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NEW REDUCTIONS OF A MATRIX GENERALIZED HEISENBERG FERROMAGNET EQUATION*

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In this report, we shall present a new $1+1$ dimensional nonlinear partial differential equation integrable through inverse scattering transform. The integrable system under consideration is a pseudo-Hermitian reduction of a matrix generalization of classical $1+1$ dimensional Heisenberg ferromagnet equation. We derive recursion operators and describe the integrable hierarchy related to that matrix equation.

1. Introduction

Heisenberg ferromagnet equation (HF)

$$\mathbf{S}_t = \mathbf{S}\times\mathbf{S}_{xx}, \qquad \mathbf{S}^2 = 1 \tag{1}$$

is one of classical equations integrable through inverse scattering transform [9]. Above, $\mathbf{S} = (S_1, S_2, S_3)$ is the spin vector of a one-dimensional ferromagnet and subscripts mean partial derivatives with respect to space variable x and time t, see [1] for more details. HF represents the compatibility condition $[L(\lambda), A(\lambda)] = 0$ of the Lax operators:

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x - \lambda S, \qquad \lambda\in\mathbb{C}, \\ A(\lambda) &= \mathrm{i}\partial_t + \frac{\mathrm{i}\lambda}{2}[S, S_x] + 2\lambda^2 S \end{aligned}$$

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where $\mathrm{i} = \sqrt{-1}$ and

$$S = \begin{pmatrix} S_3 & S_1 - \mathrm{i}S_2 \\ S_1 + \mathrm{i}S_2 & -S_3 \end{pmatrix}.$$

In a series of papers [4, 5, 11, 12, 14], the properties of the pair of nonlinear evolution equations (NLEEs)

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + (\epsilon u u_x^* + v v_x^*)u_x + (\epsilon u u_x^* + v v_x^*)_x u = 0, \qquad \epsilon = \pm 1 \\ &\mathrm{i}v_t + v_{xx} + (\epsilon u u_x^* + v v_x^*)v_x + (\epsilon u u_x^* + v v_x^*)_x v = 0 \end{aligned} \tag{2}$$

and the auxiliary spectral problem associated with it have extensively been studied. Above, $*$ denotes complex conjugation and the complex-valued functions u and v are subject to the condition $\epsilon|u|^2 + |v|^2 = 1$ (of course, we have two distinct systems here: one for $\epsilon = +1$ and another for $\epsilon = -1$).

Similarly to HF, (2) can be written into the Lax form $[L(\lambda), A(\lambda)] = 0$ for Lax operators given by:

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x - \lambda S, \qquad \lambda \in \mathbb{C}, \qquad S = \begin{pmatrix} 0 & u & v \\ \epsilon u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}, \\ A(\lambda) &= \mathrm{i}\partial_t + \lambda A_1 + \lambda^2 A_2, \qquad A_2 = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 - \epsilon|u|^2 & -\epsilon u^* v \\ 0 & -v^* u & 2/3 - |v|^2 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} 0 & a & b \\ \epsilon a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} a &= -\mathrm{i}u_x - \mathrm{i}\left(\epsilon u u_x^* + v v_x^*\right) u \\ b &= -\mathrm{i}v_x - \mathrm{i}\left(\epsilon u u_x^* + v v_x^*\right) v \end{aligned}. \end{aligned}$$

Thus, one can view (2) a formal S-integrable generalization of HF.

In the present report, we intend to consider a new matrix version of (2) and discuss some of its basic properties. The matrix NLEE we aim to study still has a Lax representation and its Lax pair is related to symmetric spaces of the type $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m)\times\mathrm{U}(n))$. We are going to describe the integrable hierarchy of NLEEs using recursion operators. Our approach will not use the notion of gauge equivalent NLEEs and Lax pairs.

2. Matrix HF Type Equations

In this section, we shall introduce a new multicomponent NLEE generalizing the coupled system (2). Since our analysis will require certain knowledge of Lie algebras and Lie groups, we refer to the classical monograph [6] for more detailed explanations. Let us start with a few remarks on the notations we intend to use.

Assume $\mathbb{F}$ is either the field of real or complex numbers. We are going to denote the linear space of all $m \times n$ matrices with entries in $\mathbb{F}$ by $M_{m,n}(\mathbb{F})$. $\mathrm{SL}_n(\mathbb{F})$ and SU_n will stand for the special linear group over $\mathbb{F}$ and the unitary group of order n respectively while $\mathfrak{sl}_n(\mathbb{F})$ and $\mathfrak{su}_n$ will denote the corresponding Lie algebras. When this does not lead to any ambiguity, we shall drop from the notation the field of scalars.

Next, we shall write $\left(X^T\right)_{ij} := X_{ji}$, $X \in M_{m,n}(\mathbb{F})$ and $\left(X^\dagger\right)_{ij} := X^*_{ji}$, $X \in M_{m,n}(\mathbb{C})$ for the transposition and Hermitian conjugation of a matrix.

It is well-known that $X \to -X^T$ and $X \to -X^\dagger$, $X \in \mathfrak{sl}_n$ are outer Lie algebra automorphisms for $\mathfrak{sl}_n$. An example of inner Lie algebra automorphism is given by the adjoint action

$$X \to \mathrm{Ad}_g X := gXg^{-1}, \qquad g \in \mathrm{SL}_n, \qquad X \in \mathfrak{sl}_n$$

of SL_n on its Lie algebra. The derivation corresponding to the adjoint group action will be denoted by $\mathrm{ad}_X(Y) := [X, Y]$, $X, Y \in \mathfrak{sl}_n$.

Also, we are going to write $\mathbb{1}_m$ for the unit matrix in $M_{m,m}(\mathbb{F})$ and $Q_m \in M_{m,m}(\mathbb{F})$ for a diagonal matrix with entries equal to ± 1 (we do not specify the number of positive entries here).

Let us introduce the following matrix NLEE

$$\mathrm{i}\mathbf{u}_t + \mathbf{u}_{xx} + \left(\mathbf{u} Q_m \mathbf{u}_x^\dagger Q_n \mathbf{u}\right)_x = 0 \tag{3}$$

for the smooth function $\mathbf{u} : \mathbb{R}^2 \to M_{n,m}(\mathbb{C})$, $m < n$. The above equation can be put into the Lax form $[L(\lambda), A(\lambda)] = 0$ where

$$L(\lambda) := \mathrm{i}\partial_x - \lambda S, \qquad S := \begin{pmatrix} 0 & \mathbf{u}^T \\ Q_n \mathbf{u}^* Q_m & 0 \end{pmatrix}, \tag{4}$$

$$A(\lambda) := \mathrm{i}\partial_t + \lambda A_1 + \lambda^2 A_2, \qquad A_2 := \frac{2m}{m+n} \mathbb{1}_{m+n} - S^2, \tag{5}$$

$$A_1 := \begin{pmatrix} 0 & \mathbf{a}^T \\ Q_n \mathbf{a}^* Q_m & 0 \end{pmatrix}, \qquad \mathbf{a} := -\mathrm{i}(\mathbf{u}_x + \mathbf{u} Q_m \mathbf{u}_x^\dagger Q_n \mathbf{u}). \tag{6}$$

We shall require that the matrix $\mathbf{u}(x,t)$ obeys the condition:

$$\mathbf{u}^T Q_n \mathbf{u}^* Q_m = \mathbb{1}_m. \tag{7}$$

Equation (3) turns into the matrix equation (1) that appeared in [4] when setting $Q_m = \mathbb{1}_m$ and $Q_n = \mathbb{1}_n$.

Constraint (7) imposes a restriction on the spectrum of $S(x,t)$. Indeed, it is easy to check that

$$S^3 = S, \tag{8}$$

hence the eigenvalues of S are 1, 0 and -1.

Example 1. Let us consider the case when $m = 1$ and $n \geq 2$, i.e. $\mathbf{u}$ is n-component vector function. Without any loss of generality we can set $Q_1 = 1$ and assume that at least one diagonal entry of Q_n is 1. Then (3) acquires the following vector form:

$$\mathrm{i}\mathbf{u}_t + \mathbf{u}_{xx} + \left(\mathbf{u}\mathbf{u}_x^{\dagger} Q_n \mathbf{u}\right)_x = 0 \tag{9}$$

where $\mathbf{u}$ must satisfy

$$\mathbf{u}^T Q_n \mathbf{u}^* = 1. \tag{10}$$

Relation (10) represents geometrically a sphere embedded in $\mathbb{R}^{2n}$ provided $Q_n = \mathbb{1}_n$ and a hyperboloid in $\mathbb{R}^{2n}$ otherwise.

In the vector case, the eigenvalues ± 1 appear once in the spectrum of S while 0 has multiplicity $n-1$, therefore one can pick up $\mathrm{diag}\,(1, 0, \ldots, 0, -1)$ as a canonical form of S. Evidently, for $n = 2$ the vector equation reduces to (2).

The form of the matrix coefficients in (4), (5) and (6) implies that the Lax operators are subject to the following symmetry conditions:

$$HL(-\lambda)H^{-1} = L(\lambda), \qquad HA(-\lambda)H^{-1} = A(\lambda), \tag{11}$$

$$Q_{m+n}L^{\dagger}(\lambda^*)Q_{m+n}^{-1} = -\tilde{L}(\lambda), \qquad Q_{m+n}A^{\dagger}(\lambda)Q_{m+n}^{-1} = -\tilde{A}(\lambda) \tag{12}$$

where $H = \mathrm{diag}\,(-\mathbb{1}_m, \mathbb{1}_n)$, $Q_{m+n} = \mathrm{diag}\,(Q_m, Q_n)$ and $\tilde{L}\psi := \mathrm{i}\partial_x\psi + \lambda\psi S$. The ajoint action of H in $\mathfrak{sl}_{m+n}(\mathbb{C})$ is involutive, hence it defines a $\mathbb{Z}_2$-grading of the Lie algebra in the following way:

$$\begin{aligned} &\mathfrak{sl}_{m+n} = \mathfrak{sl}_{m+n}^0 + \mathfrak{sl}_{m+n}^1, \\ &\mathfrak{sl}_{m+n}^{\sigma} := \{X \in \mathfrak{sl}_{m+n} :\ \mathrm{Ad}_H X = (-1)^{\sigma} X\}, \qquad \sigma = 0, 1. \end{aligned} \tag{13}$$

The eigenspace $\mathfrak{sl}_{m+n}^0$ consists of block diagonal matrices with $m \times m$ and $n \times n$ blocks on its principal diagonal, e.g. the matrix coefficient A_2, while $\mathfrak{sl}_{m+n}^1$ is spanned by matrices of a form like S and A_1. Let us also remark that the relation $S = -HSH$ shows that S is a semisimple matrix, hence it is diagonalizable.

The matrix H is deeply related to the Cartan involution underlying the definition of the symmetric space $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m)\times\mathrm{U}(n))$. This is why we say that the Lax pair (4)–(6) is related to the symmetric space $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m)\times\mathrm{U}(n))$ following the convention proposed by Fordy and Kulish [2].

Using the adjoint action of Q_{m+n} as appeared in (12), one can introduce a complex conjugation $\mathcal{I}$ in $\mathfrak{sl}_{m+n}$ through the equality:

$$\mathcal{I}(X) := -Q_{m+n}X^{\dagger}Q_{m+n}, \qquad X\in\mathfrak{sl}_{m+n}. \tag{14}$$

This complex conjugation defines the compact real form $\mathfrak{su}_{m+n}$ of $\mathfrak{sl}_{m+n}$ in case $Q_{m+n}=\mathbb{1}_{m+n}$ or the real form $\mathfrak{su}_{k,m+n-k}$ in case the matrix Q_{m+n} has k diagonal entries equal to 1. In order to treat both cases simultaneously we shall refer to condition (12) as a pseudo-Hermitian reduction when $Q_{m+n}\neq\mathbb{1}_{m+n}$ and a Hermitian one when $Q_{m+n}=\mathbb{1}_{m+n}$.

Let us consider the linear problem

$$L(\lambda)\psi(x,t,\lambda)=0 \tag{15}$$

and denote the set of its fundamental solutions by $\mathcal{F}$. Then (11) and (12) could be considered consequences of a Mikhailov-type reduction imposed on the linear spectral problem, see [8, 12]. Indeed, consider the following maps:

$$\psi_0(\lambda) \;\rightarrow\; G_1\psi_0(\lambda)=H\psi_0(-\lambda)H, \tag{16}$$

$$\psi_0(\lambda) \;\rightarrow\; G_2\psi_0(\lambda)=Q_{m+n}\left[\psi_0^{\dagger}(\lambda^*)\right]^{-1}Q_{m+n} \tag{17}$$

and assume that $\mathcal{F}$ is invariant under G_1 and G_2. Since $G_1G_2=G_2G_1$, $G_1^2=G_2^2=\mathrm{id}$, (16) and (17) define an action of the Mikhailov reduction group $\mathbb{Z}_2\times\mathbb{Z}_2$ for the spectral problem (15). As it is easily seen, this leads to (11) and (12) respectively.

3. Integrable Hierarchy and Recursion Operators

In this section, we shall describe the hierarchy of matrix integrable NLEEs associated with (3). In doing this, we shall follow ideas and methods discussed in [3, 5].

Let us consider the general flow Lax pair:

$$\begin{aligned} L(\lambda) &:= \mathrm{i}\partial_x-\lambda S,\\ A(\lambda) &:= \mathrm{i}\partial_t+\sum_{j=1}^{N}\lambda^j A_j, \qquad N\geq 2 \end{aligned} \tag{18}$$

where S is the same as given in (4) and $A_j \in \mathfrak{sl}^{\sigma}_{m+n}(\mathbb{C}) \cap \mathfrak{su}_{k,m+n-k}$, $j \equiv \sigma (\mathrm{mod}\, 2)$, $\sigma = 0, 1$ (see the comments at the end of previous section).

The zero curvature condition $[L(\lambda), A(\lambda)] = 0$ of the Lax pair (18) gives rise to the following set of recurrence relations:

$$[S, A_N] = 0, \tag{19}$$

$$\dots$$

$$\mathrm{i}\partial_x A_k - [S, A_{k-1}] = 0, \qquad k = 2, \dots, N, \tag{20}$$

$$\dots$$

$$\partial_x A_1 + \partial_t S = 0. \tag{21}$$

Each solution to these equations leads to a member of the integrable hierarchy of NLEEs. The analysis of (19)–(21) resembles very much the one we have in the case of (2) so we shall only sketch here the main steps and refer the reader to [5, 12] for more details.

In order to resolve the recurrence relations above, one introduces the splitting

$$A_j = A^{\mathrm{a}}_j + A^{\mathrm{d}}_j \,, \qquad j = 1, \dots, N$$

of the matrix coefficients of A into a S-commuting term A^{d}_j and some remainder A^{a}_j. We discussed in the previous section that $S(x,t)$ is a diagonalizable matrix. So ad_S is diagonalizable too with eigenvalues $0, \pm 1, \pm 2$. The above splitting means that A^{d}_j belongs to the zero eigenspace of ad_S and A^{a}_j — to the direct sum of all the nonzero eigenspaces. Consequently, the above splitting is unique and the operator ad^{-1}_S is properly defined on A^{a}_j. Moreover, considering the minimal polynomial of ad_S on the direct sum of all nonzero eigenspaces, one gets $(\mathrm{ad}^2_S - 1)(\mathrm{ad}^2_S - 4) = 0$. Therefore, we have

$$\mathrm{ad}^{-1}_S = \frac{1}{4}\left(5\mathrm{ad}_S - \mathrm{ad}^3_S\right). \tag{22}$$

Symmetry condition (11) requires that we have

$$A_j \in \mathfrak{sl}^{\sigma}_{m+n}, \qquad j \equiv \sigma \pmod 2, \qquad \sigma = 0, 1$$

for the coefficients of the second Lax operator. For our purposes it will be enough to take

$$A^{\mathrm{d}}_j = \begin{cases} a_j S_1, & j \equiv 0 \pmod 2 \\ a_j S, & j \equiv 1 \pmod 2 \end{cases}$$

where

$$S_1 := S^2 - \frac{2m}{m+n}\mathbb{1}_{m+n}$$

and a_j, $j = 1, \dots, N$ are some scalar functions to be determined in such a way that the recurrence relations are satisfied.

Equation (19) means that $A_N^{\mathrm{a}} = 0$ and in accordance with our previous assumption, for the highest degree coefficient we set:

$$A_N = \begin{cases} c_N S, & N \equiv 1 \pmod 2 \\ c_N S_1, & N \equiv 0 \pmod 2 \end{cases}, \qquad c_N \in \mathbb{R}.$$

Then relation (20) leads to

$$A_{j-1}^{\mathrm{a}} = \begin{cases} \Lambda A_j^{\mathrm{a}} + \mathrm{i}c_j \mathrm{ad}_S^{-1} S_{1,x}, & j \equiv 0 \pmod 2 \\ \Lambda A_j^{\mathrm{a}} + \mathrm{i}c_j \mathrm{ad}_S^{-1} S_x, & j \equiv 1 \pmod 2 \end{cases}, \qquad c_j \in \mathbb{R} \tag{23}$$

where

$$\Lambda := \mathrm{i}\,\mathrm{ad}_S^{-1}\left\{\partial_x(.) - \frac{S_x}{2m}\partial_x^{-1}\mathrm{tr}\left[S(\partial_x(.))^{\mathrm{d}}\right] - \frac{(m+n)S_{1,x}}{2m(n-m)}\partial_x^{-1}\mathrm{tr}\left[S_1(\partial_x(.))^{\mathrm{d}}\right]\right\}.$$

The symbol ∂_x^{-1} stands for a formal right inverse of ∂_x.

One can extend the action of Λ on the S-commuting part by requiring

$$\Lambda S := \mathrm{i}\,\mathrm{ad}_S^{-1} S_x, \qquad \Lambda S_1 := \mathrm{i}\,\mathrm{ad}_S^{-1} S_{1,x}. \tag{24}$$

Then an arbitrary member of the integrable hierarchy we consider can be written down as follows:

$$\mathrm{i}\,\mathrm{ad}_S^{-1} S_t + \sum_k c_{2k}\Lambda^{2k} S_1 + \sum_k c_{2k-1}\Lambda^{2k-1} S = 0. \tag{25}$$

The operator Λ^2 is called recursion operator of the above hierarchy of NLEEs. It is easy to check that (25) gives (3) after setting $N = 2$, $c_2 = -1$ and $c_1 = 0$. Thus, it is the simplest representative of the family (25).

As discussed in [5], the recursion operator of a S-integrable NLEE could also be derived through a method proposed by Gürses, Karasu and Sokolov [7]. This is a modification of the method used above. For its implementation we start from a Lax representation for two adjacent equations in the hierarchy

$$\mathrm{i}L_\tau = [L, \tilde{V}], \tag{26}$$

$$\mathrm{i}L_t = [L, V] \tag{27}$$

where $V = \sum_{k=1}^{N} \lambda^k A_k$ and $\tilde{V} = \sum_{k=1}^{N+2} \lambda^k \tilde{A}_k$ are two adjacent flows with evolution parameters t and τ respectively. Recursion operator $\mathcal{R}$ is then viewed as a mapping satisfying:

$$S_\tau = \mathcal{R} S_t. \tag{28}$$

The flows V and $\tilde{V}$ are interrelated in the following way:

$$\tilde{V}(\lambda) = \lambda^2 V(\lambda) + \lambda^2 B_2 + \lambda B_1 \tag{29}$$

and after substituting (29) into the Lax representation (26), one obtains:

$$\mathrm{i}L_\tau = \mathrm{i}\lambda^2 L_t + [L, B]. \tag{30}$$

Since relation (30) holds identically in λ, it splits into a set of recurrence relations which are resolved in a way similar to that we discussed above. The result reads:

$$S_\tau = \mathrm{ad}_S \Lambda^2 \mathrm{ad}_S^{-1} S_t. \tag{31}$$

Therefore we have that $\mathcal{R} = \mathrm{ad}_S \Lambda^2 \, \mathrm{ad}_S^{-1}$.

4. Conclusion

In the present report, we have considered new multicomponent NLEE of HF type, see (3). Equation (3) is S-integrable with a Lax pair associated with symmetric space $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$, $m < n$, see (4)–(6). Thus, (3) is a natural generalization of the coupled system (2) already studied. Moreover, we have derived recursion operators that allowed us to describe the integrable hierarchy of (3), see (25).

Due to the lack of space, we have not discussed in detail some important issues like the spectral properties of the scattering operator (4). As it is well-known, the spectrum of L depends on the asymptotic behavior of potential function $\mathbf{u}$ and the reductions imposed on the Lax pair. In the simplest case of constant boundary conditions, one can prove that the continuous spectrum of (4) coincides with the real line in $\mathbb{C}$. On the other hand, the discrete eigenvalues of L will be located symmetrically with respect to the real and imaginary lines in $\mathbb{C}$ due to reductions (11) and (12).

Another important issue concerns the solutions of (3). Similarly to the 2-component case [13], we can distinguish between solutions of soliton type and solutions of quasi-rational type. The latter correspond to a degenerate spectrum of the scattering operator [10]. We intend to address all these issues elsewhere.

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