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IMPROVED HARDY INEQUALITY AND APPLICATIONS

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New Hardy type inequality with double singular kernel in a bounded domain $\Omega \in R^n$ is proved. When Ω is an annulus or a ball, a generalization of the well known Hardy inequalities with kernels singular only on the boundary or at the origin is given. The inequality is is with optimal constant and has an additional positive term depending on $|\nabla u|$.

As an application of the improved Hardy inequality a new analytical lower bound for the first eigenvalue of the p–Laplacian is obtained.

1. Introduction

It is well known that the classical Hardy inequality, see Hardy [1, 2]

$$\int_0^\infty |u'(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty x^{-p}|u(x)|^p dx, \tag{1}$$

where $p > 1$ and $u(x)$ is absolutely continuous function on $[0, \infty)$, $u(0) = 0$, has an optimal constant $\left(\frac{p-1}{p}\right)^p$, i.e. there is no constant $C > \left(\frac{p-1}{p}\right)^p$ such that (1) holds with the constant C for all functions.

In the multidimensional case inequality (1) is generalized to

$$\int_\Omega |\nabla u(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty \frac{|u(x)|^p}{d^p(x)} dx, \tag{2}$$

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for $u \in W_0^{1,p}(\Omega)$, $p > 1$, $d(x) = dist(x, \partial\Omega)$ and Ω is a bounded domain in R^n, see Kufner [3], Neĉas [4]. However, there is no a nontrivial function $u(x) \in W_0^{1,p}(\Omega)$ for which (2) becames an equality. That is why Brezis and Marcus [5] state the question: is there an additional term $A(u) > 0$ such that the improved Hardy inequality

$$\int_\Omega |\nabla u(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty \frac{|u(x)|^p}{d^p(x)} dx + A(u), \tag{3}$$

holds. In the last two decades inequality (3) is intensively investigated, see for example Barbatis et al. [6], Dávila and Dupaigne [7], Filippas et al. [8], Filippas and Tertikas [9], Hoffmann-Ostenhof et al. [10], Kinnunen and Korte [11], Marcus and Shafrir [12], Tidblom [13], Vázquez and Zuazua [14]. In Brezis and Marcus [5] the authors find

$$A(u) = \lambda(\Omega) \int_\Omega u^2(x) dx, \quad \lambda(\Omega) = (4 \operatorname{diam} \Omega)^{-1}, \quad \text{for } p = 2,$$

while in Hoffmann-Ostenhof et al. [10] for $p = 2$ and in Tidblom [13] for $p > 2$ the constant $\lambda(\Omega) = C(p, n)(vol(\Omega))^{-p/n}$, where $C(p, n)$ is an explicitly given constant, independent of Ω.

The aim of this paper is to prove Hardy inequality (3) with optimal constant and additional term $A_1(|\nabla u|)$ depending on the gradient term. By means of Hardy type inequality, $A_1(|\nabla u|)$ is estimated from below by a term $A_2(u)$. Moreover, we consider double singular kernels, on the boundary $\partial\Omega$ and at some interior point of Ω.

As an application of the improved Hardy inequality in a ball and Faber–Krahn inequality we obtain an analytical estimate for the first eigenvalue $\lambda_{p,n}(\Omega)$ of the p-Laplacian for $p > n$ in arbitrary bounded domains $\Omega \in R^n$.

In Section 2. we prove Hardy type inequality in a general form, while in Section 3. the special case of an anulus and a ball is considered. Section 4. deals with an application of the obtained in Section 3. Hardy inequality in a ball for an analytical estimate from below of the first eigenvalue of the p-Laplacian with zero Dirichlet data.

2. Hardy–type inequality

Suppose for a fixed bounded domain Ω that there exist $C^{0,1}(\Omega)$ function F and a vector–function h with components $h_i \in C^{0,1}(\Omega)$, $i = 1, \ldots, n$, such that for all intervals $(\varepsilon, \tau) \subset (0, T)$ the strip $G_{\varepsilon,\tau} = \{x \in \Omega : |F(x)| \in (\varepsilon, \tau)\} \subset \Omega$, $\bar{G}_{0,T} = \bar{\Omega}$ and a. e. in Ω

$$-F \operatorname{div} h \geq 0, \tag{4}$$

$$\langle h, \nabla F\rangle > 0, \tag{5}$$

where $\langle\cdot,\cdot\rangle >$ is the scalar product in R^n

Proposition 1. *Let a function* $\mu(t) > 0$*, satisfy* $\int_0^T \mu^{1-p'}(t)dt < \infty$*, where* $\frac{1}{p} + \frac{1}{p'} = 1$*,* $p > 1$*. Then under conditions* (4), (5) *the inequality*

$$\begin{aligned} &\int_{G_{0,T}} \left(\int_{|F|}^T \frac{\mu(t)}{t} dt\right) \langle h, \nabla F\rangle^{1-p} \left|\langle h, \nabla u\rangle\right|^p dx \\ \geq \ &\left(\frac{1}{T}\int_0^T \mu^{1-p'}(t)dt\right)^{1-p} \int_{G_{0,T}} |F|^{-p}\langle h, \nabla F\rangle |u|^p dx \end{aligned} \tag{6}$$

holds for $u \in C_0^\infty(G_{0,T})$

P r o o f. Let us denote $L_t = \int_{G_{0,t}} \langle h, \nabla F\rangle^{1-p} |\langle h, \nabla u\rangle|^p dx$, $K_t = \int_{G_{0,t}} |F|^{-p}\langle h, \nabla F\rangle |u|^p dx$, $K_{0t} = \int_{\partial G_{0,t}} \frac{\langle h, \nabla F\rangle}{|\nabla F|} |u|^p dx$ and recall the inequality, proved in Theorem 2 of Fabricant et al. [15]

$$L_t \geq \left(\frac{1}{p}\right)^p \frac{(K_{0t} + (p-1)K_t)^p}{K_t^{p-1}}. \tag{7}$$

With the notation $k(t) = t^{1/p'} K_t^{1/p}$, since $K_{0t} = t\frac{d}{dt}K_t$, the inequality (7) can be written in a more compact form

$$L_t \geq t\left(\frac{d}{dt}k(t)\right)^p, \quad \text{or equivalently} \quad t^{-1/p} L_t^{1/p} \geq \frac{d}{dt}k(t). \tag{8}$$

Integrating (8) on t in $[0, T]$ we obtain

$$\int_0^T t^{-1/p} L_t^{1/p} dt \geq \left(T^{p-1} K_T\right)^{1/p}. \tag{9}$$

Applying Hölder inequality to *lhs* of (9) we get

$$\int_0^T t^{-1/p} L_t^{1/p} dt \leq \left(\int_0^T \frac{\mu(t)}{t} L_t dt\right)^{1/p} \left(\int_0^T \mu(t)^{1-p'} dt\right)^{1/p'}. \tag{10}$$

Since

$$\begin{aligned}\int_0^T \frac{\mu(t)}{t} L_t dt &= \int_0^T \frac{\mu(t)}{t} \int_{G_{0,t}} \langle h, \nabla F\rangle^{1-p} |\langle h, \nabla u\rangle|^p \, dx \\ &= \int_{G_{0,T}} \left(\int_{|F|}^T \frac{\mu(t)}{t} dt \right) \langle h, \nabla F\rangle^{1-p} |\langle h, \nabla u\rangle|^p \, dx\end{aligned}$$

then from (9) and (10) we obtain (6). $\square$

Remark 1. Inequality (6), is better than inequality

$$L_T \geq \left(\frac{1}{p'}\right)^p K_T^{p-1}, \tag{11}$$

proved in Theorem 2 in [15]. Indeed, for $0 < \beta < p-1$, $\mu(t) = t^\beta$ we have

$$\frac{1}{T} \int_0^T \mu^{1-p'}(t) dt = \frac{T^{\beta(1-p')}}{\beta(1-p')+1} \text{ and } \int_{|F|}^T \frac{\mu(t)}{t} dt = \frac{1}{\beta}(T^\beta - |F|^\beta)$$

so that the following inequality

$$\begin{aligned}&\int_{G_{0,T}} \left(1 - \frac{|F|^\beta}{T^\beta}\right) |\langle h, \nabla F\rangle|^{1-p} |\langle h, \nabla u\rangle|^p \, dx \\ \geq\ & \beta[1+\beta(1-p')]^{p-1} \int_{G_{0,T}} |F|^{-p} \langle h, \nabla F\rangle |u|^p dx\end{aligned} \tag{12}$$

holds. For $\beta = \dfrac{1}{p'}$ we obtain inequality (11) with smaller *lhs* term.

Moreover, with $\beta \to 0$ the result is

$$\int_{G_{0,T}} \ln \frac{T}{|F|} |\langle h, \nabla F\rangle|^{1-p} |\langle h, \nabla u\rangle|^p \, dx \geq \int_{G_{0,T}} |F|^{-p} \langle h, \nabla F\rangle |u|^p dx. \tag{13}$$

3. Hardy inequality in an annulus and in a ball

3.1. Inequality in an annulus

Inequality in an annulus is obtained with a special choice of F, h and β in (12).

Proposition 2. *Let $R > r$ and $m = \dfrac{p-n}{p-1}$, $p \neq n$ then the improved Hardy inequalities*

$$\begin{aligned}&\int_{B_R \setminus B_r} |\nabla u|^p dx \\ \geq\ & \left|\frac{p-n}{p}\right|^p \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} |R^m - |x|^m|^p} dx + \int_{B_R \setminus B_r} \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{\frac{1}{p'}} |\nabla u|^p dx.\end{aligned} \tag{14}$$

$$\begin{aligned}(15)\qquad &\int_{B_R\setminus B_r} |\nabla u|^p dx\\ \geq\ & \left|\frac{p-n}{p}\right|^p \int_{B_R\setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m-|x|^m|^p}\left[1+\left(\frac{1}{p'}\right)^p \left|\frac{R^m-|x|^m}{R^m-r^m}\right|^{\frac{1}{p'}}\right]dx.\end{aligned}$$

hold for every $u \in W_0^{1,p}(B_R)$.

P r o o f. Consider the problem

$$\Delta_p \psi = 0, \quad \text{in } B_R\setminus B_r, \quad \psi|_{\partial B_R} = 0, \quad \psi|_{\partial B_r} = 1, \tag{16}$$

which has a solution $\psi = \dfrac{R^m-|x|^m}{R^m-r^m}$. Let us chose $h_1 = |\nabla\psi|^{p-2}\nabla\psi$, $F_1 = \psi$, satisfying (4), (5) and $T = 1$, $\mu(t) = t^\beta$, $0 < \beta < p-1$. Then inequality (12) becomes

$$\begin{aligned}(17)\qquad &\int_{B_R\setminus B_r} \left(1-|\psi|^\beta\right)\left|\frac{\langle\nabla\psi,\nabla u\rangle}{|\nabla\psi|}\right|^p dx\\ \geq\ & \beta[1+\beta(1-p')]^{p-1}|m|^p \int_{B_R\setminus B_r} |\psi|^{-p}|\nabla\psi|^p|u|^p dx.\end{aligned}$$

Using the expression for ψ we get

$$\begin{aligned}(18)\qquad &\int_{B_R\setminus B_r} \left(1-\left|\frac{R^m-|x|^m}{R^m-r^m}\right|^\beta\right)|\nabla u|^p dx\\ \geq\ & \beta[1+\beta(1-p')]^{p-1}|m|^p \int_{B_R\setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m-|x|^m|^p}dx.\end{aligned}$$

If we chose $\beta = \dfrac{1}{p'} = \dfrac{p-1}{p}$, then $\beta[1+\beta(1-p')]^{p-1}|m|^p = \left|\dfrac{p-n}{p}\right|^p$ and inequality (18) becomes

$$\begin{aligned}(19)\qquad &\int_{B_R\setminus B_r} \left(1-\left|\frac{R^m-|x|^m}{R^m-r^m}\right|^{\frac{p-1}{p}}\right)|\nabla u|^p dx\\ \geq\ & \left|\frac{p-n}{p}\right|^p \int_{B_R\setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m-|x|^m|^p}dx.\end{aligned}$$

Let us denote

$$\begin{aligned} I_1(u) &= \left|\frac{p-n}{p}\right|^p \int_{B_R\setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m-|x|^m|^p}dx,\\ I_2(u) &= \int_{B_R\setminus B_r} \left|\frac{R^m-|x|^m}{R^m-r^m}\right|^{1/p'} |\nabla u|^p dx,\end{aligned}$$

then we can rewrite (19) as

$$\int_{B_R\setminus B_r} |\nabla u|^p dx \geq I_1(u) + I_2(u), \tag{20}$$

and (14) is proved.

In order to obtain (15) we estimate $I_2(u)$ using Theorem 1 in [15].

Let us chose $F_2 = \left|\frac{p}{p-n}\right| \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{1/p'}$ and $h_2 = -|x|^{-n}x$, then properties (4) and (5) for F_2 and h_2 hold.

Denoting $v^{1-p} = \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{1/p'}$, i.e. $v = \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{-1/p}$ then the vector function $f = F_2^{-p+1} h_2$ satisfy the equality

$$-\mathrm{div} f = (p-1)v|f|^{p'}. \tag{21}$$

Indeed, for the *lhs* and *rhs* of (21) we obtain:

$$\begin{aligned}
-\mathrm{div} f &= -F_2^{-p+1}\mathrm{div} h - (1-p)F_2^{-p}\langle h, \nabla F_2\rangle = (p-1)F_2^{-p}\langle h_2, \nabla F_2\rangle \\
&= (p-1)\left|\frac{p}{p-n}\right|^p |x|^{m-n}\left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{\frac{-p^2+p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
(p-1)v|f|^{p'} &= (p-1)\left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{-\frac{1}{p}} |F_2^{-p+1}h_2|^{p'} \\
&= (p-1)|x|^{m-n}\left|\frac{p}{p-n}\right|^p \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{\frac{-p^2+p-1}{p}}.
\end{aligned}$$

Applying Theorem 2 in [15] for I_2 in (20) we obtain (15). □

Following Fabricant et al. [16] the inequality (15) is improved in Proposition 3 with an additional logarithmic term in the *rhs*.

Proposition 3. *If $R > r$ and $m = \frac{p-n}{p-1}$, $p \neq n$ then the inequality*

$$\begin{aligned}
\int_{B_R\setminus B_r} |\nabla u|^p dx &\geq \left|\frac{p-n}{p}\right|^p \int_{B_R\setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'}|R^m - |x|^m|^p} \\
&\times \left\{ 1 + \left(\frac{1}{p'}\right)^p \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{\frac{1}{p'}} \right. \\
&\times \left. \left[1 + \frac{p^3}{2(p-1)^3} \frac{1}{\ln^2\left[\left(\frac{1}{e\tau}\left|\frac{p}{p-n}\right|\right)^{p'}\left|\frac{R^m - |x|^m}{R^m - r^m}\right|\right]}\right]\right\}
\end{aligned} \tag{22}$$

holds for $u \in W_0^{1,p}(B_R)$.

Proof. Let us define $f_1 = fz(\ln F_2)$ with f satisfying (21) and z be a solution of the differential inequality obtained in Lemma 1 of [16]. Then we have

$$\begin{aligned} -\mathrm{div} f_1 &= -\mathrm{f}z(\ln F_2) - \langle f, \frac{\nabla F_2}{F_2}\rangle z'(\ln F_2) \\ &= (p-1)v|f|^{p'} z(\ln F_2) - \langle f, \frac{\nabla F_2}{F_2}\rangle z'(\ln F_2) \\ &= (p-1)v|f_1|^{p'} + w. \end{aligned}$$

Here

$$\begin{aligned} w &= -(p-1)v|f|^{p'} z^{p'} - \langle f, \frac{\nabla F_2}{F_2}\rangle z' + (p-1)v|f|^{p'} z \\ &= v|f|^{p'} \left[-z' + (p-1)z - (p-1)z^{p'}\right], \end{aligned}$$

because for the coefficient of z' we obtain

$$\langle f, \frac{\nabla F_2}{F_2}\rangle = \left(\frac{p-n}{p}\right)^p |x|^{-n+m} \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{p-\frac{1}{p'}} = v|f|^{p'}.$$

Applying Lemma 1 in [16] for w we obtain the inequality

$$\begin{aligned} w &= v|f|^{p'} \left[-z' + (p-1)z - (p-1)z^{p'}\right] \\ &\geq v|f|^{p'} H(s), \end{aligned}$$

where

$$H(s) = \left(\frac{1}{p'}\right)^p \left(1 + \frac{p}{2(p-1)} \frac{1}{(1 + \ln\tau - s)^2}\right)$$

with $s = \ln F_2$. Note that

$$(1 + \ln\tau - \ln F_2)^2 = \ln^2 \frac{1}{e\tau} \frac{p}{|p-n|} \left|\frac{R^m - |x|^m}{R^m - r^m}\right|^{1/p'}$$

and for $H(\ln F_2)$ we obtain

$$H(\ln F_2) = \left(\frac{1}{p'}\right)^p \left(1 + \frac{p}{2(p-1)} \frac{1}{\left(\frac{p-1}{p}\right)^2 \ln^2 \left(\frac{1}{e\tau}\frac{p}{|p-n|}\right)^{p'} \left|\frac{R^m-|x|^m}{R^m-r^m}\right|}\right).$$

and in sum we get inequality (22). □

3.2. Inequality in a ball

For $m > 0$, i.e. $p > n$ with the limit proses $r \to 0$ in (14), (15) and (22) we obtain the following inequalities in B_R.

Proposition 4. *For $p > n$ the inequalities*

$$\tag{23} \begin{aligned} \int_{B_R} |\nabla u|^p dx \quad &\geq \quad \left(\frac{p-n}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m - |x|^m|^p} dx \\ &+ \quad \int_{B_R} \left(\frac{R^m - |x|^m}{R^m}\right)^{\frac{1}{p'}} |\nabla u|^p dx, \end{aligned}$$

$$\tag{24} \begin{aligned} \int_{B_R} |\nabla u|^p dx \quad &\geq \quad \left(\frac{p-n}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m - |x|^m|^p} \\ &\times \quad \left[1 + \left(\frac{1}{p'}\right)^p \left(\frac{R^m - |x|^m}{R^m}\right)^{\frac{1}{p'}}\right] dx, \end{aligned}$$

$$\tag{25} \begin{aligned} \int_{B_R} |\nabla u|^p dx \quad &\geq \quad \left(\frac{p-n}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m - |x|^m|^p} \\ &\times \quad \left\{1 + \left(\frac{1}{p'}\right)^p \left|\frac{R^m - |x|^m}{R^m}\right|^{\frac{1}{p'}}\right. \\ &\times \quad \left.\left[1 + \frac{p^3}{2(p-1)^3} \frac{1}{\ln^2\left[\left(\frac{1}{e\tau}\left(\frac{p}{p-n}\right)\right)^{p'}\left(\frac{R^m - |x|^m}{R^m}\right)\right]}\right]\right\} dx, \end{aligned}$$

hold for every $u \in W_0^{1,p}(B_R)$.

In order to show that the first term in (25) has an optimal constant let us prove the following Lemma.

Lemma 1. *Let $p > 1$, $m = \dfrac{p-n}{p-1} > 0$, then for $x \in B_R$ the inequality*

$$\tag{26} \left(\frac{p-n}{p}\right)^p |x|^{m-n}\,|R^m - |x|^m|^{-p} \geq \left(\frac{p-1}{p}\right)^p |R - |x||^{-p}$$

holds.

P r o o f. Inequality (26) is equivalent to

$$(p-n)(R-\rho) \geq (p-1)\rho^{\frac{n-1}{p-1}}(R^m - \rho^m),$$

for $|x| = \rho$ and $n - m = \dfrac{(n-1)p}{p-1}$. The function

$$\begin{aligned} h(\rho) &= (p-n)(R-\rho) - (p-1)\rho^{\frac{n-1}{p-1}}(R^m - \rho^m) \\ &= (p-n)(R-\rho) - (p-1)\left(R^m \rho^{\frac{n-1}{p-1}} - \rho\right), \end{aligned}$$

is decreasing one for $\rho \in [0, R]$ because

$$\begin{aligned} h'(\rho) &= (n-p) - (n-1)R^m \rho^{\frac{n-1}{p-1}-1} + p - 1 \\ &= (n-1)\left[1 - \left(\frac{R}{\rho}\right)^m\right] \leq 0, \end{aligned}$$

and $m > 0$. Since $h(R) = 0$ inequality (26) is satisfied. $\square$

Multiplying (26) by $|u|^p$ and integrating over B_R we obtain

$$\left(\frac{p-n}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{(n-1)p'}\,|R^m - |x|^m|^p} \geq \left(\frac{p-1}{p}\right)^p \int_{B_R} \frac{|u|^p}{d^p(x)}, \tag{27}$$

where $d(x) = dist(x, \partial B_R) = R - |x|$. Hence, the constant $\left(\dfrac{p-n}{p}\right)^p$ in (23), (24) and (25) is optimal because the constant $\left(\dfrac{p-1}{p}\right)^p$ is optimal for (2). Thus inequalities (24), (25) improve the Hardy inequality in Theorem 2 in [15].

4. Lower bound of the first eigenvalue of p–Laplacian

Es an application of inequalities (24) and (25) we estimate from below the first eigenvalue $\lambda_{p,n}(\Omega)$ of the p–Laplacian in a bounded smooth domain $\Omega \subset R^n$, $n \geq 2$, $p > n$. Recall that, the first eigenvalue is defined as

$$\lambda_{p,n}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p dx}{\displaystyle\int_\Omega |u|^p dx}. \tag{28}$$

and $\lambda_{p,n}(\Omega)$ is simple, i.e., the first eigenfunction $\varphi(x)$ is unique up to multiplication with nonzero constant. Moreover, φ is positive in Ω, $\varphi \in W_0^{1,p}(\Omega) \cap C^{1,s}(\bar{\Omega})$ for some $s \in (0, 1)$, see e. g. [17] and the references therein.

Due to the Faber–Krahn type inequality, see [18, 17], which says that for an arbitrary bounded domain $\Omega \subset R^n$ the estimate

$$\lambda_{p,n}(\Omega) \geq \lambda_{p,n}(\Omega^*) \tag{29}$$

holds, where Ω^* is the n–dimensional ball of the same volume as Ω, it is enough to prove lower bound of $\lambda_{p,n}$ only for a ball B_R.

4.1. Analytical estimates

Using inequality (24) the following analytical estimate is obtained.

Proposition 5. *For every ball $B_R \in R^n$, $n \geq 2$, $p > n$ the estimate*

$$\begin{aligned} \lambda_{p,n}(B_R) \;\; &\geq \;\; \left(\frac{1}{pR}\right)^p \left[\frac{(p-1)^{p-1}}{(n-1)^{n-1}}\right]^{\frac{p}{p-n}} \Bigg[1 + p^{-\frac{p(n+p-2)}{p-n}} (p-n)^{\frac{p-1}{p}} \\ &\times \;\; (n-p+p^2)^{\frac{(p-1)(n-p+p^2)}{p(p-n)}} \left(\frac{p-1}{1-p+p^2}\right)^{\frac{1-p+p^2}{p}}\Bigg]. \end{aligned} \tag{30}$$

holds.

Proof. From (28) with $\Omega = B_R$, and (24), denoting $(1-|x|^m) = \rho$, where $m = \dfrac{p-n}{p-1}$, we get

$$\begin{aligned} \lambda_{p,n}(B_R) \;\; &\geq \;\; \left(\frac{p-n}{pR}\right)^p \inf_{\rho\in(0,1)} \left[\frac{1}{\rho^p(1-\rho)^{\frac{n-m}{m}}} + \left(\frac{p-1}{p}\right)^p \frac{\rho^{\frac{p-1}{p}}}{\rho^p(1-\rho)^{\frac{n-m}{m}}}\right] \\ &\geq \;\; \left(\frac{p-n}{pR}\right)^p \left[\inf_{\rho\in(0,1)} \frac{1}{\rho^p(1-\rho)^{\frac{n-m}{m}}} + \left(\frac{p-1}{p}\right)^p \inf_{\rho\in(0,1)} \frac{\rho^{\frac{p-1}{p}}}{\rho^p(1-\rho)^{\frac{n-m}{m}}}\right] \\ &= \;\; \left(\frac{p-n}{pR}\right)^p \left[\inf_{\rho\in(0,1)} J_1(\rho) + \left(\frac{p-1}{p}\right)^p \inf_{\rho\in(0,1)} J_2(\rho)\right] \end{aligned} \tag{31}$$

As in [16] we get

$$\inf_{\rho\in(0,1)} J_1(\rho) = J_1(m) = \frac{1}{m^p(1-m)^{\frac{n-m}{m}}},$$

$$\left(\frac{p-1}{p}\right)^p \inf_{\rho\in(0,1)} J_2(\rho) = \left(\frac{p-1}{p}\right)^p J_2(\kappa m) = J_1(m) J(\kappa m)$$

with

$$\kappa = \frac{1-p+p^2}{n-p+p^2} \quad \text{and} \quad J(\kappa m) = \left(\frac{p-1}{p}\right)^p \kappa^{\frac{p-1}{p}-p} m^{\frac{p-1}{p}} \left(\frac{1-m}{1-\kappa m}\right)^{\frac{n-m}{m}}.$$

so that

(32) $$\lambda_{p,n}(B_R) \geq J_1(m)[1+J(\kappa m)].$$

Simplifying $J(\kappa m)$ we obtain (30). □

4.2. Numerical estimates

Estimates from below for $\lambda_{p,n}(B_R)$ are developed numerically in [19], analytically in [18] with Cheeger's constant and from the definition (28) via different Hardy inequalities in [13], [20] and recently in [16].

Let us illustrate with numerical examples, see Figure 1 the comparison between the estimates of $\lambda_{p,n}(B_R)$ obtained in [13], [20], [16] and present study for $R=1$ and different values of $n \geq 2$, $m = \dfrac{p-n}{p-1} > 0$, i.e. $p > n$, namely we will compare the following formulas:

a) (see [13]),

$$\lambda_{p,n}^{(a)} = \left(\frac{p-1}{p}\right)^p \left[1 + (p-1)\left(\frac{2}{n}\right)^{p/n} \pi^{1/2} \frac{\Gamma^{p/n}\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n+p}{2}\right)}{\Gamma^{p/n+1}\left(\frac{n}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}\right];$$

b) (see (22) in [20]),

$$\lambda_{p,n}^{(b)} = \left(\frac{1}{p}\right)^p \left[\frac{(p-1)^{p-1}}{(n-1)^{n-1}}\right]^{\frac{p}{p-n}};$$

c) (see (27) in [16]),

$$\begin{aligned} \lambda_{p,n}^{(c)} \quad &\geq \quad \left(\frac{1}{pR}\right)^p \left[\frac{(p-1)^{p-1}}{(n-1)^{n-1}}\right]^{\frac{p}{p-n}} \\ &\times \quad \left\{1 + \frac{p}{4(p-1)}\left[1 + \sqrt{\frac{5p-7}{3(p-1)}} - 2\ln m - 2\ln\tau\right]^{-2}\right\}, \qquad ; \\ &\tau = 1 - 4(1-m)\left[p\left(1 + \sqrt{\frac{5p-7}{3(p-1)}} - 2\ln m\right) - 4m\right]^{-1}. \end{aligned}$$

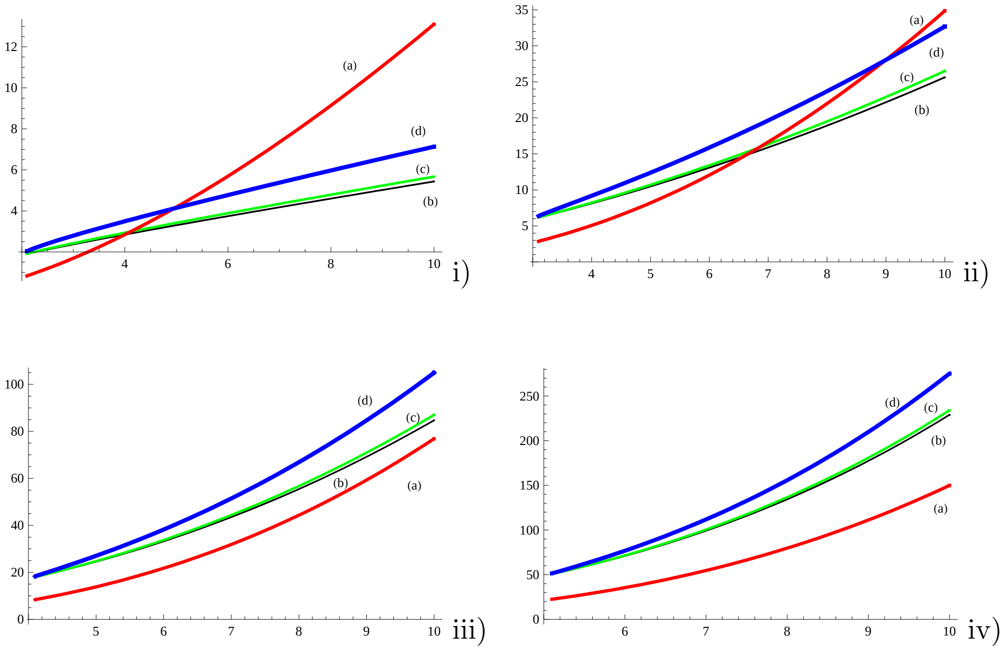

Figure 1: Comparison of $\lambda^{(a)}_{p,n} \div \lambda^{(d)}_{p,n}$: i) $n = 2, p \in (2, 10]$; ii) $n = 3, p \in (3, 10]$; iii) $n = 4, p \in (4, 10]$; i) $n = 5, p \in (5, 10]$.

d) (using (30)),

$$
\begin{aligned}
\lambda^{(d)}_{p,n} &= \left(\frac{1}{p}\right)^p \left[\frac{(p-1)^{p-1}}{(n-1)^{n-1}}\right]^{\frac{p}{p-n}} \left[1 + p^{-\frac{p(n+p-2)}{p-n}} (p-n)^{\frac{p-1}{p}}\right. \\
&\times \left.(n-p+p^2)^{\frac{(p-1)(n-p+p^2)}{p(p-n)}} \left(\frac{p-1}{1-p+p^2}\right)^{\frac{1-p+p^2}{p}}\right].
\end{aligned}
$$

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