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APPLICATION OF THE IMPROVED CONCAVITY METHOD TO SIXTH ORDER BOUSSINESQ EQUATIONS WITH ARBITRARY HIGH INITIAL ENERGY*

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Finite time blow up of the solutions to sixth order Boussinesq equation with arbitrary positive initial energy is proved. An improved variant of the concavity method of Levine is applied. This new method allows us to derive nonexistence of global solutions under conditions on the initial data which are more general than the assumptions used in the literature.

1. Introduction

The higher order nonlinear dispersive equation

$$u_{tt} - \beta_1 u_{xx} - \beta_2 u_{ttxx} + \beta_3 u_{xxxx} + \beta_4 u_{ttxxxx} + mu + f(u)_{xx} = 0, \tag{1}$$

where $\beta_i \geq 0$, $i = 1, 2, 3, 4$ and $m \geq 0$ are real constants, has been intensively studied in the last decades. For different values of the parameters β_i and m, this equation refers to good Boussinesq equation, Rosenau equation, forth and sixth order Boussinesq equation with/without linear restoring force. For example, equation (1) describes the transverse deflcation of an elastic rod on elastic foundation, see [1, 7, 9, 10]. Also equation (1) occurs in the water wave problems with nonzero surface tension, see [11].

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The aim of this paper is to investigate the global behavior of the solutions to Cauchy problem for the following sixth order Boussinesq equation:

$$u_{tt} - u_{xx} - u_{ttxx} + u_{xxxx} + u_{ttxxxx} + u + f(u)_{xx} = 0, \qquad x \in \mathbb{R},\ t \geq 0, \tag{2}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \mathbb{R}. \tag{3}$$

Here

$$u_0 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2}u_0 \in \mathrm{L}^2(\mathbb{R}),\ u_1 \in \mathrm{H}^1(\mathbb{R}),\ (-\Delta)^{-1/2}u_1 \in \mathrm{L}^2(\mathbb{R}) \tag{4}$$

and $(-\Delta)^{-s}u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right)$ for $s > 0$, $\mathcal{F}(u)$, $\mathcal{F}^{-1}(u)$ are the Fourier transformation and the inverse Fourier transformation, respectively.

We consider general power type nonlinearity $f(u)$, which has one of the following forms:

$$\begin{aligned} f(u) &= \sum_{k=1}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u, \\ f(u) &= a_1|u|^{p_1} + \sum_{k=2}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u, \end{aligned} \tag{5}$$

where the constants a_k, p_k $(k = 1, 2, \dots, l)$ and b_j, q_j $(j = 1, 2, \dots, s)$ fulfill the conditions

$$\begin{aligned} &a_1 > 0,\ a_k \geq 0,\ b_j \geq 0 \ \text{ for } \ k = 2, \dots, l, \quad j = 1, \dots, s, \\ &1 < q_s < q_{s-1} < \cdots < q_1 < p_1 < p_2 < \cdots < p_l < \infty. \end{aligned}$$

The global existence and finite time blow up of the solution to (2)-(4) is mainly investigated for initial data with positive subcritical energy $0 < E(0) < d$, see [4, 15, 17]. In this case the potential well method, introduced by Payne and Sattinger [8], gives a complete description of the behavior of the solutions with subcritical initial energy.

As for the solutions to (2)-(4) with arbitrary positive initial energy, there are only few results about their qualitative properties. The long time behavior of the solution to (1),(3),(4) when $m = 0$ is considered in [13, 14, 16] for small data by means of the contraction mapping theorem. Nonexistence of global solutions to (1),(3),(4) for $m \neq 0$ and nonlinearity $f(u) = a|u|^{p-1}u$ is obtained in [3] by means of sign preserving properties of some functionals.

The aim of this paper is to extend the results in [3] to more general nonlinear terms (5), as well as to wider class of initial data. Under very general structural conditions on the initial data in comparison with the conditions in [3], we prove finite time blow up of the solutions to (2)-(5). For this purpose we use a new method, which is an improvement of the concavity method of Levine, proposed in [5].

The paper is organized in the following way. In Section 2 some preliminary results are given and the main result is formulated. Section 3 deals with the proof of the main result.

2. Preliminaries and formulation of the main result

For functions depending on t and x we use the following short notations:

$$\|u\| = \|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}, \quad \|u\|_1 = \|u(t,\cdot)\|_{\mathrm{H}^1(\mathbb{R})},$$
$$(u,v) = (u(t,\cdot), v(t,\cdot)) = \int_{\mathbb{R}} u(t,x)v(t,x)\,dx.$$

In the space $\{u \in \mathrm{H}^1(\mathbb{R}) : (-\Delta)^{-1/2}u \in \mathrm{L}^2(\mathbb{R})\}$ we define the scalar product

$$\langle u,v\rangle = \langle u(\cdot,t), v(\cdot,t)\rangle = (u,v) + (u_x,v_x) + \left((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}v\right).$$

Let us recall the local existence result for problem (2)-(5).

Theorem 1. (Local existence) *Problem* (2)–(5) *admits a unique local solution* $u(t,x) \in \mathrm{C}^1([0,T_m);\mathrm{H}^1(\mathbb{R}))$ *on a maximal existence time interval* $[0,T_m)$, $T_m \le \infty$. *Moreover:*

(*i*) *If*

$$\limsup_{t\uparrow T_m} \langle u,u\rangle < \infty \qquad \textit{then} \qquad T_m = \infty;$$

(*ii*) *The weak solution* $u(t,x)$ *blows up for a finite time if*

$$\limsup_{t\uparrow T_m} \langle u,u\rangle = \infty \qquad \textit{for} \qquad T_m < \infty;$$

(*iii*) *For every* $t \in [0,T_m)$ *the solution* $u(t,x)$ *satisfies the conservation law* $E(0) = E(t)$, *where*

$$E(t) := E(u(\cdot,t)) = \frac{1}{2}\left(\langle u_t,u_t\rangle + \langle u,u\rangle\right) - \int_{\mathbb{R}}\int_0^u f(y)\,dy\,dx. \tag{6}$$

The proof of Theorem 1 is similar to the proof of Theorem 2.3 and Theorem 2.4 in [15] and we omit it.

Below we recall our previous result from [3] for nonexistence of global solution to (2)-(4) with a single nonlinearity $f(u) = a|u|^{p_1-1}u$. The statement is suitably altered to fit into the present context.

Theorem 2. ([3]) *Suppose $u(t,x)$ is the weak solution to* (2)–(3) *with $f(u) = a|u|^{p_1-1}u$, $p_1 > 1$, defined in the maximal existence time interval $[0, T_m)$, $T_m \leq \infty$. If*

$$\langle u_0, u_1 \rangle > 0, \qquad \langle u_0, u_0 \rangle \neq 0$$

and

$$\frac{p_1 - 1}{2(p_1 + 1)} \langle u_0, u_0 \rangle + \frac{1}{2} \frac{\langle u_0, u_1 \rangle^2}{\langle u_0, u_0 \rangle} > E(0) > 0$$

then $u(t,x)$ blows up for a finite time, i.e. $T_m < \infty$.

The proof of Theorem 2 is based on sign invariance of some new functionals introduced in [3].

In the following theorem we use a different approach based on an improved variant of the concavity method of Levine, see [5]. This new method allows us to prove stronger than Theorem 2 result for finite time blow up of the solutions to problem (2)–(4). Namely, we derive more general conditions on the initial data for nonexistence of global solutions.

Let us recall that the idea of the concavity method of Levine [6] is to reduce the finite time blow up of the solutions of nonlinear dispersive equations to proving blow up of the solution to the following ordinary differential inequality

$$\Psi''(t)\Psi(t) - \gamma \Psi'^2(t) \geq 0, \quad t \geq 0, \quad \gamma > 1. \tag{7}$$

Here $\Psi(t)$ is a nonnegative functional of the solution to the original dispersive equation. Later on, Straughan [12] and Korpusov [2] generalized (7) to

$$\Psi''(t)\Psi(t) - \gamma \Psi'^2(t) \geq -\beta \Psi(t), \quad t \geq 0, \quad \gamma > 1, \ \beta > 0. \tag{8}$$

In [5] we have improved the concavity method of Levine replacing the finite time blow up of the solutions of the equations with the blow up of the solutions to the following inequality

$$\Psi''(t)\Psi(t) - \gamma \Psi'^2(t) \geq \alpha \Psi^2(t) - \beta \Psi(t), \quad t \geq 0, \quad \gamma > 1, \ \alpha > 0, \ \beta > 0.$$

This new inequality allows us to obtain very general conditions for finite time blow up of the solutions to different nonlinear dispersion equations in comparison with the results derived by means of (7) and (8).

Theorem 3. (Main result) *Suppose $u(t,x)$ is the weak solution to* (2)–(5) *defined in the maximal existence time interval $[0,T_m)$, $T_m \le \infty$. If*

$$\langle u_0,u_1\rangle > 0, \qquad \langle u_0,u_0\rangle \neq 0 \tag{9}$$

and

$$\frac{\langle u_0,u_0\rangle}{p_1+1}\left[1-\left(1+\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2}\right)^{\frac{1-p_1}{2}}\right] + \frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle + \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} > E(0) > 0 \tag{10}$$

then $u(t,x)$ blows up for a finite time, i.e. $T_m < \infty$.

3. Proof of the main result

Proof of Theorem 3. Suppose that the weak solution of (2)-(5) is defined for every $t \ge 0$. For the function

$$\Psi(t) = \langle u,u\rangle = \|u\|^2 + \|u_x\|^2 + \|(-\Delta)^{-1/2}u\|^2$$

we have

$$\Psi'(t) = 2\langle u,u_t\rangle, \qquad \Psi''(t) = 2\langle u_t,u_t\rangle + 2\langle u,u_{tt}\rangle.$$

By means of equation (2) and equality (6) we get

$$\begin{aligned}\Psi''(t) &= 2\langle u_t,u_t\rangle - 2\|u\|_1^2 - 2\|(-\Delta)^{-1/2}u\|^2 + 2\int_{\mathbb{R}} uf(u)\,dx \\ &= 2\langle u_t,u_t\rangle - 2\langle u,u\rangle + 2\int_{\mathbb{R}} uf(u)\,dx \\ &= (p_1+3)\langle u_t,u_t\rangle - 2(p_1+1)E(0) + (p_1-1)\langle u,u\rangle + 2(p_1+1)B(t).\end{aligned}$$

Here $B(t) := B(u(t))$ is given by the expression

$$B(t) = \sum_{k=2}^{l}\frac{a_k(p_k-p_1)}{(p_k+1)(p_1+1)}\int_{\mathbb{R}}|u|^{p_k+1}\,dx + \sum_{j=1}^{s}\frac{b_j(p_1-q_j)}{(q_j+1)(p_1+1)}\int_{\mathbb{R}}|u|^{q_j+1}\,dx.$$

From (9) we have $\Psi(0) > 0$ and we suppose that $\Psi(t) > 0$ for $t \in [0, t_1)$, for some t_1, $0 < t_1 \leq \infty$.

We will show that $t_1 = \infty$. By contradiction, we suppose that $t_1 < \infty$ and hence $\Psi(t_1) = 0$. In $[0, t_1)$ the function $\Psi(t)$ satisfies the equality

$$\Psi''(t)\Psi(t) - \frac{p_1+3}{4}\Psi'^2(t) = (p_1-1)\Psi^2(t) - 2(p_1+1)E(0)\Psi(t) + H(t)\Psi(t), \tag{11}$$

where

$$H(t) := H(u(t)) = (p_1+3)\left\langle u_t - \frac{\langle u, u_t\rangle}{\langle u, u\rangle}.u, u_t - \frac{\langle u, u_t\rangle}{\langle u, u\rangle}.u \right\rangle + 2(p_1+1)B(t).$$

From (5) it follows that $B(t) \geq 0$ and hence

$$H(t) \geq 0 \qquad \text{for every} \quad t \in [0, t_1). \tag{12}$$

After the change

$$\Psi(t) = z^{\frac{4}{1-p_1}}(t) \quad \text{ or } \quad z(t) = \Psi^{\frac{1-p_1}{4}}(t)$$

we have from (11) and (12)

$$z' = \frac{1-p_1}{4}\Psi^{-\frac{3+p_1}{4}}\Psi',$$

$$\begin{aligned} z'' &= -\frac{p_1-1}{4}\left(\Psi''\Psi - \frac{p_1+3}{4}\Psi'^2\right)\Psi^{-\frac{p_1+7}{4}} \\ &= -\frac{1}{4}(p_1-1)^2 z + \frac{1}{2}(p_1^2-1)E(0)z^{\frac{p_1+3}{p_1-1}} - \frac{1}{4}(p_1-1)Hz. \end{aligned}$$

Thus $z(t)$ satisfies the inequality

$$z'' \leq -\frac{1}{4}(p_1-1)^2 z + \frac{1}{2}(p_1^2-1)E(0)z^{\frac{p_1+3}{p_1-1}} \tag{13}$$

and

$$\begin{aligned} z(0) &= \Psi^{\frac{1-p_1}{4}}(0) = \langle u_0, u_0\rangle^{\frac{1-p_1}{4}} > 0, \\ z'(0) &= \frac{1-p_1}{4}\Psi^{-\frac{p_1+3}{4}}(0)\Psi'(0) = \frac{1-p_1}{2}\langle u_0, u_0\rangle^{-\frac{p_1+3}{4}}\langle u_0, u_1\rangle < 0. \end{aligned}$$

We will show that

$$z'(t) < 0, \quad \text{or equivalently} \quad \Psi'(t) > 0, \quad \text{for every} \quad t \geq 0. \tag{14}$$

By contradiction we suppose that there exists a time t_0, $t_0 \in (0, t_1)$ such that $z'(t) < 0$ for $t \in [0, t_0)$ and $z'(t_0) = 0$.

Case 1:

$$\frac{p_1 - 1}{2(p_1 + 1)}\langle u_0, u_0\rangle + \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} \geq E(0) > 0 \tag{15}$$

First we prove (14) under more restricted condition (15) instead of (10).

From the monotonicity of $z(t)$ for $t \in [0, t_0)$ and (13) we get

$$\begin{aligned} z''(t) &\leq \frac{1}{2}(p_1^2 - 1)z^{\frac{p_1+3}{p_1-1}}(t)\left(-\frac{p_1 - 1}{2(p_1 + 1)}z^{-\frac{4}{p_1-1}}(t) + E(0)\right) \\ &\leq \frac{1}{2}(p_1^2 - 1)z^{\frac{p_1+3}{p_1-1}}(t)\left(-\frac{p_1 - 1}{2(p_1 + 1)}z^{-\frac{4}{p_1-1}}(0) + E(0)\right) \\ &= \frac{1}{2}(p_1^2 - 1)z^{\frac{p_1+3}{p_1-1}}(t)\left(-\frac{p_1 - 1}{2(p_1 + 1)}\langle u_0, u_0\rangle + E(0)\right). \end{aligned} \tag{16}$$

Multiplying (16) by $z'(t) < 0$ and integrating from 0 to t_0 we obtain from the monotonicity of $z(t)$ and (15) the following impossible chain of inequalities:

$$\begin{aligned} 0 = z'^2(t_0) \geq & z'^2(0) + \frac{(p_1 - 1)^2}{2}\left(-\frac{p_1 - 1}{2(p_1 + 1)}\langle u_0, u_0\rangle + E(0)\right) \\ & \times \left(z^{\frac{2(p_1+1)}{p_1-1}}(t_0) - z^{\frac{2(p_1+1)}{p_1-1}}(0)\right) \\ = & \frac{(p_1 - 1)^2}{4}\langle u_0, u_1\rangle^2\langle u_0, u_0\rangle^{-\frac{p_1+3}{2}} \\ & + \frac{(p_1 - 1)^2}{2}\left(E(0) - \frac{p_1 - 1}{2(p_1 + 1)}\langle u_0, u_0\rangle - \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}\right) \\ & \times \left(z^{\frac{2(p_1+1)}{p_1-1}}(t_0) - z^{\frac{2(p_1+1)}{p_1-1}}(0)\right) \\ & + \frac{(p_1 - 1)^2}{4}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}\left(z^{\frac{2(p_1+1)}{p_1-1}}(t_0) - z^{\frac{2(p_1+1)}{p_1-1}}(0)\right) \\ = & \frac{(p_1 - 1)^2}{2}\left(E(0) - \frac{p_1 - 1}{2(p_1 + 1)}\langle u_0, u_0\rangle - \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}\right) \\ & \times \left(z^{\frac{2(p_1+1)}{p_1-1}}(t_0) - z^{\frac{2(p_1+1)}{p_1-1}}(0)\right) \\ & + \frac{(p_1 - 1)^2}{4}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}z^{\frac{2(p_1+1)}{p_1-1}}(t_0) > 0. \end{aligned} \tag{17}$$

Thus our assumption $z'(t_0) = 0$ fails and $z'(t) < 0$ for $t \in [0, t_1)$. Hence $z(t) > 0$ and $\Psi(t) > 0$ for $t \in (0, t_1]$, i.e. $\Psi(t_1) > 0$ and $t_1 = \infty$ when (15) holds.

Case 2:

$$\begin{aligned}\frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle + \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} < E(0) < \frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle + \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} \\ + \frac{\langle u_0,u_0\rangle}{p_1+1}\left[1-\left(1+\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2}\right)^{\frac{1-p_1}{2}}\right]\end{aligned} \tag{18}$$

Multiplying (13) by $z'(t) < 0$ and integrating from 0 to $t \le t_0$ we get

$$z'^2(t) \ge -\frac{1}{4}(p_1-1)^2 z^2(t) + \frac{1}{2}(p_1-1)^2 E(0) z^{\frac{2(p_1+1)}{p_1-1}}(t) + C, \tag{19}$$

where

$$\begin{aligned} C &= z'^2(0) + \frac{(p_1-1)^2}{4} z^2(0) - \frac{(p_1-1)^2}{2} E(0) z^{\frac{2(p_1+1)}{p_1-1}}(0) \\ &= \frac{(p_1-1)^2}{16}\Psi^{-\frac{p_1+3}{2}}(0)\Psi'^2(0) + \frac{(p_1-1)^2}{4}\Psi^{\frac{1-p_1}{2}}(0) - \frac{(p_1-1)^2}{2}E(0)\Psi^{-\frac{p_1+1}{2}}(0) \\ &= \frac{(p_1-1)^2}{2}\Psi^{-\frac{p_1+1}{2}}(0)\left(\frac{1}{8}\frac{\Psi'^2(0)}{\Psi(0)} + \frac{1}{2}\Psi(0) - E(0)\right) \\ &= \frac{(p_1-1)^2}{2}\langle u_0,u_0\rangle^{-\frac{p_1+1}{2}}\left(\frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{1}{2}\langle u_0,u_0\rangle - E(0)\right). \end{aligned}$$

The constant C is positive since from (18) we get

$$\begin{aligned} C \ge &\frac{(p_1-1)^2}{2}\langle u_0,u_0\rangle^{-\frac{p_1+1}{2}}\left[\frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{1}{2}\langle u_0,u_0\rangle - \frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle\right. \\ &\left. -\frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} - \frac{\langle u_0,u_0\rangle}{p_1+1}\left(1-\left(1+\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2}\right)^{\frac{1-p_1}{2}}\right)\right] \\ = &\frac{(p_1-1)^2}{2(p_1+1)}\langle u_0,u_0\rangle^{\frac{1-p_1}{2}}\left(1+\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2}\right)^{\frac{1-p_1}{2}} > 0. \end{aligned}$$

Using the short notation

$$A = 1 + \frac{\langle u_0, u_1 \rangle^2}{\langle u_0, u_0 \rangle^2} > 1$$

we have from (19) the inequality

$$\frac{4z'^2(t)}{(p_1-1)^2} \geq -z^2(t) + 2E(0) z^{\frac{2(p_1+1)}{p_1-1}} + \frac{4C}{(p_1-1)^2} = P(z^2(t)), \tag{20}$$

where

$$P(\zeta) = -\zeta + 2E(0)\zeta^{\frac{p_1+1}{p_1-1}} + \frac{4C}{(p_1-1)^2}.$$

We will show that $P(\zeta) > 0$ for every $\zeta \in [0, \infty)$. Indeed,

$$P(0) = \frac{4C}{(p_1-1)^2} > 0, \quad P(\zeta) \to \infty \quad \text{for} \ \ \zeta \to \infty$$

and $P(\zeta)$ has a global minimum at the point ζ_1:

$$\zeta_1 = \left(\frac{p_1 - 1}{2(p_1+1)E(0)} \right)^{\frac{p_1-1}{2}}$$

because

$$P'(\zeta) = -1 + \frac{2(p_1+1)}{p_1-1} E(0) \zeta^{\frac{2}{p_1-1}} \quad \text{and} \quad P(\zeta) = 0 \quad \text{if and only if} \quad \zeta = \zeta_1.$$

Moreover,

$$P''(\zeta_1) = \frac{4(p_1+1)}{(p_1-1)^2} E(0) \zeta_1^{\frac{3-p_1}{p_1-1}} > 0$$

and hence ζ_1 is the unique point of minimum for the function $P(\zeta)$. Now we have to prove that $P(\zeta_1) > 0$. Tedious calculations give us

$$
\begin{aligned}
P(\zeta_1) = &-\left(\frac{p_1-1}{2(p_1+1)E(0)}\right)^{\frac{p_1-1}{2}} + 2E(0)\left(\frac{p_1-1}{2(p_1+1)E(0)}\right)^{\frac{p_1+1}{2}} + \frac{4C}{(p_1-1)^2} \\
= &-\left(\frac{p_1-1}{2(p_1+1)E(0)}\right)^{\frac{p_1-1}{2}} + \frac{p_1-1}{p_1+1}\left(\frac{p_1-1}{2(p_1+1)E(0)}\right)^{\frac{p_1-1}{2}} \\
&+ 2\langle u_0,u_0\rangle^{-\frac{p_1+1}{2}}\left(\frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{1}{2}\langle u_0,u_0\rangle - E(0)\right) \\
= &-\frac{2}{p_1+1}\left(\frac{p_1-1}{2(p_1+1)E(0)}\right)^{\frac{p_1-1}{2}} - 2\langle u_0,u_0\rangle^{-\frac{p_1+1}{2}}E(0) + \langle u_0,u_0\rangle^{\frac{1-p_1}{2}} \\
&+ \langle u_0,u_1\rangle^2\langle u_0,u_0\rangle^{-\frac{p_1+3}{2}} \\
= &\frac{2}{p_1+1}\langle u_0,u_0\rangle^{\frac{1-p_1}{2}} \\
&\times\left[-\left(\frac{(p_1-1)\langle u_0,u_0\rangle}{2(p_1+1)E(0)}\right)^{\frac{p_1-1}{2}} + \frac{p_1+1}{2}\left(1+\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2}\right) - \frac{E(0)(p_1+1)}{\langle u_0,u_0\rangle}\right] \\
= &\frac{2}{p_1+1}\langle u_0,u_0\rangle^{\frac{1-p_1}{2}}A^{\frac{p_1+1}{2}}\left(\frac{(p_1-1)\langle u_0,u_0\rangle}{2(p_1+1)E(0)}\right)^{\frac{p_1-1}{2}} \\
&\times\left[-A^{-\frac{1+p_1}{2}} + \frac{p_1+1}{2}A^{-\frac{1-p_1}{2}}\left(\frac{(p_1-1)\langle u_0,u_0\rangle}{2(p_1+1)E(0)}\right)^{\frac{1-p_1}{2}}\right. \\
&\left.-\frac{p_1-1}{2}\left(\frac{(p_1-1)\langle u_0,u_0\rangle}{2(p_1+1)E(0)}\right)^{-\frac{1+p_1}{2}}A^{-\frac{1+p_1}{2}}\right] \\
= &\frac{2}{p_1+1}\langle u_0,u_0\rangle^{\frac{1-p_1}{2}}Ay_1^{\frac{1-p_1}{2}}\left(-A^{-\frac{1+p_1}{2}} + \frac{p_1+1}{2}y_1^{\frac{p_1-1}{2}} - \frac{p_1-1}{2}y_1^{\frac{p_1+1}{2}}\right).
\end{aligned}
$$

In the above equality we use the following notation:

$$
y_1 = \frac{2(p_1+1)E(0)}{(p_1-1)\langle u_0,u_0\rangle A}.
$$

Thus we have

$$
P(\zeta_1) = \frac{2}{p_1+1}\langle u_0,u_0\rangle^{\frac{1-p_1}{2}}Ay_1^{\frac{1-p_1}{2}}Q(y_1), \tag{21}
$$

where

$$Q(y) = -A^{-\frac{1+p_1}{2}} + \frac{p_1+1}{2} y^{\frac{p_1-1}{2}} - \frac{p_1-1}{2} y^{\frac{p_1+1}{2}}.$$

We will prove that $Q(y_1) > 0$. For this purpose let us show that $Q(y)$ is a concave function for $y \geq 1$. Indeed,

$$\begin{aligned} Q'(y) &= \frac{1}{4}(p_1^2-1)\left(y^{\frac{p_1-3}{2}} - y^{\frac{p_1-1}{2}}\right), \\ Q''(y) &= \frac{1}{8}(p_1^2-1) y^{\frac{p_1-5}{2}} (p_1 - 3 - (p_1-1)y^2) < 0. \end{aligned}$$

Hence $Q(y)$ is a strictly concave function for $y \geq 1$. Moreover, we have

$$Q(1) = 1 - A^{-\frac{1+p_1}{2}} > 0 \quad \text{and} \quad Q\left(\frac{p_1+1}{p_1-1}\right) = -A^{-\frac{1+p_1}{2}} < 0.$$

Let L be the line that passes through the points $M_1 = (1, Q(1))$ and $M_2 = \left(\frac{p_1+1}{p_1-1}, Q\left(\frac{p_1+1}{p_1-1}\right)\right)$:

$$L = \left\{(y,\eta) : \eta - 1 + A^{-\frac{p_1+1}{2}} + \frac{p_1-1}{2}(y-1) = 0\right\}.$$

We denote by y_0 the intersection point of the line L with $\{\eta = 0\}$, i.e.

$$y_0 = 1 + \frac{2}{p_1-1}\left(1 - A^{-\frac{p_1+1}{2}}\right).$$

From the strict concavity of $Q(y)$ it follows that

$$Q(y) > 0 \quad \text{for every} \quad y \in [1, y_0). \tag{22}$$

From (18) we get the following estimates for y_1:

$$\begin{aligned} y_1 >& 1, \\ y_1 <& \frac{2(p_1+1)}{(p_1-1)\langle u_0, u_0\rangle A}\left[\frac{1}{2}\frac{(p_1-1)}{(p_1+1)}\langle u_0,u_0\rangle + \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{\langle u_0,u_0\rangle}{p_1+1}\left(1-A)^{\frac{1-p_1}{2}}\right)\right] \\ =& A^{-1}\left[1 + \frac{(p_1+1)}{(p_1-1)}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2} + \frac{2}{(p_1-1)}\left(1-A)^{\frac{1-p_1}{2}}\right)\right] \\ =& A^{-1}\left[\frac{p_1+1}{p_1-1} + \frac{p_1+1}{p_1-1}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2} - \frac{2}{(p_1-1)}A)^{\frac{1-p_1}{2}}\right] \\ =& \frac{p_1+1}{p_1-1} - \frac{2}{(p_1-1)}A^{-\frac{1+p_1}{2}} = 1 + \frac{2}{p_1-1}\left(1 - A^{-\frac{1+p_1}{2}}\right) = y_0. \end{aligned}$$

Since $y_1 \in (1, y_0)$ from (22) it follows that $Q(y_1) > 0$. From (21) we get $P(\zeta_1) > 0$ and hence

$$P(\zeta) \geq P(\zeta_1) > 0 \quad \text{for} \quad \zeta \in [0, \infty). \tag{23}$$

Replacing in (20) t with t_0, from (23) we have the following impossible chain of inequalities:

$$0 = \frac{4}{(p_1 - 1)^2} z'^2(t_0)) = P(z^2(t_0)) = P(\zeta(t_0)) \geq P(\zeta_1) > 0.$$

Thus $z'(t) < 0$ for every $t \geq 0$, or equivalently, $\Psi'(t) > 0$ for every $t \geq 0$, i.e. $t_0 = \infty$, $t_1 = \infty$.

In the both cases, from (17) and (20), we obtain

$$z'(t) \leq -K < 0 \quad \text{for every} \quad t \geq 0, \tag{24}$$

where K is a positive constant. After integration of (24) from 0 to t we get

$$z(t) \leq z(0) - Kt$$

and $z(t) \to 0$ for $t \to t_*$

$$t_* \leq \frac{z(0)}{K} < \infty$$

Equivalently, $\Psi(t) \to \infty$ for $t \to t_*$, which proves Theorem 3.

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