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BIFURCATIONS OF PERIODIC SOLUTIONS TO DIFFERENTIAL EQUATIONS AND MULTIVALENT GUIDING FUNCTIONS METHOD*

Valeri Obukhovskii, Pietro Zecca, Nguyen Van Loi,
Sergei Kornev

In this paper, we define a new class of multivalent guiding functions called *local multivalent guiding functions* and use it to study the global bifurcation problem of periodic soluions to a parameterized differential equation.

1. Introduction

The paper deals with the application of the method of multivalent guiding functions to the study of the global bifurcation of periodic solutions of the following parameterized family of differential equation

$$x'(t) = f\big(t, x(t), \mu\big), \tag{1.1}$$

where $f\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ $(n \geq 2)$ is a given continuous function and $\mu \in \mathbb{R}$ is a parameter.

Let us recall that the method of guiding functions was developed by A. I. Perov and M. A. Krasnosel'skii and others (see [12, 13]) and is one of effective tools for the study of periodic oscillations in systems governed by differential

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equations or inclusions. Up to now, this method is developed into many various ways (see [16] for more details).

The application of guiding functions to the study of a global bifurcation problem was suggested first by W. Kryszewski (see [14]) and the other approach based on the use of integral guiding functions was developed in the works [16, 17]. It is worth noting that the method of multivalent guiding functions was introduced by D.I. Rachinskii (see [18]). Some of its developments can be found in [10, 11].

In the present paper, we focus ourselves on the global bifurcation of periodic solutions to (1.1) by using multivalent guiding functions. The paper is organized in the following way. In the next section we recall some basic notions from the multivalued analysis, degree theory and bifurcation theory. The main result is given in Section 3.

2. Preliminaries

2.1. Multimaps

Let X, Y be metric spaces. Denote by $P(Y)$ [resp., $K(Y)$] the collection of all nonempty [respectively, nonempty compact] subsets of Y.

Definition 1. (see, e.g., [1, 4, 8, 9]) *A multivalued map (multimap) $F\colon X \to P(Y)$ is said to be:*

(*i*) *upper semicontinuous (u.s.c.), if for every open subset $V \subset Y$ the set*

$$F_+^{-1}(V) = \{x \in X : F(x) \subset V\}$$

is open in X;

(*ii*) *completely u.s.c. if it is u.s.c. and maps every bounded subset $U \subset X$ into a relatively compact subset $F(U)$ of Y;*

(*iii*) *compact, if the set*

$$F(X) := \bigcup_{x \in X} F(x)$$

is relatively compact in Y.

A set $M \in K(Y)$ is said to be *aspheric* (or UV^∞, or ∞-*proximally connected*) (see, e.g., [15, 4, 5]), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous map $\sigma\colon S^n \to O_\delta(M)$, $n = 0, 1, 2, \cdots$, can be extended to a continuous map $\widetilde{\sigma}\colon B^{n+1} \to O_\varepsilon(M)$, where $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$,

$B^{n+1} = \{x \in \mathbb{R}^{n+1} \colon \|x\| \leq 1\}$, and $O_\delta(M)$ $[O_\varepsilon(M)]$ denote the δ-neighborhood [resp., ε-neighborhood] of the set M.

Definition 2. (see [7]) *A nonempty compact space A is said to be an R_δ-set if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.*

Definition 3. (see [4]) *A u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a J-multimap ($\Sigma \in J(X,Y)$) if each value $\Sigma(x)$, $x \in X$, is an aspheric set.*

Now let us recall (see, e.g., [3]) that a metric space X is called *the absolute retract (the AR-space)* [resp., *the absolute neighborhood retract (the ANR-space)]* provided for each homeomorphism h taking it into a closed subset of a metric space X', the set $h(X)$ is the retract of X' [resp., of its open neighborhood in X']. Notice that the class of ANR-spaces is broad enough: in particular, a finite-dimensional compact set is the ANR-space if and only if it is locally contractible. In turn, it means that compact polyhedrons and compact finite-dimensional manifolds are the ANR-spaces. The union of a finite number of convex closed subsets in a normed space is also the ANR-space.

Proposition 4. (see [4]) *Let Z be an ANR-space. In each of the following cases an u.s.c. multimap $\Sigma : X \to K(Z)$ is a J-multimap:*
for each $x \in X$ the value $\Sigma(x)$ is
a) a convex set;
b) a contractible set;
c) an R_δ-set;
d) an AR-space.
In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

Definition 5. *Let X, Y be Banach spaces. By $CJ(X,Y)$ we will denote the collection of all multimaps $F \colon X \to K(Y)$ that may be represented in the form of the composition*

$$F = \varphi \circ G,$$

where, for a normed space Z, $G \in J(X,Z)$ and $\varphi \colon Z \to Y$ is a continuous map.

Let us mention that if U is an open bounded subset of a Banach space X and $F \colon \overline{U} \to K(X)$ is a compact $CJ-$multimap such that $x \notin F(x)$ for all $x \in \partial U$, then *the topological degree* $\deg(i - F, \overline{U})$ *of the corresponding multivalued vector field* $i - F$ is well-defined (here i denotes the inclusion map). This topological characteristic has all usual properties of the classical Brouwer topological degree (see, e.g., [2, 4]).

2.2. A global bifurcation theorem

Let X be a Bamach space. A ball of radius r centered at 0 in X is denoted by $B_X(0,r)$. Consider the following inclusion

$$x \in F(x,\mu), \tag{2.1}$$

where $F\colon X\times\mathbb{R}\to K(X)$ is a $CJ-$multimap and $\mu\in\mathbb{R}$ is a parameter.

Assume that:

($F1$) F is a completely u.s.c. multimap with $0\in F(0,\mu)$ for all $\mu\in\mathbb{R}$;

($F2$) there exists $\varepsilon_0>0$ such that for each μ, $0<|\mu|<\varepsilon_0$, there is $\delta_\mu>0$ such that $x\notin F(x,\mu)$ for all $x\in B_X(0,\delta_\mu)\setminus\{0\}$;

($F3$) the bifurcation index at $(0,0)$:

$$Bi(F;(0,0)) = \lim_{\mu\to 0^+} \deg(i-F, B_X(0,\delta_\mu)) - \lim_{\mu\to 0^-} \deg(i-F, B_X(0,\delta_\mu))$$

is non-zero.

A point $(0,\mu_0)$, $\mu_0\in\mathbb{R}$, is said to be a bifurcation point of (2.1) if for every open subset $U\subset X\times\mathbb{R}$ containing $(0,\mu_0)$ there is a nontrivial solution of (2.1) in U.

Let $\mathcal{S}$ be the set of all nontrivial solutions of (2.1).

Theorem 6. (see, e.g., [14]) *Let conditions* $(F1)-(F3)$ *hold. Then* $(0,0)$ *is a bifurcation point for problem* (2.1) *and there exists a connected subset* $\mathcal{R}\subset\mathcal{S}$ *such that* $(0,0)\in\overline{\mathcal{R}}$ *and either* $\mathcal{R}$ *is unbounded or* $\overline{\mathcal{R}}\ni(0,\mu_*)$ *for some* $\mu_*\neq 0$.

3. Main result

Consider again problem (1.1) under the following assumptions:

($f1$) the continuous function $f\colon\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ is $T-$periodic with respect to the first argument ($T>0$);

($f2$) there exist a constant $c>0$ and a positive function $h\colon[0,\infty)\to[0,\infty)$ such that $h(0)=0$ and

$$|f(t,x,\mu)| \le c\,h(|\mu|)\,|x|,$$

for all $(t,x,\mu)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}$;

($f3$) for each $\mu\in\mathbb{R}$ problem (1.1) admits a solution $x\colon[0,T]\to\mathbb{R}^n$ with $x(0)=x(T)=0$.

Let a two-dimensional plane $\mathbb{R}^2$ be chosen in the space $\mathbb{R}^n$ and $\mathbb{R}^{n-2}$ be its complementary space. Let $q\colon \mathbb{R}^n \to \mathbb{R}^2$ be the projection operator and $p = i - q$, where i denotes the identity operator. Let $\{\varphi, \rho\}$ be a fixed polar coordinate system on the plane $\mathbb{R}^2$ and elements of $\mathbb{R}^{n-2}$ denoted by ξ.

Problem (1.1) is equivalent to the following one:

$$\begin{cases} \dfrac{d\xi}{dt} = g(t,\xi,\varphi,\rho,\mu), \\ \dfrac{d\varphi}{dt} = h(t,\xi,\varphi,\rho,\mu), \\ \dfrac{d\rho}{dt} = w(t,\xi,\varphi,\rho,\mu), \end{cases} \tag{3.1}$$

where continuous functions g, h, w are $T-$periodic with respect to the first argument.

For each $(z,\mu) \in \mathbb{R}^n \times \mathbb{R}$ let Π_z be the solution set of the Cauchy problem

$$\begin{cases} x' = f(t,x,\mu), \\ x(0) = z. \end{cases}$$

It is well know (see, e.g., [4]) that Π_z is a $R_\delta-$set in $C([0,T];\mathbb{R}^n)$ and if we define the multimap

$$\Pi\colon \mathbb{R}^n \times \mathbb{R} \to C([0,T];\mathbb{R}^n), \quad \Pi(z,\mu) = \Pi_{(z,\mu)},$$

then Π is a $J-$multimap (see [9]).
Now we define the translation multioperator in the following way

$$U_T\colon \mathbb{R}^n \times \mathbb{R} \to K(\mathbb{R}^n),$$
$$U_T(z,\mu) = \big\{x(T)\colon x \in \Pi(z,\mu)\big\}.$$

Then we can replace problem (1.1) with the equivalent problem

$$z \in U_T(z,\mu). \tag{3.2}$$

From $(f1)-(f2)$ it follows that U_T is a completely u.s.c. $CJ-$multimap and

$$0 \in U_T(0,\mu)$$

for all $\mu \in \mathbb{R}$. Let us denote by $\mathcal{S}$ the set of all nontrivial solutions of problem (3.2), i.e.,

$$\mathcal{S} = \{(z,\mu) \in \mathbb{R}^n \times \mathbb{R}\colon z \neq 0 \text{ and } z \in U_T(z,\mu)\}.$$

Consider now the Riemann surface

$$\mathcal{R} = \{(\varphi, \rho)\colon \varphi \in (-\infty, \infty), \rho \in (0, \infty)\}.$$

Let $W\colon \mathcal{R} \times \mathbb{R} \to \mathbb{R}$ be such that for each $\mu \in \mathbb{R}$ the function $W(\cdot, \cdot, \mu)$ is continuously differentiable and

$$\frac{\partial}{\partial \varphi} W(\varphi, \rho, \mu) > 0, \quad (\varphi, \rho) \in \mathcal{R}, \tag{3.3}$$

$$W(\varphi + 2\pi, \rho, \mu) = W(\varphi, \rho, \mu) + 2\pi, \quad (\varphi, \rho) \in \mathcal{R}. \tag{3.4}$$

From the last equality it follows that

$$\nabla W(\varphi + 2\pi, \rho, \mu) = \nabla W(\varphi, \rho, \mu),$$

where $\nabla W(\varphi, \rho, \mu) = \left(\frac{\partial W}{\partial \varphi}, \frac{\partial W}{\partial \rho}\right)$.

For $r_* > 0$ put

$$G(r_*) = \{z \in \mathbb{R}^n\colon |pz| < r_*,\ |qz| < r_*\}.$$

Definition 7. *A pair of smooth functions $V(\xi, \mu), W(\varphi, \rho, \mu)$ satisfying relations* (3.3)–(3.4) *is said to be a local multivalent guiding function for problem* (3.1) *at* $(0,0)$, *if there exists $\varepsilon_0 > 0$ such that for each μ, $0 < |\mu| \le \varepsilon_0$, $\nabla V(0, \mu) = 0$ there is $r_\mu > 0$ such that:*

($a1$) *for $0 < |pz| < r_\mu$ and $|qz| < r_\mu$:*

$$\langle \nabla V(\xi, \mu), g(t, \xi, \varphi, \rho, \mu) \rangle < 0;$$

($a2$) *for $0 < |qz| < r_\mu$:*

$$\frac{\partial W(\varphi, \rho, \mu)}{\partial \varphi} h(t, 0, \varphi, \rho, \mu) + \frac{\partial W(\varphi, \rho, \mu)}{\partial \rho} w(t, 0, \varphi, \rho, \mu) < 0;$$

($a3$) *the functions*

$$\alpha(t, \mu) = \sup_{z \in G(r_\mu)} \langle \nabla W(qz, \mu), qf(t, z, \mu) \rangle$$

and

$$\beta(t, \mu) = \inf_{z \in G(r_\mu)} \langle \nabla W(qz, \mu), qf(t, z, \mu) \rangle$$

are continuous;

$(a4)$ *there is an integer number* N_μ *such that*

$$2\pi(N_\mu - 1) < \int_0^T \alpha(s,\mu)ds \quad \text{and} \quad \int_0^T \beta(s,\mu)ds < 2\pi N_\mu.$$

From the definition above it follows that for each $\mu\colon 0 < |\mu| \le \varepsilon_0$ the topological degree $\deg\big(\nabla V(\cdot,\mu), B_{\mathbb{R}^{n-2}}(0,r_\mu)\big)$ is well defined. Denote

$$\operatorname{ind} V = \lim_{\mu\to 0^+} \deg\big(\nabla V(\cdot,\mu), B_{\mathbb{R}^{n-2}}(0,r_\mu)\big) - \lim_{\mu\to 0^-} \deg\big(\nabla V(\cdot,\mu), B_{\mathbb{R}^{n-2}}(0,r_\mu)\big).$$

Theorem 8. *Let conditions* $(f1)$–$(f3)$ *hold. Assume, in addition, that there is a local multivalent guiding function for problem* (3.1) *at* $(0,0)$ *such that* $\operatorname{ind} V \neq 0$ *then* $(0,0)$ *is a bifurcation point of* (1.1) *and there exists a connected subset* $\mathcal{C} \subset \mathcal{S}$ *such that:*

(i) *each point* $(z,\mu) \in \overline{\mathcal{C}}$ *corresponds to a solution* $x\colon [0,T] \to \mathbb{R}^n$ *of* (1.1) *with* $x(0) = x(T) = z$*;*

(ii) $(0,0) \in \overline{\mathcal{C}}$ *and either* $\mathcal{C}$ *is unbounded or* $\overline{\mathcal{C}} \ni (0,\mu_*)$ *for some* $\mu_* \neq 0$.

In particular, there are a sequence $\{x_n\}$ *of solutions to* (1.1), $x_n(0) = x_n(T) = z_n$, $\{z_n\} \subset \mathcal{S}$, *such that either* $\{x_n\}$ *is unbounded or* $\{x_n\}$ *converges to a solution* x_* *of* (1.1) *with* $x_*(0) = x_*(T) = 0$.

Proof. It is clear that multimap U_T satisfies condition $(F1)$ of Theorem 6. Let us show that it satisfies also condition $(F2)$. Toward this goal, for each $\mu, 0 < |\mu| \le \varepsilon_0$, let us show that problem (3.2) has only the trivial solution on the ball $B_{\mathbb{R}^n}(0,\delta_\mu)$, where

$$\delta_\mu = \frac{1}{2e^{cTh(|\mu|)}} r_\mu.$$

In fact, assume that there exists a nontrivial solution $z \in B_{\mathbb{R}^n}(0,\delta_\mu)$. Then there exists a function x such that

$$\begin{cases} x' = f(t,x,\mu) \ \text{ for } \ t \in [0,T], \\ x(0) = x(T) = z. \end{cases}$$

From the relation

$$x(t) = x(0) + \int_0^t f(s,x(s),\mu)ds$$

it follows that

$$|x(t)| \leq |z| + \int_0^t |f(s, x(s), \mu)|ds \leq |z| + \int_0^t c\,h(|\mu|)\,|x(s)|ds.$$

By applying the Gronwall Lemma (see, e.g., [6]) we obtain

$$|x(t)| \leq |z|e^{cTh(|\mu|)} \leq \frac{1}{2}r_\mu < r_\mu.$$

If there exists $\Omega \subset (0, T)$ such that

$$\xi(t) = px(t) = 0 \quad \text{for} \quad t \in \Omega$$

and

$$\xi(t) \neq 0 \ \text{ for } \ t \in (0, T) \setminus \Omega,$$

then from Definition 7 it follows that

$$\begin{aligned} &\big\langle \nabla V(\xi, \mu), g(t, \xi, \varphi, \rho, \mu) \big\rangle = 0 \ \text{ for all } \quad t \in \Omega, \quad \text{and} \\ &\big\langle \nabla V(\xi, \mu), g(t, \xi, \varphi, \rho, \mu) \big\rangle < 0 \ \text{ for all } \quad t \in (0, T) \setminus \Omega. \end{aligned}$$

Consequently,

$$\int_0^T \big\langle \nabla V(\xi, \mu), g(t, \xi, \varphi, \rho, \mu) \big\rangle dt < 0.$$

On the other hand,

$$\begin{aligned} \int_0^T \big\langle \nabla V(\xi, \mu), g(t, \xi, \varphi, \rho, \mu) \big\rangle dt &= \int_0^T \big\langle \nabla V(\xi, \mu), \frac{d\xi}{dt} \big\rangle dt \\ &= V(\xi(T), \mu) - V(\xi(0), \mu) = 0, \end{aligned}$$

giving the contradiction.

If $\xi(t) = 0$ for all $t \in (0, T)$, e.g., $x(t) = qx(t) \in \mathbb{R}^2$ for all $t \in (0, T)$, then from the fact that $x(\cdot)$ is the nonzero function it follows that there exists $\Omega \subset (0, T)$ such that $x(t) \neq 0$ for all $t \in [0, T] \setminus \Omega$ and $x(t) = 0$ for $t \in \Omega$. Therefore,

$$\frac{\partial W(\varphi, \rho, \mu)}{\partial \varphi} h(t, 0, \varphi(t), \rho(t), \mu) + \frac{\partial W(\varphi, \rho, \mu)}{\partial \rho} w(t, 0, \varphi(t), \rho(t), \mu) < 0,$$

for $t \in [0, T] \setminus \Omega$.
From $(f2)$ it follows that $f(t, x(t), \mu) = 0$ for all $t \in \Omega$. So,

$$h(t, 0, \varphi(t), \rho(t), \mu) = w(t, 0, \varphi(t), \rho(t), \mu) = 0, \quad \forall t \in \Omega. \tag{3.5}$$

Hence,

$$\int_0^T \left(\frac{\partial W(\varphi,\rho,\mu)}{\partial \varphi} h(t,0,\varphi(t),\rho(t),\mu) + \frac{\partial W(\varphi,\rho,\mu)}{\partial \rho} w(t,0,\varphi(t),\rho(t),\mu)\right) dt < 0.$$

On the other hand,

$$\int_0^T \left(\frac{\partial W(\varphi,\rho,\mu)}{\partial \varphi} h(t,0,\varphi(t),\rho(t),\mu) + \frac{\partial W(\varphi,\rho,\mu)}{\partial \rho} w(t,0,\varphi(t),\rho(t),\mu)\right) dt \\ = W\big(\varphi(T),\rho(T),\mu\big) - W\big(\varphi(0),\rho(0),\mu\big) = 0,$$

giving the contradiction. Thus, condition $(F2)$ holds true.
So, the topological degree $\deg\big(i - U_T(\cdot,\mu), B_{\mathbb{R}^n}(0,\delta_\mu)\big)$ is well defined. For its evaluation, choose $\widetilde{r}_\mu > 0$ such that

$$\overline{G(\widetilde{r}_\mu)} \subset B_{\mathbb{R}^n}(0,\delta_\mu).$$

Since problem (3.2) has only trivial solution on $B_{\mathbb{R}^n}(0,\delta_\mu)$ we have

$$\deg\big(i - U_T(\cdot,\mu), B_{\mathbb{R}^n}(0,\delta_\mu)\big) = \deg\big(i - U_T(\cdot,\mu), \overline{G(\widetilde{r}_\mu)}\big).$$

Consider the multimap

$$\Phi(t,z,\mu) = z - pU_t(z,\mu) - qU_T(z,\mu),\ t \in (0,T],\ z \in \overline{G(\widetilde{r}_\mu)}.$$

Since $p\Phi(t,z,\mu) = p(z - U_t(z,\mu)),\ q\Phi(t,z,\mu) = q(z - U_T(z,\mu))$ we have

$$0 \notin p\Phi(t,z,\mu), \quad t \in (0,T], z \in \partial G(\widetilde{r}_\mu); \tag{3.6}$$

$$0 \notin q\Phi(t,z,\mu), \quad t \in (0,T], z \in \partial G(\widetilde{r}_\mu). \tag{3.7}$$

Hence,

$$0 \notin \Phi(t,z,\mu), \quad t \in (0,T], z \in \partial G(\widetilde{r}_\mu).$$

If $U_t(z,\mu)$ is a trajectory starting from the point z, then

$$\frac{d}{dt} U_t(z,\mu) = f(t, U_t(z,\mu)),$$

consequently,

$$\frac{d}{dt} pU_t(z,\mu) = pf(t, U_t(z,\mu)).$$

For $t = 0$, we have

$$\lim_{t\to+0}\frac{pU_t(z,\mu)-pz}{t} = pf(0,z,\mu).$$

Therefore, the vector fields $-pf(0,z,\mu)$ and $p(z-U_\varepsilon(z,\mu))$ have no opposite directions provided $\varepsilon > 0$ is sufficiently small.

From $(a1)$ it follows that

$$\langle \nabla V(pz,\mu), -pf(0,z,\mu)\rangle > 0, \quad z \in \partial G(\widetilde{r}_\mu).$$

Since

$$0 \notin q(z-U_T(z,\mu)), \quad t \in (0,T], z \in \partial G(\widetilde{r}_\mu),$$

the fields

$$-pf(0,z,\mu) + q(z-U_T(z,\mu)),$$
$$\nabla V(pz,\mu) + q(z-U_T(z,\mu)),$$

and the fields

$$-pf(0,z,\mu) + q(z-U_T(z,\mu)),$$
$$p(z-U_\varepsilon(z,\mu)) + q(z-U_T(z,\mu)) = \Phi(\varepsilon,z,\mu)$$

have no opposite directions for $z \in \partial G(\widetilde{r}_\mu)$.
Therefore,

$$\deg\big(\nabla V(pz,\mu) + q(z-U_T(z,\mu)), \overline{G(\widetilde{r}_\mu)}\big) =$$
$$= \deg\big(-pf(0,z,\mu) + q(z-U_T(z,\mu)), \overline{G(\widetilde{r}_\mu)}\big) = \deg\big(\Phi(\varepsilon,z,\mu), \overline{G(\widetilde{r}_\mu)}\big).$$

Hence,

$$\deg\big(i-U_T(\cdot,\mu), \overline{G(\widetilde{r}_\mu)}\big) = \deg\big(\nabla V(pz,\mu) + q(z-U_T(z,\mu)), \overline{G(\widetilde{r}_\mu)}\big). \tag{3.8}$$

From (3.3) it follows that the equation

$$W(\varphi,\rho,\mu) = w$$

has a unique solution for each φ.
Define the function $\Psi(\lambda,z,\mu) : [0,1]\times\overline{G(\widetilde{r}_\mu)}\times\mathbb{R}\to\mathbb{R}^2$ by

$$\begin{gathered}\varphi(\Psi(\lambda,z,\mu)) = \theta\Big((1-\lambda)\rho(qU_T(z,\mu)),\, W\big(\varphi(qU_T(z,\mu)),\rho(qU_T(z,\mu)),\mu\big)\Big),\\ \rho(\Psi(\lambda,z,\mu)) = (1-\lambda)\rho(qU_T(z,\mu)).\end{gathered} \tag{3.9}$$

Assuming $\varphi = \varphi(qU_T(z,\mu))$ and $\rho = \rho(qU_T(z,\mu))$ in the relation

$$\theta(\rho, W(\varphi,\rho,\mu)) \equiv \varphi,$$

we have

$$\theta\Big(\rho(qU_T(z,\mu)),\, W\big(\varphi(qU_T(z,\mu)),\rho(qU_T(z,\mu)),\mu\big)\Big) = \varphi\big(qU_T(z,\mu)\big)$$

or

$$\theta\big(\Psi(0,z,\mu)\big) = \varphi\big(qU_T(z,\mu)\big).$$

According to the relation $\rho(\Psi(0,z,\mu)) = \rho(qU_T(z,\mu))$, $\rho(\Psi(1,z,\mu)) \equiv 0$ we obtain

$$\Psi(0,z,\mu) = qU_T(z,\mu), \quad \Psi(1,z,\mu) \equiv 0.$$

Thus, the curve $\Gamma_z(\lambda) = \Psi(\lambda,z,\mu)$, where $0 \le \lambda \le 1$ and z,μ are fixed, connects the points $\xi_0 = qU_T(z,\mu)$ and $\xi_1 = 0$. Further, since

$$W(\theta(\rho,w),\rho,\mu) = w,$$

we have

$$W\big(\varphi(\Psi(\lambda,z,\mu)),\rho(\Psi(\lambda,z,\mu)),\mu\big) = W\big(\varphi(qU_T(z,\mu)),\rho(qU_T(z,\mu)),\mu\big),$$

i.e., the function $W(\varphi,\rho,\mu)$ has the values not depending on λ on $\Gamma_z(\lambda)$.

Put

$$\Phi_1(\lambda,z,\mu) = \nabla V(pz,\mu) + qz - \Psi(\lambda,z,\mu), \quad \lambda \in [0,1],\ z \in \overline{G(\widetilde{r}_\mu)}.$$

The continuous deformation $\Phi_1(\lambda,z,\mu)$ connects the fileds

$$\nabla V(pz,\mu) + q(z - U_T(z,\mu)) \ \text{ and } \ \nabla V(pz,\mu) + qz.$$

Let us show that it is non-degenerate on $\partial G(\widetilde{r}_\mu)$, and moreover,

$$0 \notin q\Phi_1(\lambda,z,\mu), \quad \lambda \in [0,1],\ z \in \partial G(\widetilde{r}_\mu). \tag{3.10}$$

To the contrary, assume that there exist $\nu_* \in [0,1]$, $z_* \in \partial G(\widetilde{r}_\mu)$, such that

$$0 \in q\Phi_1(\lambda_*,z_*,\mu), \ \text{ i.e., } \ 0 \in qz_* - q\Psi(\lambda_*,z_*,\mu).$$

Then

$$\varphi(\Psi(\lambda_*,z_*,\mu)) = \varphi(qz_*) + 2\pi k, \quad \rho(\Psi(\lambda_*,z_*,\mu)) = \rho(qz_*).$$

From (3.4) it follows that

$$W\big(\varphi(\Psi(\lambda_*,z_*,\mu)),\rho(\Psi(\lambda_*,z_*,\mu)),\mu\big) = W\big(\varphi(qz_*),\rho(qz_*),\mu\big) + 2\pi k.$$

By virtue of

$$W\big(\varphi(\Psi(\lambda_*, z_*, \mu)), \rho(\Psi(\lambda_*, z_*, \mu)), \mu\big) = W\big(\varphi(qU_T z_*), \rho(qU_T z_*), \mu\big)$$

we have

$$W\big(\varphi(qU_T(z_*, \mu)), \rho(qU_T(z_*, \mu)), \mu\big) - W\big(\varphi(qz_*), \rho(qz_*), \mu\big) = 2\pi k. \tag{3.11}$$

Assume that $\omega_*(t, \mu) = W\big(\varphi(qU_t(z_*, \mu)), \rho(qU_t(z_*, \mu)), \mu\big)$. Since $z_* \in G(\widetilde{r}_\mu)$, we obtain

$$U_t(z_*, \mu) \in G(r_\mu), \quad t \in (0, T].$$

Therefore

$$2\pi(N_\mu - 1) < \omega_*(T, \mu) - \omega_*(0, \mu) < 2\pi N_\mu.$$

According to the equation

$$\omega_*(0, \mu) = W(\varphi(qz_*), \rho(qz_*), \mu),\ \omega_*(T, \mu) = W(\varphi(qU_T(z_*, \mu)), \rho(qU_T(z_*, \mu)), \mu),$$

we have

$$2\pi(N_\mu - 1) < W(\varphi(qz_*), \rho(qz_*), \mu) - W(\varphi(qU_T(z_*, \mu)), \rho(qU_T(z_*, \mu)), \mu) < 2\pi N_\mu,$$

giving the contradiction to (3.11). From (3.10) and

$$p\Phi_1(\lambda, z, \mu) = \nabla V(pz, \mu) \neq 0, \quad z_* \in \partial G(\widetilde{r}_\mu),$$

it follows that

$$\deg\big(\nabla V(pz, \mu) + q(z - U_T(z, \mu)), \overline{G(\widetilde{r}_\mu)}\big) = \\ = \deg\big(\nabla V(pz, \mu) + qz, \overline{G(\widetilde{r}_\mu)}\big).$$

By virtue of (3.8)

$$\deg\big(i - U_T(\cdot, \mu), \overline{G(\widetilde{r}_\mu)}\big) = \deg\big(\nabla V(pz, \mu) + qz, \overline{G(\widetilde{r}_\mu)}\big).$$

So,

$$\deg(\nabla V(pz, \mu) + qz, \overline{G(\widetilde{r}_\mu)}) = \deg(\nabla V(pz, \mu), \overline{G(\widetilde{r}_\mu)}) \cdot \deg(qz, \overline{G(\widetilde{r}_\mu)}) \\ = \deg\big(\nabla V(pz, \mu), B_{\mathbb{R}^{n-2}}(0, r_\mu)\big).$$

Now, our proof is completed by applying Theorem 6 and the fact that $\operatorname{ind} V \neq 0$. □

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Valeri Obukhovskii
e-mail: valerio-ob2000@mail.ru
Sergei Kornev
e-mail: kornev_vrn@rambler.ru
Faculty of Physics and Mathematics
Voronezh State Pedagogical University
394043 Voronezh, Russia
and
the RUDN University
Moscow, Russia

Pietro Zecca
Dipartimento di Matematica
Università degli Studi di Firenze
Firenze, Italy
e-mail: zecca@unifi.it

Nguyen Van Loi
Division of Computational
Mathematics and Engineering
Institute for Computational Science
Ton Duc Thang University
Ho Chi Minh city, Viet Nam
and
Faculty of Civil Engineering
Ton Duc Thang University
Ho Chi Minh city, Viet Nam
e-mail: nguyenvanloi1@tdt.edu.vn