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# CONTRIBUTIONS TO THE CLASS OF BRANCHING PROCESSES IN VARYING ENVIRONMENTS*

Manuel Molina, Manuel Mota, Alfonso Ramos

In this work we provide a survey on the main probabilistic contributions derived in the literature for the class of asexual branching processes in varying environments.

## 1. Introduction

The asexual branching process in varying environments was motivated by I. J. Good in the discussion at the Symposium on Stochastic Processes organized by the Royal Statistical Society in 1949 (see [13] for details). It generalizes the classical Galton-Watson process by allowing that the offspring probability distribution, which governs the reproduction phase, can be different over time. In the same way, but considering sexual reproduction, the two-sex branching process in varying environments, introduced in [17], generalizes the bisexual Galton-Watson process, studied by Daley [5]. In both cases, asexual and sexual reproduction, the main motivation is to develop mathematical models to adequately describe

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the demographic dynamics of biological populations in which, due to various environmental or social factors, the reproduction law can be different in successive generations. Many authors have investigated this issue and have provided several interesting contributions. In particular, conditions guaranteeing almost sure extinction or positive probability of survival in populations with demographic dynamics modeled by such branching processes in varying environments have been established. In [18] a survey about the main contributions derived for sexual reproduction was provided. We focus here our attention on the main contributions derived for the asexual case. In Section 2, we formally describe the probability model. Section 3 is devoted to stating the results obtained for the possible extinction of the process. The most important results established about the asymptotic behaviour of the process are presented in Section 4.

## 2. The probability model

One of the essential characteristics of the Galton-Watson process is its extinction-explosion duality, namely, either it becomes extinct or it grows unboundedly. However, it is well-known that in many biological species, the population size stabilizes when time passes. As consequence, the demographic dynamics of such populations is not well-described through the outline of the Galton-Watson process. One of the possible reasons for the stabilization is that the probability distribution governing the reproduction phase is not the same in each generation. This reason motivated the introduction of the following branching process:

**Definition 1.** *Let* $\{X_{ni} : n = 0, 1, \ldots;\ i = 1, 2, \ldots\}$ *be a sequence of integer-valued, non-negative, and independent random variables such that for each* $n \geq 0$, $\{X_{ni}\}_{i=1}^{\infty}$ *is a sequence of identically distributed random variables with probability distribution* $\{p_{nk}\}_{k=0}^{\infty}$, $p_{nk} := P(X_{n1} = k)$. *The branching process in varying environments (BPVE), also called the inhomogeneous branching process,* $\{Z_n\}_{n=0}^{\infty}$, *is then defined in the recursive form:*

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{ni}\ , \quad n = 0, 1, \ldots \tag{1}$$

*where* $\sum_{1}^{0} = 0$. *Without loss of generality, it is assumed that the process is initiated with* $Z_0 = 1$ *individual.*

Intuitively, the variables $X_{ni}$ and $Z_n$ have the same interpretation as in the Galton-Watson process, namely, $X_{ni}$ is the number of individuals produced by the $i$-th individual at time (generation) $n$, and $Z_n$ is the total number of individuals in this generation. The crucial difference in this new branching process defined in (1) is that the probability distribution governing the reproduction phase is not necessarily the same over time. The probability distribution $\{p_{nk}\}_{k=0}^{\infty}$ is referred to as the offspring distribution in the $n$-th generation. It can be verified that the BPVE is a Markov chain with transition probabilities not necessarily stationary.

For the Galton-Watson branching process the probability generating function concerning the total numbers of individual in the $n$-th generation is determined by the $n$-fold composition of the probability generating function associated to the offspring distribution. This result was generalized for the BPVE by Fearn [9]:

**Proposition 1.** (Fearn (1971)) *For $n \geq 0$, let $g_n(s) := E[s^{X_{n1}}]$ and $f_n(s) := E[s^{Z_n}]$, $0 \leq s \leq 1$, be the probability generating functions of $X_{n1}$ and $Z_n$, respectively. Then:*

$$f_{n+1}(s) = f_n(g_n(s)),\ 0 \leq s \leq 1,\ n = 0, 1, \ldots$$

From this result, by using an iterative procedure, it is deduced that:

$$f_n(s) = (g_0 \circ \cdots \circ g_{n-1})(s),\ 0 \leq s \leq 1,\ n = 1, 2, \ldots$$

Hence, if for $n \geq 0$, $\mu_n := E[X_{n1}]$ and $\sigma_n^2 := \mathrm{Var}[X_{n1}]$, assumed to be finite, then it is derived that:

$$E[Z_n] = \prod_{j=0}^{n-1} \mu_j, \quad \mathrm{Var}[Z_n] = \prod_{j=0}^{n-1} \mu_j^2 \sum_{k=0}^{n-1} \frac{\sigma_k^2}{\mu_k^2 \prod_{j=0}^{k-1} \mu_j}, \quad n = 1, 2, \ldots$$

## 3. Extinction probability

If, for some $n \geq 1$, $Z_n = 0$ then, from (1), it is deduced that $Z_{n+m} = 0$, $m \geq 1$, which implies the extinction of the process. Let

$$Q := P(\lim_{n\to\infty} Z_n = 0 \mid Z_0 = 1)$$

be the probability of extinction. Several authors ([1], [3], [8] and [10]) have investigated conditions for the almost sure extinction of the process or for its survival with positive probability. Firstly, Church [3] established the following result concerning the extinction-explosion property of the BPVE (an alternative proof of this result was also proposed by Athreya and Karlin [2]):

**Theorem 1.** (Church (1971)) *Assume that* $p_{n0} < 1$, $n \geq 0$. *Then:*

$$P(\lim_{n\to\infty} Z_n = \infty \mid Z_0 = 1) = 1 - Q \quad \textit{if and only if} \quad \sum_{n=0}^{\infty}(1 - p_{n1}) = \infty.$$

Taking into account that $Q = \lim_{n\to\infty} f_n(0)$, Agresti [1] studied the extinction problem of the process by considering appropriate lower and upper bounds for $f_n(0)$, $n \geq 0$.

**Theorem 2.** (Agresti (1975)) *Assume that* $\sigma_n^2 < \infty$, $n \geq 0$. *Then:*

$$1 - \left(\frac{1}{m_n} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{g_j''(0)}{\mu_j m_{j+1}}\right)^{-1} \leq f_n(0) \leq 1 - \left(\frac{1}{m_n} + \sum_{j=0}^{n-1}\frac{g_j''(1)}{\mu_j m_{j+1}}\right)^{-1}$$

*where* $m_n := \prod_{j=0}^{n-1}\mu_j$.

From Theorem 2, it is derived that:

- If $\inf_{n\geq 0}\left(\frac{1}{m_n} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{g_j''(0)}{\mu_j m_{j+1}}\right)^{-1} = 0$ then $Q = 1$.
- If $Q = 1$ then $\lim_{n\to\infty}\left(\frac{1}{m_n} + \sum_{j=0}^{n-1}\frac{g_j''(1)}{\mu_j m_{j+1}}\right)^{-1} = 0$.

Agresti [1] established also the following necessary and sufficient condition for the almost sure extinction of a BPVE:

**Theorem 3.** (Agresti (1975)) *Assume that*

$$\sup_{n\geq 0}\frac{g_n''(0)}{g_n''(1)} < \infty \quad \textit{and} \quad \inf_{n\geq n_0}\frac{g_n''(0)}{g_n''(1)} > 0, \quad \textit{for some } n_0 \geq 0.$$

*Then:*

$$Q = 1 \quad \textit{if and only if} \quad \sum_{n=1}^{\infty} \frac{1}{m_n} = \infty.$$

Note that, the classical Steffensen's result on the extinction probability for the Galton-Watson process is obtained as particular case of Theorem 3 when $\mu_j$ is a constant independent of $j$.

A similar result, assuming a functional relation among the means and the variances of the offspring distribution, was obtained by Fujimagari [10]:

**Theorem 4.** (Fujimagari (1980)) *Assume the existence of* $\mu := \lim_{n\to\infty} \mu_n$.

(*a*) *If* $\mu < 1$ *then* $Q = 1$.

(*b*) *If* $\mu > 1$ *and there exits* $b \in (0, \infty)$ *such that* $\mu_n \geq b\sigma_n^2$, $n \geq 0$, *then* $Q < 1$.

Fujimagari's result was improved by D'Souza and Biggins [8], assuming weaker conditions on the offspring mean. They considered uniformly supercritical BPVE.

**Definition 2.** *A BPVE is said to be uniformly supercritical if there exist constants* $A > 0$ *and* $c > 1$ *such that:*

$$\frac{m_{n+k}}{m_k} \geq Ac^n, \quad k, n = 0, 1, \ldots \tag{2}$$

**Remarks.**

- For $k = 0$, it is deduced that $m_n \geq Ac^n$ for all $n$. In particular, when $n = 0$, using the fact that $m_0 = 1$, it is derived that $A \leq 1$.
- The inequality (2) is verified if, for example, $\liminf_{n\to\infty} \mu_n > c > 1$.
- If (2) holds then $\limsup_{n\to\infty} \mu_n > c > 1$.

**Theorem 5.** (D'Souza and Biggins (1992)) *Assume a BPVE uniformly supercritical and the existence of a random variable* $X$, *with* $E[X] < \infty$, *stochastically bigger than* $X_{n1}\mu_n^{-1}$, *for all* $n \geq 0$. *Then:*

(*a*) $Q \leq \eta < 1$ *for certain* $\eta$ *which depends of* $A$, $c$ *and* $X$.

(*b*) $1 - Q = P(\lim_{n\to\infty} Z_n = \infty \mid Z_0 = 1)$.

The second part of Theorem 5 is an application of the criteria given in Theorem 1 of [3]. Note that to this end, it is sufficient to verify that $\limsup_{n\to\infty} p_{n1} > 0$ which implies that $\sum_{n=0}^{\infty} p_{n1} = \infty$.

## 4. Asymptotic behaviour

In order to present the main results derived about the asymptotic behaviour of the BPVE, we will distinguish two periods, clearly differentiated. The first one, corresponding to those works developed in the seventies ([3], [9], [11], [12], [14]). In those works, the asymptotic behaviour was investigated by using some analytic results based on probability generating functions and martingale theory. The second period corresponding to works obtained in the nineties ([4], [6], [7], [8] and [15]). In those works, the methodology used was different. It was based on some conditions about the growth of the offspring means and appropriate conditions about the tails of the offspring distributions.

### 4.1. Results developed in the 70s

First, Church [3] proved the convergence in distribution of $\{Z_n\}_{n=0}^{\infty}$ to a random variable $Z$ (possibly null or infinite). Unlike the extinction-explosion property of the Galton-Watson process, it was showed that the BPVE can eventually become stable over time. Later on, Lindvall [14] proved the almost sure convergence of $\{Z_n\}_{n=0}^{\infty}$ to $Z$.

On the other hand, Fearn [9] studied the limiting behaviour of the sequence $\{W_n\}_{n=0}^{\infty}$, $W_n := Z_n(E[Z_n])^{-1}$. He showed that it is a non-negative martingale, relative to the sequence of $\sigma$-algebras $\{\mathcal{F}_n\}_{n=0}^{\infty}$, $\mathcal{F}_n := \sigma(Z_0, \dots, Z_n)$. Hence, he proved its almost sure convergence to a random variable $W$ such that $P(0 \leq W < \infty) = 1$. The interest was then focussed on the research of necessary and sufficient conditions which guarantee that $P(W = 0) < 1$. This issue was also studied in [9], along with the convergence of $\{W_n\}_{n=0}^{\infty}$ to $W$ in quadratic mean.

**Theorem 6.** (Fearn (1971)) *The following statements are equivalent:*

(*a*) $W_n$ *converges in* $L^2$ *to* $W$ *with* $E[W] = 1$ *and* $\mathrm{Var}[W] = \sum_{k=0}^{\infty} \frac{\sigma_k^2}{\mu_k^2\, m_k} < \infty$.

(*b*) $\lim_{n\longrightarrow\infty} \mathrm{Var}[W_n] = \sum_{k=0}^{\infty} \frac{\sigma_k^2}{\mu_k^2\, m_k} < \infty.$

Clearly, if $\lim_{n\to\infty} \mathrm{Var}[W_n] < \infty$ then $W$ is a non-degenerate at 0 random variable. Also, if for some $n$, $\sigma_n^2 > 0$, then $W$ is, almost surely, a non-degenerate variable.

Other authors have looked for conditions for $P(W = 0) < 1$. In this sense, we can highlight during this same period the work by Goettge [11] who proved that $E[W] = 1$, under strictly weaker conditions than those of Fearn. To this end, he used analytic techniques similar to those considered in [2]. However, he was not able to prove that $\{W > 0\} = \{Z_n \to \infty\}$ almost surely.

It is worth mentioning two important contributions provided by Jagers [12] in this period, which are the homologous to the results for the critical and subcritical branching processes, respectively, due the Russian School. The proofs of both results are based on a formula for $\mathrm{Var}[W]$ and under certain additional conditions on the growth of the means associated to the process $\{Z_n\}_{n=0}^{\infty}$.

**Theorem 7.** (Jagers (1974)) *Assume that* $E[Z_n^2] < \infty$, $n \geq 1$. *Assume also that* $0 < \inf_{n\geq 0} m_n \leq \sup_{n\geq 0} m_n < \infty$ *and* $\inf_{n>0} g_n''(1) > 0$. *Then:*

(*a*) $\lim_{n\to\infty} d_n^{-1} P(Z_n > 0) = 1.$

(*b*) $\lim_{n\to\infty} P\left( \frac{Z_n}{d_n m_n} \leq u \,\middle|\, Z_n > 0 \right) = 1 - e^{-u}, \quad u \geq 0.$

*where* $d_n := \sum_{k=0}^{n-1} (2 m_k \mu_k^2)^{-1} g_n''(1).$

**Theorem 8.** (Jagers (1974)) *Assume* $\limsup_{n\to\infty} m_n \sum_{k=1}^{n} (m_{k+1})^{-1} g_n''(1) < \infty.$ *Then:*

(*a*) $\lim_{n\to\infty} m_n^{-1} P(Z_n > 0)$ *exists and it is finite and positive.*

(*b*) $b_k := \lim_{n\to\infty} P(Z_n = k | Z_n > 0)$ *exists,* $\sum_{k=1}^{\infty} k\, b_k = \lim_{n\to\infty} (P(Z_n > 0))^{-1} m_n$ *and* $\sum_{k=1}^{\infty} b_k = 1.$

### 4.2. Results developed in the 90s

Lyons [15] investigates some properties of the limit variable $W$. More specifically, using techniques based on random walks this author provides new conditions under which $P(W = 0) < 1$, i.e. $W$ is non-degenerated at 0. Following this line, D'Souza and Biggins [6] studied the asymptotic behaviour of the BPVE, looking for a certain regularity in the behaviour of the offspring means and also in the tails of the offspring distributions. They provided the following result, which requires a very similar condition to the logarithmic of Kesten-Stigum for the supercritical Galton-Watson branching process. They were the first authors that determined sufficient conditions which guarantee that $\{W > 0\} = \{Z_n \to \infty\}$ almost surely.

**Theorem 9.** (D'Souza and Biggins (1992)) *If the BPVE is uniformly supercritical and there exists a random variable $X$ with $E[X \log^+ X] < \infty$ stochastically bigger than $X_{n1}(\mu_n)^{-1}$, for all $n \geq 0$, then:*

$$E[W] = 1 \quad \textit{and} \quad \{W > 0\} = \{Z_n \to \infty\} \ a.s.$$

Later on, D'Souza[8] investigated the asymptotic behaviour of the sequence $\{P(Z_n > 0)\}_{n=0}^{\infty}$ under conditions which guarantee the existence of a positive probability for the extinction of the process. Such a behaviour is very similar to the one of the subcritical Galton-Watson branching process, as we can check in the following result:

**Theorem 10.** (D'Souza (1994)) *Given a BPVE $\{Z_n\}_{n=0}^{\infty}$ such that $\{m_n\}_{n=0}^{\infty}$ converges to $0$ and for all $n$, $k \geq 0$ it is verified that:*

$$\frac{m_{n+k}}{m_k} \leq \frac{A}{g(n)}$$

*being $A > 0$ and $g : \mathbb{R} \to \mathbb{R}$ a differentiable function with $g(0) = 1$ and strictly increasing to infinity. If there exists a random variable $X$ with $E[X] < \infty$ such that for all $x$, $P(X \geq x) \geq P(X_{n1} \geq x | X_{n1} > 0)$, $n = 0, 1, \ldots$, then:*

$$\lim_{n\to\infty} \left( \frac{P(Z_n > 0)}{m_n} \right)^{1/n} = 1.$$

**Remark.** To show that the asymptotic behaviour of the BPVE is similar to that one corresponding to the subcritical Galton-Watson branching process, it is sufficient to consider in Theorem 10 the particular function $g(x) = m^{-x}$ with $m < 1$, where $m$ denotes the corresponding offspring mean of the Galton-Watson process.

Other interesting question investigated for the BPVE concerns its rates of growth.

**Definition 3.** *A sequence of constants $\{C_n\}_{n=0}^{\infty}$ is a rate of growth of a BPVE $\{Z_n\}_{n=0}^{\infty}$ if the sequence $\{C_n^{-1} Z_n\}_{n=0}^{\infty}$ converges almost surely to a finite limit which is strictly positive on the set $\{Z_n \to \infty\}$.*

Macphee and Schuh [16] presented an example where $\lim_{n\to\infty} \mu_n > 4$, $\sigma_n^2 < \infty$, $n \geq 1$, $E[W] = 1$, and nevertheless, $\{Z_n\}_{n=0}^{\infty}$ has two rates of growth. This example motivated the study by D'Souza [7] searching sufficient conditions in order that the BPVE had only one rate of growth. He also investigated what processes admit more than one rate of growth. To this end, he introduced the two following conditions:

$(A)$ There exists $A > 0$ and $\gamma > 1$ such that for all $n \geq 0$ and $k \geq 1$,

$$\frac{m_{n+k}}{m_k} \geq A \left( \frac{n+k}{k} \right)^{\gamma} \tag{3}$$

$(B)$ Provided $\gamma > 1$, there exist $p > 1 + \gamma^{-1}$ and $K < \infty$ such that

$$E \left[ \frac{X_{n1}}{\mu_n} \right]^p \leq K.$$

Taking $n = 0$ in (3), it is deduced that $A \leq 1$. Also, from (3), it is deduced that $m_n \geq A m_1 n^{\gamma}$, $n \geq 0$ . On the other hand, from condition (B) and Jensen's inequality, it is derived that $K \geq 1$ and that $E[X_{n1}(\mu_n)^{-1}]^s \leq K$ for $1 < s < p$.

**Theorem 11.** (D'Souza (1994)) *Let $\gamma > 1$ and $p > 1 + \gamma^{-1}$ be given. For any $\alpha \in (0, \beta)$, where $\beta = (p-1)^{-1}$, it is possible to construct a BPVE satisfying conditions (A) and (B) such that $\{n^{\alpha}\}_{n=0}^{\infty}$ is a rate of growth of $\{Z_n\}_{n=0}^{\infty}$. Furthermore, if $\alpha > 1$ (this requires $p < 2$) we can construct the process so that $\liminf_{n\to\infty} P(X_{n1} \geq 2)(\mu_n)^{-2} > 0$ is satisfied.*

The next result provides a sufficient condition for the BPVE to have an only rate of growth.

**Theorem 12.** (D'Souza (1995)) *Assume that* $\{Z_n\}_{n=0}^{\infty}$ *is an BPVE satisfying conditions (A) and (B) (for* $p = 2$*) and* $\liminf_{n\to\infty} P(X_{n1} \geq 2)(\mu_n)^{-2} > 0$*. Then* $\{m_n\}_{n=0}^{\infty}$ *is the only rate of growth of the process.*

**Remarks.**

- Under the conditions in Theorem 9, the sequence $\{m_n\}_{n=0}^{\infty}$ is a rate of growth of the BPVE. In fact, under conditions in Theorem 12, it is the only rate of growth of the process.
- Theorem 11 shows that the last conditions in Theorem 12 are not sufficient to ensure that the process has only one rate of growth.
- D'Souza and others authors have presented some examples of BPVE's with infinitely many rates of growth. This happens because the conditions (A) and/or (B) are not verified.

Finally it is worth mentioning here the work by Cohn [4], which investigates the continuity of the distribution of the limit variable $W$ on the set of non extinction, providing also some conditions for such a distribution to have some jumps in the positive real numbers. In this work plays a essential role the quantity $M = \lim_{n\to\infty} \max_{i>0} P(Z_n = i)$.

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## REFERENCES

[1] A. Agresti. On the extinction times of random and varying environment branching processes. *J. Appl. Probab.*, **12** (1975), 39–46.

[2] K. B. Athreya, S. Karlin. On branching processes with random environments: I. extinction probabilities. *Ann. Math. Statist.*, **42** (1971), 1499–1520.

[3] J. D. Church. On infinite composition products of probability generating functions. *Z. Wahrscheinlichkeitsth. und Verw. Gebiete.*, **19** (1971), 243–256.

[4] H. Cohn. On the asymptotic patterns of supercritical branching processes in varying environments. *Annl. Appl. Probab.*, **6** (1996), 896–902.

[5] D. J. Daley. Extinction conditions for certain bisexual Galton-Watson branching processes. *Z. Wahrscheinlichkeitsth. und Verw. Gebiete.*, **9** (1968), 315–322.

[6] J. C. D'Souza. The rates of growth of the Galton-Watson process in varying environments. *Adv. Appl. Probab.*, **26** (1994), 698–714.

[7] J. C. D'Souza. The extinction time of the inhomogeneous branching process. *Lectures Notes in Statistics, Springer*, **99** (1995), 106–117.

[8] J. C. D'Souza, J. D. Biggins, The supercritical Galton-Watson process in varying environments. *Stochastic Process. Appl.*, **42** (1992), 39–47.

[9] D. H. Fearn. Galton-Watson processes with generation dependence. *Proc. 6th Berkeley Symp. Math. Statist. Probab.*, **4** (1971), 159–172.

[10] T. Fujimagari. On the extinction time distribution of a branching process in varying environments. *Adv. Appl. Probab.*, **12** (1980), 350–366.

[11] R. T. Goettge. Limit theorems for the supercritical Galton-Watson process in varying environments. *Math. Biosci.*, **28** (1976), 171–190.

[12] P. Jagers. Galton-Watson processes in varying environments. *J. Appl. Probab.*, **11** (1974), 174–178.

[13] D. G. Kendall. Stochastic processes and population growth. *J. Roy. Statist. Soc. Ser. B*, **11** (1949), 230–282.

[14] T. Lindvall. Almost sure convergence of branching processes in varying and random environments. *Ann. Probab.*, **2** (1974), 344–346.

[15] R. Lyons. Random walks, capacity and percolation on trees. *Ann. Probab.*, **20** (1992), 2043–2088.

[16] I. M. Macphee, H. J. Schuh. A Galton-Watson branching process in varying environments with essentially constant offspring means and two rates of growth. *Austral. J. Statist.*, **25** (1983), 329–338.

[17] M. Molina, M. Mota, A. Ramos. Bisexual Galton-Watson branching process in varying environments. *Stochastic Anal. Appl.*, **21** (2003), 1353–1367.

[18] M. Molina, M. Mota, A. Ramos. Discrete time bisexual branching processes in varying environments. *Pliska Stud. Math. Bulgar.*, **16** (2004), 147–157.

*Manuel Molina*
*University of Extremadura*
*mmolina@unex.es*

*Manuel Mota*
*University of Extremadura*
*mota@unex.es*

*Alfonso Ramos*
*University of Extremadura*
*aramos@unex.es*