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SEMILINEAR DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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We define limit solutions for semilinear differential inclusions in arbitrary Banach space. The main properties of the limit solution set is then studied. In particular it is shown that the limit solution set is R_δ.

1. Introduction

In this paper we study a class of semilinear differential inclusions, having the form

$$(1) \qquad \dot{y} \in A(t)y + F(t, y), t \in I = [t_0, T]y(t_0) = y_0$$

Many papers and books are devoted to theory of the theory of multivalued maps and differential inclusions (see for example [1, 6, 8, 12, 13, 18]).

Notice that the problem (1) covers a large class of parabolic partial differential equations (inclusions). Many papers are devoted to (1) when A does not depend on t. We refer the reader to the books [7, 10, 14] for the general theory of such kind of systems. The theory of semilinear evolution inclusions is also fast developed due to the applications in optimal control problems. For the theory we refer [2, 3, 4, 5, 7] and for the applications [10].

Evolution inclusion (1) is studied in the literature under some compactness or dissipative type assumptions. We refer the reader [2, 3, 4, 7, 12, 16, 18] for the

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theory of such kind of inclusions, the some properties of the set of solutions, and [4, 9, 17] for the so-called generalized solutions. The compactness type assumptions are mainly in two directions.

(I) Assume that $A(\cdot)$ generates an equicontinuous evolution operator, while $F(\cdot,\cdot)$ satisfies condition with respect to some measure of noncompactness. In this case the solution set of (1) is nonempty and compact.

(II) Assume that the evolution operator $K(\cdot,\cdot)$ is compact. In this case one has to use some conditions of the image of $F(\cdot,\cdot)$ has weak compactness of the values or the Banach space Y is assumed to be reflexive. We refer the reader to [2, 12]. If such conditions are not assumed then it is impossible practically to prove existence of mild solution (for the mild solutions see for example [2, 3]).

In this thesis we assume that $A(\cdot)$ generates a compact operator $K(\cdot,\cdot)$, however, the values of $F(\cdot,\cdot)$ are only closed bounded. We define a new type of a solutions (limit solutions) which have similar properties of the mild solutions. The advantage here is that the limit solution set is nonempty and compact.

The dissipative type of assumptions are used commonly when $A(\cdot)$ generates an equicontinuous operator and $F(\cdot,\cdot)$ is (locally) Lipschitz or more general locally Perron. In this case the solution set is nonempty but not closed even $F(\cdot,\cdot)$ has closed convex and bounded values. Its closure is the set of the limit solutions defined and studied here. Notice that these results are proved in [4] in case $A(t) \equiv A$, i.e. autonomous system.

In this chapter give the main definitions and notations used further.

Let Y be Banach space with dual Y^* and let $A \subset Y$ be nonempty, closed and bounded. The support function of the set A is

$$\sigma(l, A) = \sup_{a \in A} \langle l, a \rangle.$$

Definition 1. *Let $A, B \subset Y$ the Hausdorff distance between A and B is defined by*

$$D_H(A, B) = max\{Ex(A, B), Ex(B, A)\}$$

where $Ex(A, B) = \sup_{b \in B} \inf_{a \in A} d(a, b)$.

The multifunction $F : I \times Y \to Y$ is said to be lower semicontinuous (LSC) if for any $(t, y) \in I \times Y$, any $v \in F(t, y)$ and any sequence $(t_n, y_n)_n$ with $t_n \to t$ and $y_n \to y$ there exists a sequence $(v_n)_n$ with $(v_n) \in F(t_n, y_n)$ such that $v_n \to v$. Further $F(\cdot,\cdot)$ is continuous if it is continuous w.r.t the Hausdorff metric.

$F(\cdot,\cdot)$ is called upper hemicontinuous if the support function $\sigma(l, F(\cdot,\cdot))$ is upper semicontinuous as real valued function.

The multifunction $F : I \times Y \to Y$ is said to be almost lower semicontinuous (almost continuous, etc) if for any $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with meas$(I \backslash I_\varepsilon) < \varepsilon$ such that $F(\cdot, \cdot)$ is lower smicontinuous (continuous, etc) on $I_\varepsilon \times Y$.

Definition 2. *Let E be a closed subset of Y, w be a real number and $\Delta = \{ (t, s) : 0 \le s \le t\}$. A family of mappings*

$$K = \{K(t, s) : E \to E \ ; 0 \le t \le s\}$$

is said to be an evolution operator of type w if the following properties are satisfied:
(i) $K(t, t) = I$ for all $x \in E$ and $t \ge 0$;
(ii) $K(t, s)K(s, r) = K(t, r)$ for all $x \in E$ and $0 \le r \le s \le t$;
(iii) For each $x \in E$, the mapping $(t, s) \to K(t, s)x$ is continuous from Δ into Y;
(iv) $\|K(t, s)x - K(t, s)y\| \le \|x - y\|e^{w(t-s)}$ for all $x, y \in E$ and $0 \le s \le t$.

For every evolution system, we can consider the respective evolution operator $K : \Delta \to \mathcal{L}(Y)$, where $\mathcal{L}(Y)$ is the space of all bounded linear operators in Y. Since the evolution operator K is strongly continuous on the compact set Δ, by the uniform boundedness theorem there exist a constant $D = D_\Delta > 0$ such that

$$\|K(t, s)\|_{\mathcal{L}(Y)} \le D, \quad (t, s) \in \Delta.$$

Definition 3. *An evolution operator is said to be compact when $K(t, s)$ is a compact operator for all $t - s > 0$ i.e. $K(t, s)$ maps bounded sets into relatively compact sets.*

We refer the reader to chapter 5 of [16].
Let $x, y \in Y$, where Y is a Banach space with norm $\| \cdot \|$.

$$[x, y]_+ = \lim_{h \to 0^+} \frac{\|x + hy\| - \|x\|}{h}.$$

It is well known that [13]
(i) $|[x, y]_+ - [x, z]_+| \le \|y - z\|$
(ii) $[\cdot, \cdot]_+$ is upper semicontinuous as a real valued function.

2. Limit Solutions

In this section we define the limit solutions of (1) and study their main properties.

Let Y be a Banach space and $I = [t_0, T] \subset \mathbf{R}$. We consider (1), i.e.

$$\dot{y} \in A(t)y + F(t, y(t)),\ t \in I, y(t_0) = y_0.$$

Here $\{A(t)\}_{t\in[t_0,T]}$ is a family of densely defined linear operators which generates a strongly continuous evolution operator $K : \Delta \to \mathcal{L}(Y)$, and $F : I \times Y \to Y$ is a multivalued map with nonempty closed values.

Definition 4. *The continuous function $z(\cdot)$ is said to be a (mild) solution of (1) if*

$$z(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s)f_z(s)ds,$$

where $f_z(\cdot)$ is a measurable selection of $F(\cdot, z(\cdot))$, i.e. $f_z(t) \in F(t, z(t))$.

We will call $f_z(\cdot)$ a pseudo derivative of $z(\cdot)$.

We will use the following assumptions:

A1. $\{A(t)\}_{t\in[t_0,T]}$ is the family of densely defined linear operators which generates a strongly continuous evolution operator $K : \Delta \to \mathcal{L}(Y)$.

A2. The operator $K(t, s)$ is compact for all $t > s$.

F1. $F(\cdot, \cdot)$ satisfies a growth condition, i.e. there exist a constant (Lebesgue integrable) C such that$|F(t, y)| \le C(1 + |y|)$.

F2. $F(\cdot, y)$ is measurable.

Definition 5. *The continuous function $y : I \longrightarrow Y$ is called ε-solution of (1) if it is a solution on I of the problem*

$$\dot{y} \in A(t)y + F(t, y + \varepsilon\mathbf{B}), y(t_0) = y_0, \quad t \in I. \tag{2}$$

Lemma 1. *Under* **A1, F1, F2** *there exist two constants M and N such that $|y(t)| \le M$ and $|F(t, y(t) + \mathbf{B})| \le N - 1$ for every solution $y(\cdot)$ of*

$$\dot{y}(t) \in A(t)y + \bar{co}F(t, y(t) + \mathbf{B}) + \mathbf{B}. \tag{3}$$

Proof. Since

$$y(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s)((f(s) + \mathbf{B}) + \mathbf{B})ds$$

then

$$\begin{aligned} |y(t)| &\le D(|y_0| + \textstyle\int_{t_0}^t f(s)ds + (D+1)(t-t_0)) \\ &\le D|y_0| + (D+1)(T-t_0) + D.C(T-t_0) + DC \textstyle\int_{t_0}^t |y(s)|ds, \text{ i.e.} \\ |y(t)| &\le a + b \textstyle\int_{t_0}^t |y(s)|ds, \end{aligned}$$

where $a = D|y_0| + (D+1)(T-t_0) + D.C(T-t_0)$ and $b = DC$. Due to Gronwall,s inequality there exist a constant M such that $|y(t)| \le M$. Since $|F(t, y(t)| \le C(1+|y|)$ then

$$|F(t, y + \mathbf{B}) \le C(2+M) = N - 1.$$

The proof is therefore complete. □

Lemma 2. *Under* **A1, F1, F2** *for every* $\varepsilon > 0$ *there exist* ε*-solution of* (1). *If* **A2** *also hold then the set of all* ε*-solutions is* $C(I, Y)$ *precompact.*

P r o o f. Let $f_0(t) \in F(t, y_0)$ be measurable selection. We define

$$y(t) = K(t, t_0)y_0 + \int_{t_0}^t K(t,s) f_0(s)ds, \quad t \in [t_0, T].$$

We have that

$$|y(t) - y_0| \le |K(t, t_0)y_0 - y_0| + Me^{w(t-t_0)}C(1 + |y_0|)(t - t_0) \tag{4}$$

for any $t \in [t_0, T]$. Thus for every $\varepsilon > 0$ there exist $t_1 > t_0$ such that $|y(t) - y_0| < \varepsilon$ for any $t \in [t_0, t_1]$, i.e.

$$F(t, y_0) \subset F(t, y(t) + \varepsilon\mathbf{B}).$$

Suppose that the required ε-solution $y(\cdot)$ exists on $[t_0, \tau)$ with $\tau < T$. Due to Lemma 1 $\lim_{t \nearrow \tau} y(t) = y_\tau$ exist. We study (2) on $[\tau, T]$ replacing t_0 by τ and y_0 by y_τ. Dealing as before we can show that there exist $\tau_1 > \tau_0$ such that

$$y(t) = K(t, \tau)y_\tau + \int_\tau^t K(t,s) f_y(s)ds, \quad f_y(t) \in F(t, y(t))$$

is ε-solution on $[\tau, \tau_1]$. Using trivial modification of Zorn's Lemma one can show that $y(\cdot)$ exists on $[t_0, T]$. Therefore there exists ε-solution. Let $y(\tau) = y_\tau$. Replacing y_0 by y_τ and t_0 by τ in (4) we derive

$$|y(t) - y_\tau| \le |K(t, \tau)y_\tau - y_\tau| + Me^{w(t-\tau)}C(1 + |y_\tau|)(t - \tau)).$$

Let **A2** hold. Since $K(t, s)$ is strongly continuous and compact, one has that it is equicontinuous If $\{y^n(\cdot)\}_{n=1}^{\infty}$ is a sequence of ε-solutions, then it is equicontinuous. Furthermore $K(t, s)(N\mathbf{B})$ is precompact and hence Arzela-Ascoli theorem applies. Consequently $\{y_n(\cdot)\}_{n=1}^{\infty}$ is precompact and hence the set of ε-solutions is precompact, i.e. its closure is $C(I, Y)$ compact set. □

Remark. Notice that the conclusion of Lemma 2 holds also for $\varepsilon = 1$. Furthermore $F(t, x) + \mathbf{B}$ also satisfies the conditions of Lemma 1. Therefore the solution set of (3) is precompact.

Definition 6. *The continuous function $y : I \longrightarrow Y$ is called a limit solution of* (1) *if there exist a sequence of positive numbers $(\varepsilon_n)_n \to 0$ and ε_n-solution $y^n(\cdot)$ such that $\lim_{n\to\infty} y^n(t) = y(t)$ uniformly on I.*

Theorem 1. *The limit solution set of* (1) *is nonempty and compact.*

Proposition 1. *Let $f(\cdot,\cdot)$ be Caratheodary with $f(t,\cdot)$ locally Lipschitz. If $f(\cdot,\cdot)$ satisfies* **H1** *and* **A1**, *then the evolution equation*

$$\dot{y} = A(t)y + f(t, y(t)), y(t_0) = \eta, \quad t \in I. \tag{5}$$

has a unique solution $y(\cdot, \eta)$, which is defined on I and depends continuously on the initial condition.

Along with (1) we consider the system

$$\dot{y} \in A(t)y + W(t, y), y(t_0) = y_0, \tag{6}$$

Where $W(t, y) = \bigcap_{\varepsilon>0} \bar{co}F(t, y + \varepsilon\mathbf{B})$.

Definition 7. *Let Y be a complete metric space.*

(i) A set $A \subset Y$ is said to be contractible if there exists a continuous function $H : [0, 1] \times A \to A$ and $\tilde{y} \in A$ such that $H(0, y) = y$ and $H(1, y) = \tilde{y}$ on A.

(ii) The set $A \subset Y$ is said to be compact R_δ if there exist a decreasing sequence of compact contractible sets A_n such that $A = \bigcap_{n=1}^{\infty} A_n$.

Theorem 2. *Under the assumptions* **A1, F1, F2** *the limit solution set of* (6) *is nonempty R_δ.*

Proof. The proof will be given by locally Lipschitz approximations of W. Let us denote **LS** the set of limit solutions of (6). Let $r_n = \frac{1}{3^n}$ and $(V_\nu)_{\nu\in\mathcal{M}}$ be a locally finite refinement of the open covering $Y = \bigcup_{y\in Y}(y + r_n\mathbf{B})$. Let $(\psi_\nu)_{\nu\in\mathcal{M}}$

be a locally Lipschitz partition of unity subordinate to $(V_\nu)_{\nu\in\mathcal{M}}$ and take y_ν such that $V_\nu \subset y_\nu + r_n\mathbf{B}$. Consider the approximations

$$W_n(t,y) = \sum_{\nu\in\mathcal{M}} \psi_\nu(y) C_\nu(t),$$

where $C_\nu(t) = \bar{co} W(t, y_\nu + 2r_n\mathbf{B})$. Then we have that

$$W(t,y) \subset W_{n+1}(t,y) \subset W_n(t,y) \subset W(t, y + 3r_n\mathbf{B}) \quad \textit{on } I\times Y \tag{7}$$

Denote by S_n the mild solution of

$$\dot{y} \in A(t)y + W_n(t,y), y(t_0) = y_0, \tag{8}$$

By (7) we have $S_{n+1} \subset S_n$. Moreover, the solution set S_n is compact for every n with $r_n < \frac{1}{3}$. We shall prove that S_n is contractible. Let $\tilde{f}_\nu$ be a measurable selection of $F(\cdot, y_\nu)$ for every $\nu \in \mathcal{M}$. We define

$$f(t,y) = \sum_{\nu\in\mathcal{M}} \psi_\nu(y)\tilde{f}_\nu(t) \ \textit{on } I\times Y.$$

Since for a.e. $t\in I$, $\tilde{f}_\nu \in F(t,y_\nu) \subset C_\nu(t)$, we have that $f(t,y) \in W_n(t,y)$ for a.e. $t \in I$ and $y \in Y$. Also $(V_\nu)\nu \in \mathcal{M}$ is locally finite then $f(\cdot, y)$ is measurable and $f(t,\cdot)$ is locally Lipschitz. Due to Proposition 1 the equation

$$\dot{y} \in A(t)y + f(t, y(t)),\ t \in [s,T] y(s) = \rho, \tag{9}$$

has a unique solution $\tilde{y}(\cdot, s, \rho)$ which depend continuously on ρ. Take $\tau \in [0,1]$ and denote $b_\tau = \tau(T - t_0) + t_0$. We define the homotopy $H : [0,1] \times \overline{S_n} \to \overline{S_n}$ as

$$H(\tau, v)(t) = \begin{cases} v(t), & \text{t} \in [t_0, b_\tau] \\ \tilde{y}(t; b_\tau, v(b_\tau)), & \text{t} \in (b_\tau, T] \end{cases}$$

Let $u(\cdot), v(\cdot) \in \overline{S_n}$ and let

$$\begin{aligned} \dot{z} &= A(t)z + f(t, z(t)), \quad z(s) = u(s) \\ \dot{y} &= A(t)y + f(t, y(t)), \quad y(\tau) = u(\tau). \end{aligned}$$

Suppose $\tau > s$. Then

$$z(\tau) = K(\tau, s)u(s) + \int_\tau^s K(\tau,\mu) f(\mu, z(\mu)) d\mu.$$

Thus

$$z(t) = K(t,\tau)z(\tau) + \int_\tau^t K(t,\mu)f(\mu, z(\mu))d\mu.$$

Consequently

$$y(t) - z(t) = K(t,\tau)(y(\tau) - z(\tau)) + \int_\tau^t K(t,\mu)[f(\mu, y(\mu)) - f(\mu, z(\mu))]d\mu,$$

i.e.

$$|y(t) - z(t)| \le |K(t,\tau)(y(\tau) - z(\tau))| + \int_\tau^t K(t,\mu)l(\mu)|y(\mu) - z(\mu)|d\mu.$$

Assume that $|u(t)-v(t)| \le \varepsilon$. Fix $\delta > 0$ and $|\tau - s|$ so small that $|u(\tau)-u(s)| < \delta$ and $|K(t,s)u(\xi) - K(\tau,\tau)| < \delta$ for any $|\xi| \le M$ and $ND(t-s) < \delta$.

Consequently

$$|z(\tau) - v(\tau)| \le |v(\tau) - v(s)| + |z(\tau) - z(s)|$$

$$\le \delta + |K(\tau,s)u(s)_K(\tau,\tau)u(s)| + |\int_s^\tau K(\tau,\mu)f(\mu, y(\mu))d\mu$$

$$\le \delta + \delta + D.M(t-s) \le 3\delta.$$

Let $|u(s) - v(s)| \le \varepsilon \quad \forall s \in [t_0, T]$, then

$$|y(\tau) - z(\tau)| \le |u(s) - v(s)| + |y(\tau) - u(s)| + |z(\tau) - z(s)|$$
$$\le \varepsilon + 4\delta.$$

Consequently

$$|y(t) - z(t)| \le |K(t,\tau)||y(\tau) - z(\tau)| + \int_\tau^t |K(t,s)|l(s)|y(s) - z(s)|ds,$$

i.e.

$$|y(t) - z(t)| \le D(\varepsilon + 4\delta) + D\int_\tau^t l(s)|y(s) - z(s)|ds.$$

Thus $|u(t) - z(t)| \le r(t)$, where $r(t) = D(\varepsilon + 4\delta) + D\int_\tau^t l(s)r(s)ds$. Gronwall's inequality then applies and hence $H(\cdot,\cdot)$ is continuous map from $[0,1] \times \overline{S_n} \to \overline{S_n}$.

Furthermore $H(0, v) = \tilde{y}$ and $H(1, v) = v$. So, we find a decreasing sequence of compact contractible sets $(\overline{S_n})_n \subset C(I, Y)$. By Definition 7, we have to only show that

$$\mathbf{LS} = \bigcap_{n=1}^{\infty} \overline{S_n}.$$

Notice that $\mathbf{LS} \subset \overline{S_n}$ for any $n \in \mathbf{N}$. Let $y \in \mathbf{LS}$ and fix n. Then, there exist a decreasing $(\varepsilon_m)_m \downarrow 0$ and $(y_m)_m$ a sequence of ε_m-solution for (6) such that $y_m \to y$. Let m_n be such that $\varepsilon_{m_n} < r_n$. For any $m \geq m_n$ we have

$$W(t, y_m(t) + \varepsilon_m \mathbf{B}) \subset W(t, y_m(t) + r_n \mathbf{B}) \subset W_n(t, y_m(t)),$$

because, if $\psi_\nu(y_m(t)) > 0$, then $y_m(t) \in V_\nu \subset y_\nu + r_n \mathbf{B}$. Hence, $y_m \in S_n$, for any $m \geq m_n$. Now, let $s \in \bigcap_{n=1}^{\infty} \overline{S_n}$ so $s \in \overline{S_n}$, for any n. Then for any n there exist a sequence $(z_n^m)_m \subset S_n$ such that $z_m^n \to s(t)$ uniformly when $m \to \infty$. By (7) z_m^n is a solution of

$$\dot{y} \in A(t)y + W(t, y + 3r_n \mathbf{B}).\dot{y} \in A(t)y + W(t, y + 3r_n \mathbf{B}).$$

Thus z_m^n is $\varepsilon_n = 3r_n$-solution of (6). Let $s_n = z_n^n$. Then s_n is ε_n-solution of $\dot{y} \in A(t) + W(t, y)$ and $s_n(t) \to s(t)$ uniformly on I as $n \to \infty$, i.e. $s(\cdot)$ is a limit solution for $\dot{y} \in A(t)y + W(t, y)$. $\square$

Proposition 2. *Let* **A1, F1, F2** *hold true. If $F(\cdot, \cdot)$ is (jointly) $\mathcal{L} \bigotimes \mathbf{B}$ (Lebesgue, Borel) measurable then for any $\varepsilon > 0$ and every $\delta > 0$ the solution set of*

$$\dot{y} \in A(t)y + \bar{co}F(t, y + \varepsilon \mathbf{B}), y(t_0) = y_0 \tag{10}$$

is contained in the closure of the $(\varepsilon + \delta)$-solution set of (1).

Proof. Let $y(\cdot)$ be a solution of (10) with pseudo derivative $f_y(\cdot)$. That is

$$y(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s) f_y(s) ds.$$

Furthermore, for any $t_0 \leq \tau \leq t \leq T$ one has that

$$y(t) = K(t, \tau)y(\tau) + \int_{\tau}^{t} K(t, s) f_y(s) ds.$$

Since $F(\cdot,\cdot)$ is measurable and $y(\cdot)$ is continuous then $t \to F(t, y(t) + \varepsilon\mathbf{B})$ is measurable.

$$\overline{\int_\tau^t K(t,s)\bar{co}F(s,y(s)+\varepsilon\mathbf{B})ds} = \overline{\int_\tau^t K(t,s)F(s,y(s)+\varepsilon\mathbf{B})ds}.$$

Fix $\nu > 0$. The solution set of (10) is equicontinuous, i.e. there exists a uniform subdivision

$$t_0 < t_1 < \cdots < t_k < \cdots < t_n < t_{n+1} = T$$

such that $|y(t) - y(\tau)| < \frac{\nu}{5} \quad \forall t, s \in [t_k, t_{k+1}], \quad k = 0, 1, 2, 3, \ldots, n$. Clearly for any $[t_k, t_{k+1}]$ there exist a measurable selection $f_z(t) \in F(t, y(t) + \varepsilon\mathbf{B})$ such that

$$\left| \int_{t_k}^{t_{k+1}} K(t_{k+1}, s)|f_y(s) - f_x(s)|ds \right|$$
$$< \frac{\nu}{n+1} \cdot \left| \int_{t_k}^{t_{k+1}} K(t_{k+1}, s)[f_y(s) - f_z(s)]ds \right| < \frac{\nu}{5(n+1)}.$$

Define

$$z(t) = K(t, t_0)y_0 + \int_{t_0}^t f_z(s)ds.$$

Consequently

$$|y(t) - z(t)| < |y(t_k) - z(t_k)| + |y(t) - y(t_k)| + |z(t) - z(t_k)|$$

$$\leq \frac{\nu}{5(n+1)}(n+1) + \frac{\nu}{5} + \frac{\nu}{5} \leq \nu,$$

i.e. $|y(t) - z(t)| \leq \nu$ and hence $y(\cdot)$ is $(\varepsilon + \nu)$-solution for which $|y(t) - z(t)| < \nu$. Taking $\nu < \delta$ we have that for any δ and any ν there exists $(\varepsilon + \delta)$-solution $z(\cdot)$ with $|y(t) - z(t)| < \nu$. $\square$

It follows from Proposition 2 that under **A1, F1, F2** the set of limit solutions of (1) and (6) coincide.

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