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GLOBAL SOLVABILITY TO DOUBLE DISPERSION EQUATION WITH BERNOULLI TYPE NONLINEARITY VIA ONE PARAMETRIC FAMILY OF POTENTIAL WELLS*

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One parametric family of potential wells for double dispersion equation with Bernoulli type nonlinearity is introduced. Sign preserving properties of the Nehari functionals are obtained. Global existence of the weak solution to the Cauchy problem is proved for wider class of initial data than the corresponding ones in the classical potential well method.

1. Introduction

In this paper we study the global solvability of the Cauchy problem to the double dispersion equation

$$(1)\qquad u_{tt} - u_{xx} - u_{ttxx} + u_{xxxx} = (f(u))_{xx} \qquad \text{for } x \in \mathbb{R},\ \ t \in [0,T),\ T \le \infty,$$

$$(2)\qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \qquad \text{for } x \in \mathbb{R},$$

where

$$(3)\qquad u_0(x) \in \mathrm{H}^1(\mathbb{R}), \quad u_1(x) \in \mathrm{L}^2(\mathbb{R}), \quad (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R})$$

Here $(-\Delta)^{-s} u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right)$ for $s > 0$, $\mathcal{F}(u)$ and $\mathcal{F}^{-1}(u)$ are the Fourier and the inverse Fourier transform, respectively.

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The nonlinear term $f(u)$ has one of the following forms:

$$\begin{aligned} &f(u) = a|u|^p u + b|u|^{2p} u \quad \text{or} \\ &f(u) = a|u|^{p+1} + b|u|^{2p} u, \quad p > 0, \quad a, b = \text{const}, \ b < 0. \end{aligned} \tag{4}$$

Nonlinearity (4) covers the cases of quadratic-cubic nonlinearity $f(u) = u^2 + u^3$ and cubic-quintic nonlinearity $f(u) = u^3 + u^5$, which appear in the theory of atomic chains [1] and shape-memory alloys [2].

We call nonlinearity (4) generalized Bernoulli type nonlinearity (or generalized Lenard nonlinearity) because the stationary problem corresponding to (1)

$$\psi''(x) = \psi(x) + f(\psi(x)), \qquad x \in \mathbb{R}$$

is known in the literature as generalized Bernoulli equation.

For the time being equation (1) has been intensively investigated for single nonlinear terms

$$f(u) = a|u|^p, \quad f(u) = a|u|^{p-1}u, \quad p > 1, \quad a = \text{const}, \tag{5}$$

see [3, 4, 5, 6] and references therein. In [7] the author considers the combined power-type nonlinearities with the following special sign conditions on the coefficients a_k:

$$\begin{aligned} &f(u) = \sum_{k=1}^{m} a_k |u|^{p_k - 1} u, \qquad 1 < p_1 < \cdots < p_m \ \text{ and } \ \exists\, \bar{s} : \ \bar{s} \in [1, m-1] : \\ &a_i \geq 0 \ \text{ for } \ i \in [1, \bar{s}]; \qquad a_i \leq 0 \ \text{ for } \ i \in [\bar{s}+1, m-1]; \ \ a_m < 0. \end{aligned} \tag{6}$$

In the papers cited above the global existence or finite time blow up of the solution to (1),(2) with nonlinearity (5) or (6) is proved by means of the classical potential well method, suggested by Sattinger and Payne in [8]. An extension of the classical potential well method is introduced in [9], where the global solvability is studied for (1),(2) with Bernoulli type nonlinearity (4) for special choice of a and b. In this case the classical potential well method is not applicable.

Another improvement of the potential well method is suggested in [10] for nonlocal nonlinear wave equations with a single nonlinear term (5). In [10] the authors consider one-parametric family of functionals which are analogue of the Nehari functional $I(u)$. This approach is motivated by the study in [11] of stability and strong instability of the solitary waves to (1).

The aim of this paper is to introduce one parametric family of potential wells to (1), (2) with generalized Bernoulli nonlinearity (4), depending on a parameter

c, $0 \leq c^2 < 1$. Global solvability to (1)–(4) is proved by means of the newly introduced one parametric potential wells. The critical energy constant $d(c)$ is explicitly calculated for nonlinear term (4) in the case $p = 1$. Moreover, the global existence result is obtained for initial data u_0, u_1 with energy $E(u_0, u_1)$ greater than the energy allowed by the classical potential well method.

The paper is organized in the following way. In Section 2 we recall some preliminary results. Moreover, one parametric family of potential wells, the functionals J_c, I_c and the depth $d(c)$ are defined. Sign invariance properties of the Nehari functionals I_c and global existence of the solutions to (1)–(4) are proved in Section 3. Section 4 deals with the comparison of the potential wells for different values of the parameter c for a single nonlinearity (5).

2. Preliminaries

Throughout the paper we denote $\mathrm{L}^2(\mathbb{R})$ and $\mathrm{H}^s(\mathbb{R})$ by L^2 and H^s respectively, the norm $\|u\|_{\mathrm{L}^2}$ by $\|u\|$ and $\|u\|_{\mathrm{H}^1}$ by $\|u\|_1$. We define the inner product (u,v) as $(u,v) = \displaystyle\int_{\mathbb{R}} uv\,dx$.

Theorem 1. *Problem* (1)–(4) *admits a unique local weak solution* $u(t,x) \in \mathrm{C}([0,T_m);\mathrm{H}^1) \cap \mathrm{C}^1([0,T_m);\mathrm{L}^2)$, *where* $[0,T_m)$ *is the maximal existence time interval. If*

$$\limsup_{t\uparrow T_m} (\|u\|_1 + \|u_t\|) < \infty$$

then $T_m = \infty$.

Moreover, the solution $u(t,x)$ *satisfies the conservation laws*

$$E(0) = E(t), \quad M(0) = M(t) \quad \textit{for every} \quad t \in [0,T_m), \tag{7}$$

where the full energy functional $E(t)$ *and the momentum* $M(t)$ *are defined by*

$$\begin{aligned} E(t) :=& E(u(t,\cdot), u_t(t,\cdot)) \\ =& \frac{1}{2}\left(\left\|(-\Delta)^{-1/2}u_t\right\|^2 + \|u_t\|^2 + \|u\|_1^2\right) + \int_{\mathbb{R}}\int_0^u f(s)\,ds\,dx, \end{aligned} \tag{8}$$

$$M(t) := M(u(t,\cdot), u_t(t,\cdot)) = \int_{\mathbb{R}} \left(u.(-\Delta)^{-1/2}u_t + u_x u_t\right)\,dx.$$

The proof of Theorem 1 is similar to the proof in [3, Theorem 2.1 and Lemma 3.1] and we omit it.

Let us define the following important functionals $J_c(u)$, $I_c(u)$ and the Nehari manifold $\mathcal{N}_c$ associated with equation (1) for $c^2 < 1$:

$$J_c(u) = \frac{1}{2}(1-c^2)\|u\|_1^2 + \int_{\mathbb{R}}\int_0^u f(s)\,ds\,dx,$$

$$I_c(u) = (1-c^2)\|u\|_1^2 + \int_{\mathbb{R}} uf(u)\,dx,$$

$$\mathcal{N}_c = \{u \in \mathrm{H}^1 :\ I_c(u) = 0,\ \|u\|_1 \neq 0\}.$$

Let us mention that for $c = 0$ $J_0(u)$, $I_0(u)$ and $\mathcal{N}_0$ coincide with $J(u)$, $I(u)$ and $\mathcal{N}$, respectively, defined in the classical potential well method, see [7, 9]. When the argument of the functionals I_c and J_c is a function u of t and x, i.e. $u = u(t,x)$, we use the short notations $I_c(t) = I_c(u(t,\cdot))$ and $J_c(t) = J_c(u(t,\cdot))$.

In the following lemma the properties of functionals $J_c(\lambda u)$ and $I_c(\lambda u)$ as functions of λ are given.

Lemma 1. *Suppose $f(u)$ satisfies* (4) *and $u \in \mathrm{H}^1$, $\|u\|_1 \neq 0$. Then:*

- $\lim\limits_{\lambda\to 0} J_c(\lambda u) = 0,\ \lim\limits_{\lambda\to+\infty} J_c(\lambda u) = -\infty;$
- *There exists a unique $\lambda_c \in (0,\infty)$ such that $\frac{\partial}{\partial\lambda} J_c(\lambda_c u) = 0$, where*

$$\lambda_c^p = \left\{ -a\int_{\mathbb{R}} |u|^{p+2}\,dx - \sqrt{\left(a\int_{\mathbb{R}} |u|^{p+2}\,dx\right)^2 - 4b(1-c^2)\|u\|_1 \int_{\mathbb{R}} |u|^{2p+2}\,dx} \right\} \times \left(2b\int_{\mathbb{R}} |u|^{2p+2}\,dx\right).$$

- *$J_c(\lambda u)$ is increasing for $\lambda \in (0,\lambda_c)$, decreasing for $\lambda \in (\lambda_c,\infty)$ and takes its maximum at $\lambda = \lambda_c$;*
- *$I_c(\lambda u) = \lambda \frac{d}{d\lambda} J_c(\lambda u)$ and $I_c(\lambda u) > 0$ for $\lambda \in (0,\lambda_c)$, $I_c(\lambda u) < 0$ for $\lambda \in (\lambda_c,\infty)$ and $I_c(\lambda_c u) = 0$;*
- *$\lambda_c < \lambda_0$ for $0 < c^2 < 1$.*

The proof of Lemma 1 is identical with the proof of the corresponding result for $J_0(\lambda u)$, $I_0(\lambda u)$ in the classical potential well method and we omit it (for more details see [7].

We define the function $d(c)$ as

$$d(c) = \inf_{I_c(u)=0} J_c(u), \tag{9}$$

which has a crucial role in the framework of the potential well method. By means of Euler-Lagrange equations it follows that

$$d(c) = J_c(\varphi_c), \tag{10}$$

where φ_c is a solution to the equation

$$\varphi_c''(\xi) - \varphi_c(\xi) - \frac{1}{1-c^2} f(\varphi_c(\xi)) = 0, \quad c^2 < 1. \tag{11}$$

Simple computations show that $d(c) < d(0)$ for $0 < c^2 < 1$. Indeed, from (10) and the identity

$$E(t) + cM(t) = \frac{1}{2}\left\|(-\Delta)^{-1/2}u_t + cu\right\|^2 + \frac{1}{2}\|u_t + cu_x\|^2 + J_c(u) \tag{12}$$

it follows that

$$\begin{aligned} d'(c) =& \frac{d}{dc}(E(\varphi_c) + cM(\varphi_c)) = \left(E'(\varphi_c), \frac{d\varphi_c}{dc}\right) + c\left(M'(\varphi_c), \frac{d\varphi_c}{dc}\right) + M(\varphi_c) \\ =& \left(E'(\varphi_c) + cM'(\varphi_c), \frac{d\varphi_c}{dc}\right) + M(\varphi_c) = M(\varphi_c) = -c\|\varphi_c\|^2 - c\|\varphi_c'\|^2. \end{aligned}$$

Thus, $d'(c) < 0$ for $c \in (0,1)$, $d'(c) > 0$ for $c \in (-1,0)$ and consequently $d(c)$ attains its maximum at $c = 0$, i.e. $d(c) < d(0)$ for $0 < c^2 < 1$. The explicit formula for the solution $\varphi_c(\xi)$ of equation (11) is given in [12]. As in [12], we obtain the following closed-form expression for $d(c)$ when $p = 1$:

$$\begin{aligned} d(c) =& \frac{4}{9b^2}(1-c^2) \\ &\times \left\{ a^2 - 3b(1-c^2) + \frac{\sqrt{2}a\left(2a^2 - 9b(1-c^2)\right)}{6\sqrt{-b(1-c^2)}} \left(\frac{\pi}{2} + \arctan\frac{\sqrt{2}a}{3\sqrt{-b(1-c^2)}}\right)\right\}. \end{aligned}$$

In order to formulate the sign preserving properties of the Nehari functionals $I_c(u)$ we introduce the following one parametric family of potential wells for the parameter $0 \le c^2 < 1$:

$$\begin{aligned} W_c &= \{(u,u_t) : \quad E(u,u_t) + cM(u,u_t) < d(c), \quad I_c(u) > 0\}, \\ V_c &= \{(u,u_t) : \quad E(u,u_t) + cM(u,u_t) < d(c), \quad I_c(u) < 0\}. \end{aligned} \tag{13}$$

3. Global existence

The global existence of the solution to problem (1)–(4) is based on the sign preserving properties of the Nehari functionals $I_c(u)$.

Theorem 2. (Sign invariance of $I_c(u)$) *Suppose $u(t,x)$ is the weak solution to* (1)–(4) *in the maximal existence time interval* $[0,T_m)$, $0 < T_m \le \infty$ *and* $c^2 < 1$.

(*i*) *If $(u_0,u_1) \in W_c$ then $(u,u_t) \in W_c$ for every $t \in [0,T_m)$;*

(*ii*) *If $(u_0,u_1) \in V_c$ then $(u,u_t) \in V_c$ for every $t \in [0,T_m)$.*

P r o o f. (i) Suppose by contradiction that $(u,u_t) \in \partial W_c$ for some $t_1 \in (0,T_m)$. From the conservation laws (7) it follows that $E(t_1) + cM(t_1) < d(c)$ and hence $I_c(t_1) = 0$. From (9) and (12) we get the following impossible chain of inequalities:

$$
\begin{aligned}
d(c) > E(t_1)+cM(t_1) &= \frac{1}{2}\left\|(-\Delta)^{-1/2}u_t(t_1,\cdot) + cu(t_1,\cdot)\right\|^2 \\
&+\frac{1}{2}\left\|u_t(t_1,\cdot) + cu_x(t_1,\cdot)\right\|^2 + J_c(t_1) \ge \inf_{I_c(u)=0} J_c(u) = d(c).
\end{aligned}
\tag{14}
$$

Thus $(u,u_t) \in W_c$ for every $t \in [0,T_m)$.

(ii) Suppose by contradiction that $(u,u_t) \in \partial V_c$ for some $t_2 \in (0,T_m)$. As in the proof of (i) it follows that $I_c(t_2) = 0$. From (14) we obtain the same impossible chain of inequalities, i.e. $(u,u_t) \in V_c$ for every $t \in [0,T_m)$. □

Theorem 3. (Global existence) *Suppose $u(t,x)$ is the weak solution to* (1)–(4) *in the maximal existence time interval* $[0,T_m)$, $0 < T_m \le \infty$. *If $c^2 < 1$ and $(u_0,u_1) \in W_c$ then $T_m = \infty$, i.e. $u(t,x)$ is defined for every $t \ge 0$.*

P r o o f. From Theorem 2 it follows that $(u,u_t) \in W_c$ for every $t \in [0,T_m)$. From the identity

$$
J_c(u) = \frac{I_c(u)}{p+2} + \frac{p(1-c^2)}{2(p+2)}\|u\|_1^2 - \frac{bp}{2(p+1)(p+2)}\int_{\mathbb{R}} |u|^{2p+2}\,dx,
$$

and (12), $b < 0$, $I_c(u) > 0$, we get the inequalities

$$
\frac{p(1-c^2)}{2(p+2)}\|u\|_1^2 \le J_c(u) \le E(t) + cM(t) < d(c).
$$

Hence

$$\|u\|_1^2 \le \frac{2(p+2)}{p(1-c^2)} d(c) < \infty \qquad \text{for every} \quad t \in [0, T_m).$$

In addition, from the conservation law (8) and the embedding theorem $\mathrm{H}^1(\mathbb{R}) \hookrightarrow \mathrm{L}_p(\mathbb{R})$, $p > 2$, we obtain $\|u_t\|^2 \le \text{const} < \infty$ for every $t \in [0, T_m)$. From the local existence result in Theorem 1 it follows that $T_m = \infty$, i.e. $u(t, x)$ is defined for every $t \ge 0$. $\square$

The following corollary generalizes the classical potential well method extending the set of initial data for which problem (1)–(4) has a global solution.

Corollary 1. *Suppose $u(t, x)$ is the weak solution to* (1)–(4) *in the maximal existence time interval* $[0, T_m)$, $0 < T_m \le \infty$. *If* $(u_0, u_1) \in W_* = \bigcup\limits_{c^2<1} W_c$ *then* $T_m = \infty$, *i.e. $u(t, x)$ is defined for every* $t \ge 0$.

4. Comparison of the potential wells W_c

In this section we compare the global existence result to (1)–(4), obtained by the classical potential well method ($c = 0$ or $M(u_0, u_1) = 0$) with the corresponding result for $0 < c^2 < 1$ and $M(u_0, u_1) \ne 0$ in Theorem 3. For simplicity we consider only the case of a single nonlinearity, i.e.

$$f(u) = b|u|^{p-1}u, \quad b = \text{const} < 0, \quad p > 1. \tag{15}$$

For nonlinearity (15) we get the following expression for $d(c)$, see [13],

$$d(c) = d(0)(1 - c^2)^{\frac{p+1}{p-1}} \tag{16}$$

Without loss of generality we suppose that $M(u_0, u_1) < 0$ and consider the potential wells W_c for $c \in [0, 1)$. The case $M(u_0, u_1) > 0$ can be analyzed analogously for $c \in (-1, 0]$.

In the classical potential well method the solution to (1), (2), (15) is globally defined if $I(u_0) = I_0(u_0) > 0$ and the initial energy $E(u_0, u_1)$ satisfies the condition

$$E(u_0, u_1) < d = d(0).$$

Now, from (13) we define the new critical energy constant

$$D(c) = d(c) - cM(u_0, u_1) \qquad c \ne 0. \tag{17}$$

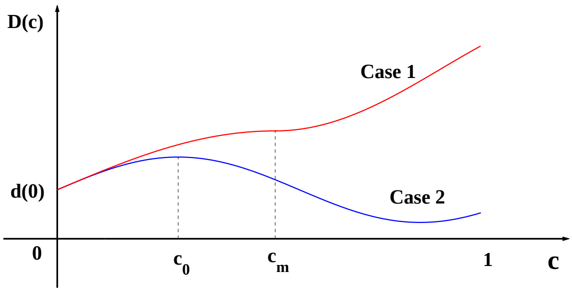

Figure 1: Graph of $D(c)$, defined in (17); $c_m = \sqrt{\dfrac{p-1}{p+3}}$, $c_0 \in (0, c_m)$

From Theorem 3 the solution to (1), (2), (15) exists for every $t \geq 0$ if $I_c(u_0) > 0$ and

$$E(u_0, u_1) < D(c).$$

Below we prove that $D(c) > d(0)$ for $c \in [0, c_0)$, $c_0 \leq 1$. This means that the result in Theorem 3 for $c \in (0, c_0)$ is true for initial energy $E(u_0, u_1)$ greater than the energy allowed for $c = 0$.

Simple computations give us

$$D'(c) = -\frac{2(p+1)}{p-1} d(0) c (1-c^2)^{\frac{2}{p-1}} - M(u_0, u_1), \tag{18}$$

$$D''(c) = d''(c) = \frac{2(p+1)(p+3)}{(p-1)^2} d(0) (1-c^2)^{\frac{3-p}{p-1}} \left(c^2 - \frac{p-1}{p+3} \right). \tag{19}$$

From (18) and (19) it follows that

$$D'(0) > 0, \quad D''(c_m) = 0 \quad \text{for } c_m = \sqrt{\frac{p-1}{p+3}},$$
$$D''(c) < 0 \ \text{ for } \ 0 < c < c_m, \qquad D''(c) > 0 \ \text{ for } \ c_m < c < 1.$$

The function $D'(c)$ has minimum at the point $c_m = \sqrt{\dfrac{p-1}{p+3}}$ and

$$D'(c_m) = -\left(\frac{4}{p+3} \right)^{\frac{p+3}{2(p-1)}} \frac{p+1}{\sqrt{p-1}} d(0) - M(u_0, u_1).$$

$$\textbf{Case 1}: \text{ If } \quad M(u_0,u_1) \le -\left(\frac{4}{p+3}\right)^{\frac{p+3}{2(p-1)}} \frac{p+1}{\sqrt{p-1}} d(0), \quad \text{i.e. } D'(c_m) \ge 0,$$

then $D(c)$ is an increasing function for $c \in [0,1)$ and consequently $D(c) > d(0)$ for every $c \in (0,1)$.

$$\textbf{Case 2}: \text{ If } \quad M(u_0,u_1) > -\left(\frac{4}{p+3}\right)^{\frac{p+3}{2(p-1)}} \frac{p+1}{\sqrt{p-1}} d(0), \quad \text{i.e. } D'(c_m) < 0,$$

then $D(c)$ is an increasing function for $c \in [0,c_0)$, where $c_0 \in (0,c_m)$ and $D'(c_0) = 0$, see Fig 1. Hence $D(c) > d(0)$ for every $c \in (0,c_0)$.

Thus for $c > 0$ Theorem 3 holds for initial data with energy larger than the initial energy in the case $c = 0$. In this way the one parametric potential well method, introduced in the present paper, essentially extends the classical potential well method.

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