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PARTIAL REGULARITY OF HOPF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS, WHICH SATISFY A SUITABLE EXTRA-CONDITION

Jmmy Alfonso Mauro

We estimate the Hausdorff dimension of the set $\mathcal{S}$ of the possible singular points, associated to a Hopf weak solution which satisfies a suitable extra-condition.
According to what is known at the moment, the extra-conditions which we consider doesn't assure the regularity of the Hopf weak solution.

1. Introduction

We consider the non-stationary Navier-Stokes equations with unit viscosity and zero body force

$$
(1)\qquad \begin{aligned} v_t - \Delta v + (v\cdot\nabla)v &= -\nabla\pi \qquad && \forall\ (x,t) \in \Omega\times(0,T),\\ \nabla\cdot v &= 0 && \forall\ (x,t) \in \Omega\times(0,T), \end{aligned}
$$

where v and π represent the unknown velocity and pressure, respectively. In our notation $(v\cdot\nabla)v = (\nabla v)v$.

In addition to (1) we require the following initial and boundary conditions

$$
(2)\qquad \begin{aligned} v(x,t) &= 0 \qquad && \forall\ (x,t) \in \partial\Omega\times(0,T),\\ v(x,0) &= v_0(x) && \forall\ x \in \Omega, \end{aligned}
$$

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If $n = 3$, the system (1)–(2) describes the motion of a Newtonian fluid with a nonslip boundary condition.

The initial data v_0 should satisfy the compatibility conditions $\nabla\cdot v_0 = 0$ in Ω and $v_0\cdot\nu_{|\partial\Omega} = 0$, with $\nu(x)$ the outward pointing unit normal vector at $x \in \partial\Omega$, at least in weak form. on Moreover, if the domain Ω is unbounded, we also assume the following condition at infinity

$$\lim_{|x|\to\infty} v(x,t) = 0 \qquad \forall\ t\ \in\ [0,T).$$

For the Cauchy problem, the existence of weak solutions for the initial-boundary value problem (1)–(2) was proved by J. Leray in [13]; in particular, he introduced the first notion of weak solution for the Navier-Stokes system (cf. Definition 2).
In [12] E. Hopf proved the existence of weak solutions on any smooth enough domain $\Omega \subset R^n$, with $n \geq 2$; nevertheless, such solutions are slightly different to Leray's ones (cf. Definition 3).

Ever since, much effort has been made to establish results on the uniqueness and regularity of weak solutions; however, such questions remain mostly open so far. In particular, till now, it is not known whether or not a Leray weak solution or a Hopf weak one can develop singularities in a finite time, even if the initial data are smooth. The uniqueness problem is strictly related to the regularity one. Indeed, it is well-known that if the solution is smooth enough, then it is unique.

In a series of papers (e.g. see [21, 22]), where he introduced the notions of *suitable* weak solution (see Definition 4) and of *generalized energy inequality* (7), V. Sheffer began to study the partial regularity theory of the Navier-Stokes system. Let us call a point (x,t) *singular* if the velocity v is not essentially bounded in any neighbourhood of (x,t); the remaining points are called *regular*. By a partial regularity theorem, we mean an estimate of the Hausdorff dimension of the set $\mathcal{S}$ of singular points.

In [5], L. Caffarelli, R. Kohn, and L. Nirenberg proved a local partial regularity theorem for suitable weak solutions. Improving a previous result of Sheffer, they showed that, for any such weak solution, the associated singular set $\mathcal{S}$ satisfies $\mathscr{P}^1(\mathcal{S}) = 0$, where $\mathscr{P}^1$ denotes a measure on $\mathbb{R}^3_x \times \mathbb{R}_t$ analogous to one-dimensional Hausdorff measure $\mathscr{H}^1$, but defined using parabolic cylinders instead of Euclidean balls (cf. Section 1.2.).

As far as we know, there are no contributions that have improved this result. In this paper (see Theorem 3), we prove that if v is a Hopf weak solution of problem (1)–(2) such that $v \in L^p\big(0,T;L^q(\mathbb{R}^3)\big)$ for some pair (p,q) satisfying condition (8) (respectively $\nabla v \in L^{\bar{p}}\big(0,T;L^{\bar{r}}(\Omega)\big)$ for some pair $(\bar{p},\bar{r})$ satisfying condition (9)), then $\mathscr{P}^k(\mathcal{S}) = 0$ with $k = p\left(\frac{3}{q}+\frac{2}{p}-1\right)$ (respectively with $k = \bar{p}\left(\frac{3}{\bar{r}}+\frac{2}{\bar{p}}-2\right)$). In particular, if $\frac{3}{q}+\frac{1}{p} < 1$ or $\frac{3}{\bar{r}}+\frac{1}{\bar{p}} < 2$, then we get $0<k<1$.

The result presented in this paper is based on the Ph.D. Thesis [16] that the author defended at the University of Pisa, under the supervision of Prof. Vladimir Georgiev.

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1.1. Notations

Throughout this paper, we assume that Ω is a domain in $\mathbb{R}^n$, with $n \geq 2$, which satisfies one of the following conditions:

Assumption 1.

$(D1)$ $\Omega \equiv \mathbb{R}^n$*;*

$(D2)$ Ω *is a bounded domain in* $\mathbb{R}^n$*;*

$(D3)$ Ω *is an exterior domain in* $\mathbb{R}^n$*.*

Moreover, if Ω *satisfies condition* $(D2)$ *or* $(D3)$*, its bounded boundary* $\partial\Omega$ *is required to be (at least) of class* C^m*, where* m *is an even positive integer such that* $2m > n$*.*

For $1 \leq p \leq \infty$, let $L^p(\Omega)$ be the Lebesgue space of vector valued functions on Ω. The norm in $L^p(\Omega)$ is indicated by $\|\cdot\|_p$ and we use the notation $\langle u,v\rangle = \int_\Omega u\cdot v\,dx$ for any vector fields u,v for which the right hand side makes sense.

For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, let $W^{m,p}(\Omega)$ be the Sobolev space of functions $u:\Omega \to \mathbb{R}^n$ in $L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$ up to order m included; the norm in $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{m,p}(\Omega)}$.

By $C_0^\infty(\Omega)$ we denote the space of all infinitely differentiable vector valued functions with compact support in Ω. By $\mathscr{C}_0(\Omega)$ we denote the class of all solenoidal vector fields $\varphi(x) \in C_0^\infty(\Omega)$; for $1 < p < \infty$, $J^p(\Omega)$ and $J^{1,p}(\Omega)$ are the closure of $\mathscr{C}_0(\Omega)$ in $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively. If Ω satisfies condition (D2) or (D3), we can give the following characterization of the spaces $J(\Omega) \equiv J^2(\Omega)$ and $J^{1,2}(\Omega)$ (see Theorems 1.4 and 1.6 in [26])

$$\begin{aligned} J(\Omega) &= \big\{u \in L^2(\Omega) \;\;:\;\; \nabla \cdot u = 0, \quad \gamma_\nu(u) = 0\big\} \\ J^{1,2}(\Omega) &= \big\{u \in \overset{\circ}{W}{}^{1,2}(\Omega) \;\;:\;\; \nabla \cdot u = 0, \quad \gamma_0(u) = 0\big\}, \end{aligned} \tag{3}$$

where γ_0 is the trace operator from $W^{1,2}(\Omega)$ into $W^{\frac{1}{2},2}(\partial\Omega)$, whereas γ_ν is a linear continuous operator from $E(\Omega) = \big\{u \in L^2(\Omega) \;:\; \nabla \cdot u \in L^2(\Omega)\big\}$ [1] into $W^{-\frac{1}{2},2}(\partial\Omega)$, such that $\gamma_\nu(u) = u \cdot \nu_{|\partial\Omega}$ for every vector field $u \in C^\infty(\overline{\Omega})$, with $\nu(x)$ the outward pointing unit normal vector at $x \in \partial\Omega$.

For $T \in (0,\infty)$ and for a given Banach space $\mathbb{X}$, with associated norm $\|\cdot\|_{\mathbb{X}}$, $L^p\big(0,T;\mathbb{X}\big)$ is the linear space of functions $f : (0,T) \to \mathbb{X}$ such that $\displaystyle\int_0^T \|u(\tau)\|_{\mathbb{X}}^p \, d\tau < \infty$, if $1 \le p < \infty$, or $\operatorname*{ess\,sup}_{\tau\in(0,T)} \|u(\tau)\|_{\mathbb{X}} < \infty$, if $p = \infty$.

For every $T \in (0,\infty)$, we set $\Omega_T = \Omega\times[0,T)$ and we define

$$\mathscr{C}_0(\Omega_T) = \{\varphi \in C_0^\infty\big(\Omega_T;\mathbb{R}^n\big) \;:\; \nabla\cdot\varphi = 0 \;\text{ in }\; \Omega_T\}.$$

In this work, we use the same symbol to denote functional spaces of scalar or vector valued functions. Moreover, the symbol c denotes a generic positive constant whose numerical value is not essential to our aims. It may assume several different values in a single computation.

1.2. The parabolic metric and measure

Let $\mathrm{d}(x,y) = |x-y|$ the Euclidean metric in $\mathbb{R}^n$; in $\mathbb{R}^n_x \times \mathbb{R}_t$ we consider the following *parabolic metric*

$$\delta\big((x,t),(y,\tau)\big) = \max\Big\{\mathrm{d}(x,y)\,;\,\sqrt{|t-\tau|}\Big\} \qquad \forall\;(x,t),\,(y,\tau) \in \mathbb{R}^n_x \times \mathbb{R}_t.$$

We denote by

$$Q_r(x,t) = B_r(x) \times (t-r^2, t+r^2)$$

the ball of radius $r > 0$, centered at $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t$, with respect to the metric δ, which we also call *parabolic cylinder*. We have

$$\mu\big(Q_r(x,t)\big) = r^{n+2}\,\mu\big(Q_1(x,t)\big) = 2\,\omega_n\, r^{n+2}.$$

[1] $E(\Omega)$ is a Hilbert space with respect to the inner product $\langle u,v\rangle_{E(\Omega)} = \langle u,v\rangle + \langle\nabla\cdot u, \nabla\cdot v\rangle$.

The following covering Lemma is the analogue for parabolic metric δ of the well-known Vitali lemma for Euclidean balls. For its proof see [5, Lemma 6.1].

Lemma 1. *Let $\mathcal{I}$ be any family of parabolic cylinders $Q_r(x,t)$ contained in some bounded subset of $\mathbb{R}^n_x \times \mathbb{R}_t$. Then, there exists a finite or denumerable subfamily $\mathcal{I}' = \{Q_{r_i}(x_i,t_i)\}$ such that*

$$Q_{r_i}(x_i,t_i) \cap Q_{r_j}(x_j,t_j) = \emptyset \qquad \textit{for} \quad i \neq j,$$

$$\forall \;\; Q_r(x,t) \in \mathcal{I} \;\;\; \exists \;\;\; Q_{r_i}(x_i,t_i) \in \mathcal{I}' \;\mid\; Q_r(x,t) \subset Q_{5r_i}(x_i,t_i).$$

We introduce the measure $\mathscr{P}^k$ on $\mathbb{R}^n_x \times \mathbb{R}_t$, defined in a similar way to Hausdorff measure $\mathscr{H}^k$, but using parabolic metric δ instead of Euclidean one (cf. [5, Section 2D]).

Definition 1. *For any $X \subset \mathbb{R}^n_x \times \mathbb{R}_t$ and $k \geq 0$ we define*

$$\mathscr{P}^k(X) = \lim_{\varepsilon \to 0^+} \mathscr{P}^k_\varepsilon(X),$$

with

$$\mathscr{P}^k_\varepsilon(X) = \inf \left\{ \sum_{i=1}^{\infty} r_i^k \;\mid\; X \subset \bigcup_{i=1}^{\infty} Q_{r_i}, \;\; r_i < \varepsilon \right\}.$$

$\mathscr{P}^k$ is an outer measure, for which all Borel sets are measurable; on its σ-algebra of measurable sets, $\mathscr{P}^k$ is a Borel regular measure.

Hausdorff measure $\mathscr{H}^k$ is defined in the same way, but replacing Q_{r_i} by an arbitrary closed subset of $\mathbb{R}^n_x \times \mathbb{R}_t$ of diameter at most r_i. Of course,

$$\mathscr{H}^k(X) \leq C(k)\, \mathscr{P}^k(X).$$

Remark 1. For any $X \subset \mathbb{R}^n_x \times \mathbb{R}_t$ and $k \geq 0$, $\mathscr{P}^k(X) = 0$ if and only if, for each $\varepsilon > 0$, there exists a sequence $\{Q_{r_i}\}$ such that $X \subset \cup_i Q_{r_i}$ and $\sum_i r_i^k < \varepsilon$.

2. Weak solutions: definitions and properties

We give three different definitions of weak solutions of the initial-boundary value problem (1)–(2) and we collect some their properties which will be used afterwards.

Definition 2. *Let $v_0 \in J(\Omega)$. A vector field $v : \Omega \times (0,\infty) \to \mathbb{R}^n$ is said a* Leray weak solution *of problem* (1)–(2) *with initial data v_0, if it satisfies the following conditions for all $T \in (0,\infty)$*

1. $v \in L^\infty\big(0,T;J(\Omega)\big) \cap L^2\big(0,T;J^{1,2}(\Omega)\big)$;
2. $\forall\, \varphi \in \mathscr{C}_0(\Omega_T)$

$$\int_0^T \big[\langle v,\varphi_t\rangle - \langle \nabla v, \nabla\varphi\rangle - \langle (v\cdot\nabla)\, v,\varphi\rangle\big]\, dt = -\langle v_0,\varphi_0\rangle \; ; \tag{4}$$

3. *there holds the following energy inequality*

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2\, d\tau \le \|v(s)\|_2^2 \tag{5}$$

for $s = 0$, a.e. $s > 0$ and $\forall\, t \ge s$.

Definition 3. *Let $v_0 \in J(\Omega)$. A vector field $v : \Omega \times (0,\infty) \to \mathbb{R}^n$ is said a* Hopf weak solution *of problem* (1)–(2) *with initial data v_0, if it satisfies, for all $T \in (0,\infty)$, conditions* 1, 2 *of Definition* 2 *and if the energy inequality* (5) *holds only for $s = 0$ and for all $t \ge 0$.*

If Ω is a domain in $\mathbb{R}^n$ (with $n = 2, 3, 4$) satisfying Assumption 1, for any initial data $v_0 \in J(\Omega)$ there exists at least a Leray weak solution of problem (1)–(2). Whereas, if Ω is an arbitrary domain in $\mathbb{R}^n$ (with $n \ge 2$), for any initial data $v_0 \in J(\Omega)$ there exists at least a Hopf weak solution (cf. [13, 12, 9, 18], see also [8, Section 3]).

Obviously, every Leray weak solution is a Hopf weak one too.

Definition 4. *Let $v_0 \in J(\Omega)$ and $T \in (0,\infty]$. A pair (v,π), having as first component a vector field $v : \Omega \times (0,T) \to \mathbb{R}^n$ and as second component a scalar function $\pi : \Omega \times (0,T) \to \mathbb{R}$, is said a* suitable weak solution *of problem* (1)–(2), *in $\Omega \times (0,T)$, with initial data v_0, if the following conditions are satisfied*

1. $v \in L^\infty\big(0,T;J(\Omega)\big) \cap L^2\big(0,T;J^{1,2}(\Omega)\big)$;
2. *the energy inequality* (5) *holds, at least, for $s = 0$ and for all $t \in (0,T)$;*
3. $\forall\, \phi \in C_0^\infty\big(\Omega_T;\mathbb{R}^n\big)$

$$\int_0^T \big[\langle v,\phi_t\rangle - \langle \nabla v, \nabla\phi\rangle - \langle (v\cdot\nabla)\, v,\phi\rangle\big]\, dt = -\int_0^T \langle \pi, \nabla\cdot\phi\rangle\, dt - \langle v_0,\phi_0\rangle \; ; \tag{6}$$

4. *for every non-negative, scalar valued function* $\sigma \in C_0^\infty(\Omega_T;\mathbb{R})$ *there holds the following generalized energy inequality*

$$\int_\Omega |v(t)|^2\sigma(t)\,dx + 2\int_s^t\int_\Omega |\nabla v|^2\sigma\,dx\,d\tau \le \int_\Omega |v(s)|^2\sigma(s)\,dx$$

$$(7)\qquad + \int_s^t\int_\Omega |v|^2(\sigma_\tau + \Delta\sigma)\,dx\,d\tau + \int_s^t\int_\Omega (|v|^2 + 2\pi)v\cdot\nabla\sigma\,dx\,d\tau$$

for $s = 0$, *a.e.* $s \in (0,T)$ *and* $\forall\, t \in (s,T)$.

Definition 5. *A point* $(x,t) \in \Omega \times (0,T)$ *is called* singular *for a solution* v *of system* (1) *iff the vector field* v *is not essentially bounded* [*i.e.* $v \notin L^\infty(I_{(x,t)})$] *on any neighborhood* $I_{(x,t)}$ *of* (x,t).

For a suitable weak solution (v,π), there holds the following result (cf. [5, Proposition 1, Proposition 2], [14, Theorem 3.1, Theorem 3.3], [28, Theorem 2]).

Theorem 1. *If* $n = 3$, *there exist universal constants* δ_1^*, δ_2^* *such that the following property holds for any suitable weak solution* (v,π) *of problem* (1)–(2), *in* $\Omega \times (0,T)$, *with* $\pi \in L^{\frac{3}{2}}(\Omega \times (0,T))$. *Let* $(\bar{x},\bar{t})$ *be in* $\Omega \times (0,T)$ *and such that*

$$\limsup_{r\to 0^+} \frac{1}{r} \int_{\bar{t}-r^2}^{\bar{t}+r^2} \int_{B_r(\bar{x})} |\nabla v|^2\,dx\,dt \le \delta_1^*,$$

or

$$\limsup_{r\to 0^+} \frac{1}{r^2} \int_{\bar{t}-r^2}^{\bar{t}+r^2} \int_{B_r(\bar{x})} \left[|v|^3 + |\pi|^{\frac{3}{2}}\right] dx\,dt \le \delta_2^*,$$

then, v *is bounded in a neighborhood of* $(\bar{x},\bar{t})$ *(i.e.* $(\bar{x},\bar{t})$ *is a regular point).*

Some local regularity results for suitable weak solutions are also obtained in [27], with slightly different hypothesis.

As a consequence of Theorem 1, for a suitable weak solution (v,π), there holds the following local partial regularity result (cf. [5, Theorem B] and [14]).

Theorem 2. *Let Ω be an arbitrary domain in $\mathbb{R}^3$ and let $T \in (0,\infty]$; for any suitable weak solution (v,π) of problem (1)–(2) in $\Omega \times (0,T)$, with $\pi \in L^{\frac{3}{2}}\big(\Omega\times(0,T)\big)$, the associated set $\mathcal{S}$ of possible singular points satisfies $\mathscr{P}^1(\mathcal{S}) = 0$.*

In the previous theorem, the hypothesis $\pi \in L^{\frac{3}{2}}\big(\Omega \times (0,T)\big)$ can be weakened to $\pi \in L^{\frac{5}{4}}\big(0,T; L^{\frac{5}{4}}_{\mathrm{loc}}(\Omega)\big)$ (cf. [5, Section 2C] and [25]).

3. Estimate of the Hausdorff dimension of the set $\mathcal{S}$

In this section we consider the question of the Hausdorff dimension of the set $\mathcal{S}$ of the possible singular points, associated to a Hopf weak solution v which satisfies a suitable extra-condition. According to what is known at the moment, the extra-conditions which we considerdoesn't assure the regularity of the Hopf weak solution.

The following Theorem is a generalization of Theorem B in [5].

Theorem 3. *Let $\Omega \subseteq \mathbb{R}^3$ be a domain satisfying Assumption 1, $T \in (0,\infty)$ and $v_0 \in J(\Omega)$. Let v be a Hopf weak solution of problem (1)–(2) with initial data v_0 and associated set $\mathcal{S}$ of singular points;*

1. *if $\Omega \equiv \mathbb{R}^3$ and $v \in L^p\big(0,T; L^q(\mathbb{R}^3)\big)$ for some pair (p,q) such that*

$$\frac{3}{q} + \frac{2}{p} > 1, \quad \frac{3}{q} + \frac{1}{p} \le 1 \quad \textit{and} \quad \frac{1}{q} + \frac{1}{p} \le \frac{1}{2} \tag{8}$$

then, for every bounded domain $\widetilde{\Omega} \subset \mathbb{R}^3$, $\mathscr{P}^k\big(\mathcal{S} \cap (\widetilde{\Omega} \times (0,T))\big) = 0$, with $k = p\left(\frac{3}{q} + \frac{2}{p} - 1\right)$;

2. *if $\nabla v \in L^{\bar{p}}\big(0,T; L^{\bar{r}}(\Omega)\big)$ for some pair $(\bar{p},\bar{r})$ such that*

$$\frac{3}{\bar{r}} + \frac{2}{\bar{p}} > 2, \quad \frac{3}{\bar{r}} + \frac{1}{\bar{p}} \le 2 \quad \textit{and} \quad \frac{6}{\bar{r}} + \frac{5}{\bar{p}} \le 5 \tag{9}$$

then, for every bounded domain $\widetilde{\Omega} \subseteq \Omega$, $\mathscr{P}^k\big(\mathcal{S} \cap (\widetilde{\Omega} \times (0,T))\big) = 0$, with $k = \bar{p}\left(\frac{3}{\bar{r}} + \frac{2}{\bar{p}} - 2\right)$.

Remark. If v is a Hopf weak solution such that $v \in L^p\big(0,T; L^q(\mathbb{R}^3)\big)$, for some pair (p,q) such that $\frac{3}{q} + \frac{2}{p} = 1$ and $q > 3$, or $\nabla v \in L^{\bar{p}}\big(0,T; L^{\bar{r}}(\Omega)\big)$, for some pair $(\bar{p}, \bar{r})$ such that $\dfrac{3}{\bar{r}} + \dfrac{2}{\bar{p}} = 2$ and $\bar{r} > \dfrac{3}{2}$, then v is regular in $\mathbb{R}^3 \times (0,T)$ (respectively in $\Omega \times (0,T)$) (cf. [19, 20], [24, Theorem 3.1], [10, Theorem 5–ii], [2], [3], [4]); for a survey of regularity results see also [8, Section 5].

According to what is known at the moment, the extra-conditions which we consider in Theorem 3 doesn't assure the regularity of the Hopf weak solution v.

Remark. In Theorem 3, if $\dfrac{3}{q} + \dfrac{1}{p} < 1$ in case *1.* or $\dfrac{3}{\bar{r}} + \dfrac{1}{\bar{p}} < 2$ in case *2.*, then we get $0 < k < 1$: as far as we know, the best partial regularity result for suitable weak solutions is $\mathscr{P}^1(\mathcal{S}) = 0$, proved in [5].

Proof. *1.* Let v be a Hopf weak solution of the Cauchy problem (1)–(2) with initial data $v_0 \in J(\mathbb{R}^3)$, such that $v \in L^p\big(0,T; L^q(\mathbb{R}^3)\big)$, for some pair (p,q) satisfying condition (8) and for some $T \in (0,\infty)$. By Theorem 1 in [17] and Corollary 1.1 in [16], there exists a scalar field $\pi : \Omega \times (0,\infty) \to \mathbb{R}$, associated to v, which is in $L^{\frac{p}{2}}\big(0,T; L^{\frac{q}{2}}(\mathbb{R}^3)\big)$. From Theorem 2.3 in [25] there follows that v and the associated pressure π satisfy the generalized energy *equality* (7) for every $0 \le s \le t \le T$, i.e. (v,π) is a suitable weak solution in $\mathbb{R}^3 \times (0,T)$.

Let $\mathcal{S}$ be the set of singular points associated to the weak solution v. For an arbitrary bounded domain $\widetilde{\Omega} \subset \mathbb{R}^3$, let $\widetilde{\mathcal{S}} = \mathcal{S} \cap (\widetilde{\Omega} \times (0,T))$. We denote by $\widetilde{\mathcal{S}}_t$ the projection of $\widetilde{\mathcal{S}}$ onto the t-axis

$$\widetilde{\mathcal{S}}_t = \Big\{ \tau \in (0,T) \ \text{ for which } \ \exists\, x \in \Omega \ \mid \ (x,\tau) \in \widetilde{\mathcal{S}} \ \Big\}$$

and by $\widetilde{\mathcal{S}}_x$ the projection of $\widetilde{\mathcal{S}}$ onto $\mathbb{R}^3_x$

$$\widetilde{\mathcal{S}}_x = \Big\{ y \in \Omega \ \text{ for which } \ \exists\, t \in (0,T) \ \mid \ (y,t) \in \widetilde{\mathcal{S}} \ \Big\}.$$

We recall, as pointed out in Remark 1 of [5, Section 6], that $\mathscr{H}^{\frac{1}{2}}(\widetilde{\mathcal{S}}_t) \le C\, \mathscr{P}^1(\widetilde{\mathcal{S}})$ and $\mathscr{H}^1(\widetilde{\mathcal{S}}_x) \le C\, \mathscr{P}^1(\widetilde{\mathcal{S}})$; since by Theorem 2 we have $\mathscr{P}^1(\widetilde{\mathcal{S}}) = 0$, we get $\mathscr{H}^{\frac{1}{2}}(\widetilde{\mathcal{S}}_t) = \mathscr{H}^1(\widetilde{\mathcal{S}}_x) = 0$.

By Theorem 1, if $(\bar{x},\bar{t}) \in \widetilde{\mathcal{S}}$ then

$$\limsup_{r\to 0^+} \ \frac{1}{r^2} \iint\limits_{Q_r(\bar{x},\bar{t})} \big[|v|^3 + |\pi|^{\frac{3}{2}} \big]\, dx\, dt > \delta_2^*.$$

Let $V \subset (0,T)$ be an open neighborhood of $\widetilde{\mathcal{S}}_t$ and let $U \subset \widetilde{\Omega}$ be an open neighborhood of $\widetilde{\mathcal{S}}_x$. So, $U \times V \subset \widetilde{\Omega} \times (0,T)$ is an open neighborhood of $\widetilde{\mathcal{S}}$. Let

$\varepsilon > 0$; for each $(\bar{x}, \bar{t}) \in \widetilde{\mathbb{S}}$ we can choose $Q_r(\bar{x}, \bar{t}) = B_r(\bar{x}) \times (\bar{t} - r^2, \bar{t} + r^2)$ such that

$$0 < r < \varepsilon, \qquad \frac{1}{\delta_2^*} \iint\limits_{Q_r(\bar{x},\bar{t})} \left[|v|^3 + |\pi|^{\frac{3}{2}} \right] dx\, dt > r^2,$$
$$B_r(\bar{x}) \subset U, \qquad (\bar{t} - r^2, \bar{t} + r^2) \subset V.$$

Applying Lemma 1 to this family of parabolic cylinders, we obtain a (finite or denumerable) disjoint subfamily $\{Q_{r_i}(x_i, t_i)\}_{i \in \mathscr{I}}$ such that

$$\widetilde{\mathbb{S}} \subset \bigcup_{i \in \mathscr{I}} Q_{5r_i}(x_i, t_i). \tag{10}$$

Using Hölder's inequality, for every $i \in \mathscr{I}$ we have

$$\begin{aligned} r_i^2 &< \frac{1}{\delta_2^*} \iint\limits_{Q_{r_i}(x_i,t_i)} \left[|v|^3 + |\pi|^{\frac{3}{2}} \right] dx\, dt \\ &\leq c \, \frac{r_i^{3(1-\frac{3}{q})+2(1-\frac{3}{p})}}{\delta_2^*} \left[\left\{ \int_{t_i - r_i^2}^{t_i + r_i^2} \|v(t)\|^p_{L^q(B_{r_i}(x_i))} \, dt \right\}^{\frac{3}{p}} \right. \\ &\qquad\qquad \left. + \left\{ \int_{t_i - r_i^2}^{t_i + r_i^2} \|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(B_{r_i}(x_i))} \, dt \right\}^{\frac{3}{p}} \right]; \\ &\leq c \, \frac{r_i^{5-3(\frac{3}{q}+\frac{2}{p})}}{\delta_2^*} \left[\left\{ \int_{t_i - r_i^2}^{t_i + r_i^2} \|v(t)\|^p_{L^q(U)} \, dt \right\}^{\frac{3}{p}} + \left\{ \int_{t_i - r_i^2}^{t_i + r_i^2} \|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)} \, dt \right\}^{\frac{3}{p}} \right]; \end{aligned}$$

from which

$$r_i^{p(\frac{3}{q}+\frac{2}{p}-1)} < \frac{c}{(\delta_2^*)^{\frac{p}{3}}} \int_{t_i - r_i^2}^{t_i + r_i^2} \left[\|v(t)\|^p_{L^q(U)} + \|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)} \right] dx\, dt, \qquad \forall \ i \in \mathscr{I},$$

where the positive constant c is independent of (x_i, t_i), r_i and δ_2^*.

Since $\{(t_i - r_i^2, t_i + r_i^2)\}_{i \in \mathscr{I}}$ is a family of disjoint intervals and

$\bigcup_{i\in\mathscr{I}}(t_i-r_i^2,t_i+r_i^2)\subseteq V$ by construction, then we have

$$\begin{aligned}\sum_{i\in\mathscr{I}} r_i^{p(\frac{3}{q}+\frac{2}{p}-1)} &< \frac{c}{(\delta_2^*)^{\frac{p}{3}}}\sum_{i\in\mathscr{I}}\int_{t_i-r_i^2}^{t_i+r_i^2}\Big[\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\Big]\,dx\,dt\\ &= \frac{c}{(\delta_2^*)^{\frac{p}{3}}}\int\limits_{\bigcup_{i\in\mathscr{I}}(t_i-r_i^2,t_i+r_i^2)}\Big[\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\Big]\,dx\,dt\\ &\le \frac{c}{(\delta_2^*)^{\frac{p}{3}}}\int\limits_V\Big[\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\Big]\,dx\,dt.\end{aligned}$$

Since $r_i<\varepsilon$ for every $i\in\mathscr{I}$, recalling (10) and Definition 1, we have

$$\mathscr{P}^k_\varepsilon(\widetilde{\mathbb{S}})\le\sum_{i\in\mathscr{I}}(5r_i)^k\le\frac{c}{(\delta_2^*)^{\frac{p}{3}}}\int\limits_V\Big[\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\Big]\,dx\,dt,$$

with $k=p\left(\dfrac{3}{q}+\dfrac{2}{p}-1\right)$. By arbitrariness of $\varepsilon>0$, we may conclude that

$$\mathscr{P}^k(\widetilde{\mathbb{S}})\le c\int\limits_V\Big[\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\Big]\,dx\,dt$$

Since $\mathscr{H}^{\frac{1}{2}}(\widetilde{\mathbb{S}}_t)=0$ and $\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\in L^1(0,T)$, using Lebesgue integral properties (cf. Section 12.34 in [11]), for every $\eta>0$ we can choose the open neighborhood V of the projection $\widetilde{\mathbb{S}}_t$ so that

$$\mathscr{P}^k(\widetilde{\mathbb{S}})\le c\int\limits_V\Big[\|v(t)\|^p_{L^q(U)}+\|\pi(t)\|^{\frac{p}{2}}_{L^{q/2}(U)}\Big]\,dx\,dt<\eta.$$

From Remark 1, then, there follows $\mathscr{P}^k(\widetilde{\mathbb{S}})=0$, with $k=p\left(\dfrac{3}{q}+\dfrac{2}{p}-1\right)$.

2. Let $\Omega\subseteq\mathbb{R}^3$ be a domain satisfying Assumption 1; let v be a Hopf weak solution of problem (1)–(2) with initial data $v_0\in J(\Omega)$, such that $\nabla v\in L^{\bar p}\big(0,T;L^{\bar r}(\Omega)\big)$, for some pair $(\bar p,\bar r)$ satisfying condition (9) and for some $T\in(0,\infty)$. By Theorem 1 in [17], there exists a scalar field $\pi:\Omega\times(0,\infty)\to\mathbb{R}$, associated to v. From Theorem 1.3 and Remark 1.5 in [16] there follows that v

and the associated pressure π satisfy the generalized energy *equality* (7) for every $0 \le s \le t \le T$, i.e. (v, π) is a suitable weak solution in $\Omega \times (0,T)$.

Let $\mathcal{S}$ be the set of singular points associated to the weak solution v. For an arbitrary bounded domain $\widetilde{\Omega} \subseteq \Omega$, let $\widetilde{\mathcal{S}} = \mathcal{S} \cap (\widetilde{\Omega} \times (0,T))$. We denote by $\widetilde{\mathcal{S}}_t$ the projection of $\widetilde{\mathcal{S}}$ onto the t-axis and by $\widetilde{\mathcal{S}}_x$ the projection of $\widetilde{\mathcal{S}}$ onto $\mathbb{R}^3_x$; we have $\mathscr{H}^{\frac{1}{2}}(\widetilde{\mathcal{S}}_t) = \mathscr{H}^1(\widetilde{\mathcal{S}}_x) = 0$.

By Theorem 1, if $(\bar{x}, \bar{t}) \in \widetilde{\mathcal{S}}$ then

$$\limsup_{r \to 0^+} \frac{1}{r} \iint\limits_{Q_r(\bar{x},\bar{t})} |\nabla v|^2 \, dx \, dt > \delta_1^*.$$

Let $V \subset (0,T)$ be an open neighborhood of $\widetilde{\mathcal{S}}_t$ and let $U \subset \widetilde{\Omega}$ be an open neighborhood of $\widetilde{\mathcal{S}}_x$. So, $U \times V \subset \widetilde{\Omega} \times (0,T)$ is an open neighborhood of $\widetilde{\mathcal{S}}$. Let $\varepsilon > 0$; for each $(\bar{x}, \bar{t}) \in \widetilde{\mathcal{S}}$ we can choose $Q_r(\bar{x}, \bar{t}) = B_r(\bar{x}) \times (\bar{t} - r^2, \bar{t} + r^2)$ such that

$$0 < r < \varepsilon, \qquad \frac{1}{\delta_1^*} \iint\limits_{Q_r(\bar{x},\bar{t})} |\nabla v|^2 \, dx \, dt > r,$$

$$B_r(\bar{x}) \subset U, \qquad (\bar{t} - r^2, \bar{t} + r^2) \subset V.$$

Applying Lemma 1 to this family of parabolic cylinders, we obtain a (finite or denumerable) disjoint subfamily $\{Q_{r_i}(x_i, t_i)\}_{i \in \mathscr{I}}$ such that there holds (10). Using Hölder's inequality, for every $i \in \mathscr{I}$ we have

$$r_i < \frac{1}{\delta_1^*} \iint\limits_{Q_{r_i}(x_i,t_i)} |\nabla v|^2 \, dx \, dt \le c \, \frac{r_i^{3(1-\frac{2}{\bar{r}})+2(1-\frac{2}{\bar{p}})}}{\delta_1^*} \left\{ \int_{t_i - r_i^2}^{t_i + r_i^2} \|\nabla v(t)\|^{\bar{p}}_{L^{\bar{r}}(B_{r_i}(x_i))} \, dt \right\}^{\frac{2}{\bar{p}}}$$

$$\le c \, \frac{r_i^{5-2(\frac{3}{\bar{r}}+\frac{2}{\bar{p}})}}{\delta_1^*} \left\{ \int_{t_i - r_i^2}^{t_i + r_i^2} \|\nabla v(t)\|^{\bar{p}}_{L^{\bar{r}}(U)} \, dt \right\}^{\frac{2}{\bar{p}}};$$

from which

$$r_i^{\bar{p}(\frac{3}{\bar{r}}+\frac{2}{\bar{p}}-2)} < \frac{c}{(\delta_1^*)^{\frac{\bar{p}}{2}}} \int_{t_i - r_i^2}^{t_i + r_i^2} \|\nabla v(t)\|^{\bar{p}}_{L^{\bar{r}}(U)} \, dt, \qquad \forall \ i \in \mathscr{I},$$

where the positive constant c is independent of (x_i, t_i), r_i and δ_1^*.

Since $\{(t_i - r_i^2, t_i + r_i^2)\}_{i\in\mathscr{I}}$ is a family of disjoint intervals and $\underset{i\in\mathscr{I}}{\cup}(t_i - r_i^2, t_i + r_i^2) \subseteq V$ by construction, then we have

$$\sum_{i\in\mathscr{I}} r_i^{\bar{p}(\frac{3}{\bar{r}}+\frac{2}{\bar{p}}-2)} < \frac{c}{(\delta_1^*)^{\frac{\bar{p}}{2}}} \sum_{i\in\mathscr{I}} \int_{t_i - r_i^2}^{t_i + r_i^2} \|\nabla v(t)\|_{L^{\bar{r}}(U)}^{\bar{p}}\, dt \le \frac{c}{(\delta_1^*)^{\frac{\bar{p}}{2}}} \int_V \|\nabla v(t)\|_{L^{\bar{r}}(U)}^{\bar{p}}\, dt.$$

Analogously at item *1*, since $\|\nabla v(t)\|_{L^{\bar{r}}(U)}^{\bar{p}} \in L^1(0,T)$, we may conclude $\mathscr{P}^k(\widetilde{S}) = 0$, with $k = \bar{p}\left(\dfrac{3}{\bar{r}} + \dfrac{2}{\bar{p}} - 2\right)$. □

Remark. From Remark 1 and Theorem 3, there follows that for every $\varepsilon > 0$, there exists a sequence $\{Q_{r_i}(x_i, t_i)\}_{i\in\mathscr{I}}$ such that $\widetilde{S} \subset \underset{i\in\mathscr{I}}{\cup} Q_{r_i}(x_i, t_i)$ and $\sum\limits_{i\in\mathscr{I}} r_i^k < \varepsilon$ (with $k = p\big(\dfrac{3}{q} + \dfrac{2}{p} - 1\big)$ and $k = \bar{p}\left(\dfrac{3}{\bar{r}} + \dfrac{2}{\bar{p}} - 2\right)$ in item *1.* and item *2.* respectively). Since

$$\begin{aligned} \underset{i\in\mathscr{I}}{\cup} Q_{r_i}(x_i, t_i) &= \underset{i\in\mathscr{I}}{\cup} \Big[B_{r_i}(x_i) \times (t_i - r_i^2, t_i + r_i^2)\Big] \\ &\subseteq \Big[\underset{i\in\mathscr{I}}{\cup} B_{r_i}(x_i)\Big] \times \Big[\underset{i\in\mathscr{I}}{\cup} (t_i - r_i^2, t_i + r_i^2)\Big], \end{aligned}$$

then, we obtain that the the projection $\widetilde{\mathcal{S}}_t \subset \underset{i\in\mathscr{I}}{\cup} (t_i - a_i, t_i + a_i)$, with $a_i = r_i^2$. Since $\sum\limits_{i\in\mathscr{I}} a_i^{k_i/2} = \sum\limits_{i\in\mathscr{I}} r_i^k < \varepsilon$, by Remark 1 we get $\mathscr{H}^{\frac{k}{2}}(\widetilde{\mathcal{S}}_t) = 0$.

Therefore, as a consequence of case 1 in Theorem 3, we obtain Theorem 5-i in [10], for the Cauchy problem with $n = 3$.

If $\Omega \subseteq \mathbb{R}^3$ satisfies condition $(D1)$ or $(D2)$, it is known that, for any Hopf weak solution of problem (1)–(2), $\mathscr{H}^{\frac{1}{2}}(\mathcal{S}_t) = 0$ (cf. [21, 6]).

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