$

we established the equivalence in uniform norm [12]

$$\|f-G_{s,n}f\| \sim K(f,\frac{1}{n^2};C[-1,1],C^2,H) \sim \Omega_2(f,\frac{1}{n}). \tag{9}$$

The $K-$functional

$$K(f,t;C[-1,1],C^2,H) := \inf\left\{\|f-g\| + t\,\|Hg\| \ : \ g\in C^2\right\},$$

where

$$H := (H_1)^2, (H_1 g)(x) := \sqrt{1-x^2}\frac{d}{dx}g(x).$$

The modulus

$$\Omega_2(f,t) := \sup_{0<h\le t}\sup_{-1\le x\le 1}\left|f\left(\cos(\arccos x + h)\right) + f\left(\cos(\arccos x - h)\right) - 2f(x)\right|.$$

It is interesting to notice that the modulus $\Omega_2(f,t)$ is the same considered by Potapov [9] in the case $\nu = \mu = -\frac{1}{2}$. Using the relation (9) we gave a solution of the problem of characterization of the best algebraic approximation, but after that we found out that this has been done by Potapov in relations (6) and (7) for $\nu = \mu = -\frac{1}{2}$. We would like to mention as advantage the strong converse theorem in our rezults with the modulus and related K-functional in respect of Potapov's result.

For $1 \le p < \infty$ and weighted $L_p(u)$ space with the norm

$$\|f\|_{p,u} = \left(\int_{-1}^{1}\left|\left(1-x^2\right)^{-\frac{1}{2p}} f(x)\right|^p dx\right)^{1/p}, u(x) = \left(1-x^2\right)^{-\frac{1}{2p}},$$

an analogue of equivalence (9) also holds [13]

$$\|f - G_{s,n}f\|_{p,u} \sim K(f, \frac{1}{n^2}; L_p(u), C^2, H) \sim \Omega_2(f, \frac{1}{n})_{p,u}, \tag{10}$$

where the $K-$functional and the modulus are defined by

$$K(f,t;L_p(u),C^2,H) := \inf\left\{\|f-g\|_{p,u} + t\,\|Hg\|_{p,u} \,:\, g\in C^2\right\},$$

$$\Omega_2(f,t)_{p,u} := \sup_{0<h\le t}\left(\int_{-1}^{1}\left|\left(f\left(\cos(\arccos x + h)\right) + f\left(\cos(\arccos x - h)\right) - 2f(x)\right)u(x)\right|^p dx\right)^{1/p}.$$

Thus we observe that results (6) and (7) for $\nu = \mu = -\frac{1}{2}$ and relations (9) and (10) yield another characterizations

$$\|f - G_{s,n}f\| = O(n^{-\alpha}) \iff E_n(f) = O(n^{-\alpha})(n\to\infty), \quad \text{for } 0<\alpha<2,$$

in uniform norm and

$$\|f - G_{s,n}f\|_{p,u} = O(n^{-\alpha}) \iff E_n(f)_{p,u} = O(n^{-\alpha})(n \to \infty), \quad \text{for } 0 < \alpha < 2$$

in weighted integral metric for $1 \leq p < \infty$.

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