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MODIFIED FVK MODEL

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This paper considers a generalization of Föppl–von Kärmán model for elastic plate depending on a parameter σ. We prove global well posedness for the Cauchy problem with small initial data and $\sigma = 1$ via Strichartz estimate for vibrating plate and Riesz transform. We generalize this result to a perturbed version of the original case ($\sigma = 2$) with the use of Yukawa potential.

1. Introduction

Our starting point is elastic plate model discussed in [9]. The model is based on two works [10] and [11] and for this is named Föppl–von Kärmán (FvK hereafter). We begin with a slightly more general variational formulation than the one proposed in [12], so we can introduce the following modified action functional involving vertical amplitude of the deformation $v(t,x)$ and the Airy stress function $u(t,x)$, where $x \in \Omega$ with Ω being open domain in $\mathbb{R}^2$ with sufficiently regular boundary $\partial\Omega$.

$$A(u,v) = \frac{1}{2}\iint_{I\times\Omega} \Big(|\partial_t v|^2 - |\Delta v|^2 + |(-\Delta)^{\sigma/2}u|^2 + u\{v,v\}\Big)\, dxdt \tag{1}$$

where $\{f,g\}$ is the quadratic form

$$\{f,g\} = Q(\nabla^2 f, \nabla^2 g) = \sum_{|\alpha|=|\beta|=2} q_{\alpha,\beta}\partial_x^\alpha f \partial_x^\beta g. \tag{2}$$

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A typical choice of the quadratic form Q is the following one

$$Q(\nabla^2 f, \nabla^2 g) = \partial^2_{x_1} f \partial^2_{x_2} g + \partial^2_{x_2} f \partial^2_{x_1} - 2\partial_{x_1 x_2} f \partial_{x_1 x_2} g. \tag{3}$$

The corresponding Euler - Lagrange equations give the system

$$\begin{cases} \partial_t^2 v + \Delta^2 v = \{v, u\} \\ (-\Delta)^\sigma u = \dfrac{1}{2}\ \{v, v\} \end{cases} \tag{4}$$

Remark 1.1. (Energy) Multiplying the first equation of (4) by $\partial_t v$ and the second by $\partial_t u$, after integration we use the relations

$$\int_{\mathbb{R}^2} w(t,x)\{u,v\}(t,x)dx = \int_{\mathbb{R}^2} u(t,x)\{w,v\}(t,x)dx$$

as well as

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_t v(t,x)\{u,v\}(t,x)dx &= \frac{1}{2}\int_{\mathbb{R}^2} u(t,x)\partial_t\{v,v\}(t,x)dx \\ &= \frac{d}{2dt}\int_{\mathbb{R}^2} u(t,x)\{v,v\}(t,x)dx - \frac{1}{2}\int_{\mathbb{R}^2}\{v,v\}(t,x)\partial_t u(t,x)dx \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} u(t,x)\{v,v\}(t,x)dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |(-\Delta)^{\sigma/2}u(t,x)|^2 dx \end{aligned}$$

obtain that the energy given by

$$\begin{aligned} E(t) = E(u,v)(t) &= \frac{1}{2}\|\partial_t v(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\|\Delta v(t)\|^2_{L^2(\mathbb{R}^2)} - \\ &-\frac{1}{2}\int_{\mathbb{R}^2} u(t,x)\{v,v\}(t,x)dx - \frac{1}{2}\|(-\Delta)^{\sigma/2}u(t)\|^2_{L^2(\mathbb{R}^2)} \end{aligned} \tag{5}$$

is a conserved quantity.

In the first part of this work we study the case $\sigma = 1$ obtaining a result of global well posedness for small initial data via Strichartz estimates for vibrating plate and the properties of Riesz Transform.

In the second part we generalize the result to a perturbed version of the case $\sigma = 2$, the original problem in [11], obtaining again a result of global well posedness for small initial data.

2. Case $\sigma = 1$

In this section we prove a result of global well-posedness for small initial data using a contraction theorem argument. We first reduce the system (4) to a nonlocal scalar plate equation and introduce the Strichartz estimate for plate equation that we will use in the proof.

2.1. Reduction to nonlocal scalar plate equation

Using the second equation in the FvK system (4) we find

$$u = \frac{1}{2}(-\Delta)^{-1}\{v, v\}$$

so the system (4) is reduced to the scalar FvK equation

$$\partial_t^2 v + \Delta^2 v = \frac{1}{2}\{v, (-\Delta)^{-1}\{v, v\}\}. \tag{6}$$

with nonlocal cubic nonlinearity term.

Remark 2.1. We can further obtain the conservation of the energy expressed only in the term of v, i.e.

$$E(v)(t) = E\left(\frac{1}{2}(-\Delta)^{-1}\{v, v\}, v\right)(t)$$

so that after integration by parts we get

$$\begin{aligned} \int_{\mathbb{R}^2} u(t,x)\{v,v\}(t,x)dx = -\int_{\mathbb{R}^2} \Delta u(t,x)(-\Delta)^{-1}\{v,v\}(t,x)dx = \\ -2\int_{\mathbb{R}^2} |(-\Delta)^{1/2}u(t,x)|^2 dx = -\frac{1}{2}\int_{\mathbb{R}^2} |(-\Delta)^{-1/2}\{v,v\}(t,x)|^2 dx \end{aligned} \tag{7}$$

and hence

$$\begin{aligned} E(v)(t) = \frac{1}{2}\|\partial_t v(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\|\Delta v(t)\|^2_{L^2(\mathbb{R}^2)} + \\ + \frac{1}{8}\int_{\mathbb{R}^2} |(-\Delta)^{-1/2}\{v,v\}(t,x)|^2 dx. \end{aligned} \tag{8}$$

2.2. Strichartz estimates for vibrating plate

Definition 2.1. [6] *The pair* (q, r) *is admissible if*

$$2 \le q, r \le \infty, \frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \ (q, r) \ne (2, \infty)$$

Proposition 2.2. [6] *Let* $I \subseteq \mathbb{R}$, (q, r) *and* $(\tilde{q}, \tilde{r})$ *admissible pairs and* $s \in \mathbb{R}$, *then:*

$$\left\| e^{it\Delta} v_0 \right\|_{\mathcal{L}^q_I \dot{W}^{s,r}} \lesssim \| v_0 \|_{\dot{H}^s}$$

$$\left\| \frac{e^{it\Delta}}{\Delta} v_1 \right\|_{\mathcal{L}^q_I \dot{W}^{s,r}} \lesssim \| v_1 \|_{\dot{H}^{s-2}}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{\mathcal{L}^q_I \dot{W}^{s,r}} \lesssim \| F \|_{\mathcal{L}^{\tilde{q}'}_I \dot{W}^{s-2,\tilde{r}'}}$$

The two formulas

$$\cos(t\Delta) = \frac{e^{it\Delta} + e^{-it\Delta}}{2}, \ \frac{\sin(t\Delta)}{\Delta} = \frac{e^{it\Delta} - e^{-it\Delta}}{2i\Delta}$$

show that proposition (2.2) could be applied to obtain the following estimate

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} F(u)(s) ds \right\|_{\mathcal{L}^q_I \dot{W}^{s,r}} \lesssim \| F \|_{\mathcal{L}^{\tilde{q}'}_I \dot{W}^{s-2,\tilde{r}'}} \tag{9}$$

In the proof of (2.3) we will also use the estimate

$$\begin{aligned} &\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} F(s) ds \right\|_{\mathcal{L}^\infty_I \dot{W}^{s,2}} + \left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} F(s) ds \right\|_{\mathcal{L}^{\tilde{q}}_I \dot{W}^{s,\tilde{r}}} \\ &\lesssim \| F \|_{\mathcal{L}^{\tilde{q}'}_I \dot{W}^{s-2,\tilde{r}'}} \end{aligned} \tag{10}$$

that follows directly from proposition 2.2 observing that both (r, q) and $(\tilde{r}, \tilde{q})$ are admissible and that the right side of (10) does not depend from the left side, so both terms can be upper bounded by the same value.

2.3. GWP for FvK with small initial data

Theorem 2.3. *Given* $(v_0, v_1) \in \dot{H}^2 \times \mathcal{L}^2$ *small enough, the Cauchy problem*

$$\begin{cases} \partial_t^2 v + \Delta^2 v = \dfrac{1}{2}\left\{v, (-\Delta)^{-1}\{v, v\}\right\} \\ v(0,x) = v_0(x) \\ \partial_t v(0,x) = v_1(x) \end{cases}$$

with $(t,x) \in I \times \Omega$ *where* Ω *being open domain in* $\mathbb{R}^2$ *with sufficiently regular boundary* $\partial\Omega$ *has an unique solution for every* $I \subseteq \mathbb{R}$.

P r o o f. In this proof we use a contraction theorem argument.
By Duhamel's formula [12] we can rewrite the problem in the following integral form

$$v = \cos(t\Delta)v_0 + \frac{\sin(t\Delta)}{\Delta}v_1 + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}F(u)(s)ds \tag{11}$$

and define the map

$$T(v)(t) = \cos(t\Delta)v_0 + \frac{\sin(t\Delta)}{\Delta}v_1 + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}F(u)(s)ds \tag{12}$$

We want to control the norm of $T(v)$, for the first two terms on the right side of (11) follow directly from Strichartz estimates for elastic plate (proposition 2.2) for small initial data:

$$\|\cos(t\Delta)v_0\|_{\mathcal{L}_I^q \dot{W}^{s,r}} \lesssim \|v_0\|_{\dot{H}^s}, \qquad \left\|\frac{\sin(t\Delta)}{\Delta}v_1\right\|_{\mathcal{L}_I^q \dot{W}^{s,r}} \lesssim \|v_1\|_{\dot{H}^{s-2}}$$

For the nonlinear term we can still use the Strichartz estimate but we could choose admissible parameters $\tilde{q}, \tilde{r}$ and s to obtain a good estimate. We put $s = 2$ and:

$$\begin{cases} \tilde{q} = 2 + \alpha \\ \tilde{r} = \dfrac{2(2+\alpha)}{\alpha} \end{cases} \implies \begin{cases} \tilde{q}' = \dfrac{2+\alpha}{1+\alpha} \\ \tilde{r}' = \dfrac{2(2+\alpha)}{4+\alpha} \end{cases} \tag{13}$$

with $\alpha \in (0, \infty)$ to preserve Strichartz admissibility condition in proposition 2.2, obtaining the following inequality

$$\left\|\int_0^t \frac{\sin((t-s)\Delta)}{\Delta}F(s)ds\right\|_{\mathcal{L}_I^q \dot{H}^r} \lesssim \|F\|_{\mathcal{L}_I^{\frac{2+\alpha}{1+\alpha}} \mathcal{L}^{\frac{2(2+\alpha)}{4+\alpha}}}$$

and, for the equation (10),

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} F(s)ds \right\|_{\mathcal{L}_I^\infty \dot{H}^2} + \left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} F(s)ds \right\|_{\mathcal{L}_I^{2+\alpha} \dot{H}^{\frac{2(2+\alpha)}{\alpha}}} \lesssim \|F\|_{\mathcal{L}_I^{\frac{2+\alpha}{1+\alpha}} \mathcal{L}^{\frac{2(2+\alpha)}{4+\alpha}}} \tag{14}$$

Thanks to the Strichartz estimate we just need to study the norm of the nonlinear term F instead the whole integral.

The reason of this preliminary result with $\sigma = 1$ is that we can use Riesz Transform to simplify the problem to study this integral. From [4], [5], we could define the Riesz Transform as R_j with the property that $\partial_x u = R_x D u$ and $\partial_y u = R_y D u$, where $D := \sqrt{-\Delta}$. These transforms are invariant in the norm $\|\cdot\|_{\mathcal{L}^q \mathcal{L}^r}$ (with $(q,r) \neq (2,\infty)$) so we could write the nonlinear part as

$$\begin{aligned} &\left\{v, (-\Delta)^{-1}\{v,v\}\right\} = \\ &= \partial_x^2 v \partial_y^2 (-\Delta)^{-1}\{v,v\} + \partial_y^2 v \partial_x^2 (-\Delta)^{-1}\{v,v\} - 2\partial_{xy} v \partial_{xy} (-\Delta)^{-1}\{v,v\} \\ &= R_x^2 D^2 v R_y^2 \{v,v\} + R_y^2 D^2 v R_x^2 \{v,v\} - 2 R_x R_y D^2 v R_x R_y \{v,v\} \end{aligned} \tag{15}$$

but

$$\begin{aligned} \{v,v\} &= \partial_x^2 v \partial_y^2 v + \partial_y^2 v \partial_x^2 v - 2\partial_{xy} v \partial_{xy} v \\ &= R_x^2 D^2 v R_y^2 D^2 v + R_y^2 D^2 v R_x^2 D^2 v - 2 R_x R_y D^2 v R_x R_y D^2 v \end{aligned} \tag{16}$$

and, by substituting (16) in (15), we obtain:

$$\begin{aligned} &\left\{v, (-\Delta)^{-1}\{v,v\}\right\} = R_x^2 D^2 v R_y^2 R_x^2 D^2 v R_y^2 D^2 v + R_x^2 D^2 v R_y^2 R_y^2 D^2 v R_x^2 D^2 v \\ &- 2 R_x^2 D^2 v R_y^2 R_x R_y D^2 v R_x R_y D^2 v + R_y^2 D^2 v R_x^2 R_x D^2 v R_y^2 D^2 v \\ &+ R_y^2 D^2 v R_x^2 R_y^2 D^2 v R_x^2 D^2 v - 2 R_y^2 D^2 v R_x^2 R_x R_y D^2 v R_x R_y D^2 v \\ &- 2 R_x R_y D^2 v R_x R_y R_x^2 D^2 v R_y^2 D^2 v - 2 R_x R_y D^2 v R_x R_y R_y^2 D^2 v R_x^2 D^2 v \\ &+ 4 R_x R_y D^2 v R_x R_y R_x R_y D^2 v R_x R_y D^2 v \end{aligned} \tag{17}$$

By the property of subadditivity of norms, we could also write

$$\left\|\left\{v, (-\Delta)^{-1}\{v,v\}\right\}\right\|_{\mathcal{L}_I^{\tilde{q}'} \mathcal{L}^{\tilde{r}'}} \lesssim \left\| R_x^2 D^2 v R_y^2 R_x^2 D^2 v R_y^2 D^2 v \right\|_{\mathcal{L}_I^{\tilde{q}'} \mathcal{L}^{\tilde{r}'}} + \cdots$$

where on the right of this inequality there are all the terms of (17).

To study each terms on the right we use the generalized Hölder inequality and obtain (here in case of the first term)

$$\begin{aligned}(18)\qquad & \left\| R_x^2 D^2 v R_y^2 R_x D^2 v R_y^2 D^2 v \right\|_{\mathcal{L}_I^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}} \\ & \leq \left\| R_x^2 D^2 v \right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\| R_y^2 R_x^2 D^2 v \right\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}} \left\| R_y^2 D^2 v \right\|_{\mathcal{L}^{q_3}\mathcal{L}^{r_3}} \\ & \lesssim \left\| D^2 v \right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\| D^2 v \right\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}} \left\| D^2 v \right\|_{\mathcal{L}^{q_3}\mathcal{L}^{r_3}}\end{aligned}$$

where the last inequality is given by Riesz Transform's property [5]. Notice that every terms on the right of (17) have the same structure, this lead to the following

$$(19)\qquad \|F(v)\|_{\mathcal{L}_I^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}} \lesssim \left\| D^2 v \right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\| D^2 v \right\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}} \left\| D^2 v \right\|_{\mathcal{L}^{q_3}\mathcal{L}^{r_3}}$$

The last problem that must be solved to complete the proof is to control if exists (q_i, r_i) that satisfy admissibility request. We could easily give an example of admissible quadruple but we want find the more general set of all admissible elements with the assumption that $(q_1, r_1) = (\infty, 2)$ and $(q_3, r_3) = (q_2, r_2)$:

$$\|F(v)\|_{\mathcal{L}_I^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}} \lesssim \left\| D^2 v \right\|_{\mathcal{L}^{\infty}\mathcal{L}^{2}} \left\| D^2 v \right\|^2_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}}$$

that must satisfy the following

$$(20)\qquad \begin{cases} \dfrac{1}{\tilde{q}'} = \dfrac{1}{q_1} + \dfrac{2}{q_2} \\ \dfrac{1}{\tilde{r}'} = \dfrac{1}{r_1} + \dfrac{2}{r_2} \\ \dfrac{1}{\tilde{q}'} + \dfrac{1}{\tilde{r}'} = \dfrac{3}{2} \\ \dfrac{1}{q_1} + \dfrac{1}{r_1} = \dfrac{1}{2} \\ \dfrac{1}{q_2} + \dfrac{1}{r_2} = \dfrac{1}{2} \end{cases} \implies \begin{cases} \dfrac{1}{\tilde{q}'} = \dfrac{2}{q_2} \\ \dfrac{1}{\tilde{r}'} = \dfrac{1}{2} + \dfrac{2}{r_2} \end{cases}$$

where the first two rows follow from Hölder's condition, the third is the condition of admissibility for Strichartz estimate and the last two are chosen to obtain also on the right side the same norm. Notice that the fourth equation is always fulfilled with the choice $(q_1, r_1) = (\infty, 2)$ and that the third could be obtained by the first summed to the second.

We've already given in (13) a description of $\tilde{q}', \tilde{r}'$ in function of a parameter $k \in (0, 2)$, we want to write also q_2, r_2 with that parameter to obtain the explicit range of k that could be used. Solving (20) we obtain

$$\begin{cases} \dfrac{1}{q_2} = \dfrac{1+\alpha}{2(2+\alpha)} \\ \dfrac{1}{r_2} = \dfrac{4+\alpha}{4(2+\alpha)} - \dfrac{1}{4} \end{cases}$$

Notice that we want to keep valid the admissibility condition, so $\dfrac{1}{q_2}, \dfrac{1}{r_2} < \dfrac{1}{2}$ and this leads to $\alpha > -1$; $\alpha > -2$ that is less restrictive then $\alpha > 0$.
This mean that for every $\alpha \in (0, \infty)$ we could find a solution of the system in the form that we want.

To complete the proof we can define the norm

$$\|v\|_X := \sum_{j=1}^{3} \left\|D^2 v\right\|_{\mathcal{L}^{q_j}\mathcal{L}^{r_j}} = \left\|D^2\right\|_{\mathcal{L}^\infty\mathcal{L}^2} + 2\left\|D^2\right\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}}$$

and the problem could be seen as a fix point one $T(v) = v$ with T defined in (12). From the previous steps we obtain:

$$\|T(v)\|_X \leq \underbrace{C\varepsilon}_{small\ initial\ data} + C\,\|v\|_X^3$$

and then, if $T(v_k) = v_{k+1}$,

$$\|T(v_{k+1}) - T(v_k)\|_X \leq C \max\left\{\|v_{k+1}\|_X^2, \|v_k\|_X^2\right\} \cdot \|v_{k+1} - v_k\|_X$$

and then $T : B(C_1\varepsilon) \to B(C_1\varepsilon)$ with $C_1 = 2C$ and ε small enough. For contraction theorem, then $\exists! v \in B(C_1\varepsilon)$ such that $v = T(v)$ and this complete the proof of GWP for our Cauchy problem. □

As said during the proof, another way to solve the problem of the parameters in Hölder inequality (20) is to give an explicit solution. We have to remember that, for inequality (14), a good choice of α is really near to 0, so we can obtain a control over the norm on the interval $\frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2}$ for $\tilde{q} \in (2+\varepsilon, \infty)$. Choosing $\tilde{q} = 2+\varepsilon$ and $\tilde{r} = \frac{2(2+\varepsilon)}{\varepsilon}$ we obtain that, for ε small enough

$$\begin{cases} q_1 = q_2 = q_3 = \dfrac{3(2+\varepsilon)}{1+\varepsilon} \\ r_1 = r_2 = r_3 = \dfrac{6(2+\varepsilon)}{4+\varepsilon} \end{cases}$$

is an admissible choice for (19). Notice that this solution is not contained in the open set described above.

3. A generalized result

We want to generalize the result obtained in previous sections. To do that we take the problem described by the following equation

$$(\xi - \Delta)(-\Delta)u = \frac{1}{2}\{v, v\} \tag{21}$$

that is the one obtained from (1) choosing $\sigma = 2$ and perturbing the first term with a l that grants regularity.

By formally inverting the operator $(\xi - \Delta)$ we obtain

$$(-\Delta)u = (\xi - \Delta)^{-1}\left(\frac{1}{2}\{v, v\}\right) = Y * \frac{1}{2}\{v, v\} \tag{22}$$

With Y the Yukawa (or Bessel) potential [5].

3.1. Study of Yukawa potential

We need to study the term on the right of (22) and so we write the explicit form of Y, that is a radial function, and move to polar coordinate

$$\begin{aligned} Y(x) &= \int_{\mathbb{R}^2} \frac{e^{i\xi x}}{x+\xi^2} d\xi \\ &= \int_0^\infty \left(\int_0^\pi e^{i\rho|x|\cos(\theta)} d\theta \right) \frac{\rho}{x+\rho^2} d\rho \\ &:= \int_0^\infty J_0(\rho|x|) \frac{\rho}{x+\rho^2} d\rho, \end{aligned} \tag{23}$$

where $J_0(\rho|x|)$ is a Bessel function [1] because Bessel functions J_n admit an explicit integral form [8]

$$\pi J_n(z) = i^{-n} \int_0^\pi e^{iz\cos(\theta)} \cos(n\theta) d\theta$$

and so

$$J_0(\rho|x|) = \frac{1}{\pi} \int_0^\pi e^{i\rho|x|\cos(\theta)} d\theta$$

We want to obtain some upper bounds to $Y(x)$ in $\mathcal{L}^p$ with appropriate p, to do that we use an explicit way to write our integral in terms of modified Bessel's functions.

From [8] (pag. 95, rel. (51)) we can use the following formula

$$\begin{aligned} &\int_0^\infty J_\mu(bt)(t^2+z^2)^{-\nu} t^{\mu+1} dt = (b/2)^{\nu-1} z^{1+\mu-\nu} K_{\nu-\mu-1}(bz)/\Gamma(\nu) \\ &\text{if } \ Re(2\nu - 1/2) > Re(\mu) > -1 \ ; \ Re(z) > 0 \end{aligned} \tag{24}$$

If we specialize 24 in our case (i.e. $\mu = 0$ and $\nu = 1$) we obtain

$$\int_0^\infty J_0(|x|\rho) \frac{\rho}{\rho^2 + x} d\rho = K_0(|x|\sqrt{x}) \tag{25}$$

and the conditions 24 are always fulfilled because

$$Re(3/2) > Re(0) > -1 \ ; \ Re(z) > 0$$

this means that we could study $K_0(x)$, the modified Bessel function, instead of $Y(x)$.

We subdivide the problem in two parts, $x < 1$ and $x > 1$. From [8] (pag. 86 rel. 7), specialized per $\nu = 0$ we obtain for large variable

$$K_0(z) = e^{-z}\left(\frac{\pi}{2z}\right)^{\frac{1}{2}}\left[\sum_{m=0}^{M-1} c_m(2z)^{-2m} + O(|z|^{-M})\right] \lesssim |z|^{-\frac{1}{2}}e^{-z} \tag{26}$$

with

$$c_m = \frac{\Gamma(1/2+m)}{\Gamma(1/2-m)}$$

For the neighborhood of the origin, instead, we obtain ([8], pag. 9 rel. (38))

$$K_0(z) = I_0(z)\ln\left(\frac{z}{2}\right) + \sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^2\frac{\psi(m+1)}{(m!)^2} \tag{27}$$

with ([8], pag. 5 rel. (13))

$$I_0 = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{2m}\frac{1}{m!\Gamma(m+1)}$$

and then $K_0(z) \lesssim \ln(z/2)$ for $z < 1$.
By the combination of the two previous results (26) and (27) we obtain

$$Y(x) = K_0(|x|\sqrt{x}) \lesssim \begin{cases} \ln(|x|\sqrt{x}) = \frac{3}{2}\ln(|x|) & \text{if } \ x < 1 \\ |x|^{-\frac{3}{4}}e^{-|x|\sqrt{x}} & \text{if } \ x > 1 \end{cases} \tag{28}$$

and then $Y(x) \in \mathcal{L}^1$ (i.e. $\|Y\|_{\mathcal{L}^\infty\mathcal{L}^1} < C_Y$).

Remark 3.1. We could obtain the same result of (28) in another less evident way. We could use the result of [2] to link the study of K_0 with the study of the free covariance that in our case (i.e. $d = 2$, $m = 1$ and $y = 0$, see rel. 7.2.2 pag. 162 [2])is

$$C(x) = K_0(|x|)$$

Notice that for the free covariance we have the following propositions that lead again to (28).

Proposition 3.1. [2] *For x bounded away from* 0

$$C(x) \lesssim |x|^{-\frac{1}{2}} e^{-|x|}$$

For $|x|$ in a neighborhood of zero

$$C(x) \sim -\ln(|x|)$$

3.2. GWP for generalized case

With the result of the previous section we're ready to prove the GWP theorem for the perturbed $\sigma = 2$ problem.

Theorem 3.2. *Given $(v_0, v_1) \in \dot{H}^2 \times \mathcal{L}^2$ small enough, the Cauchy problem*

$$\begin{cases} \partial_t^2 v + \Delta^2 v = \{v, u\} \\ (\xi - \Delta)(-\Delta) u = \frac{1}{2} \{v, v\} \\ v(0, x) = v_0(x) \\ \partial_t v(0, x) = v_1(x) \end{cases} \tag{29}$$

with $(t, x) \in I \times \Omega$ where Ω being open domain in $\mathbb{R}^2$ with sufficiently regular boundary $\partial\Omega$ has an unique solution for every $I \subseteq \mathbb{R}$ and for every $\xi \neq 0$.

Proof. This proof is similar to the one of the previous GWP theorem 2.3. As seen in (22) we take the second equation and obtain the formal solution

$$u = (-\Delta)^{-1} \left(Y * \frac{1}{2} \{v, v\} \right)$$

and substituting this in the first one of (29) we obtain

$$\partial_t^2 u + \Delta^2 u = F(u) := \frac{1}{2} \left\{ v, (-\Delta)^{-1} (Y * \{v, v\}) \right\}$$

We can write again the iteration with the Duhamel's formula

$$T(v)(t) = \cos(t\Delta) v_0 + \frac{\sin(t\Delta)}{\Delta} v_1 + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} F(u)(s) ds$$

And for Strichartz estimates for VP (14) we obtain the usual upper bound of the three terms, in particular we have to control the norm of $\|F\|_{\mathcal{L}^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}}$

This is the only difference between the two proofs, notice that as in (15) we can write

$$
\begin{aligned}
&\left\{v,(-\Delta)^{-1}(Y*\{v,v\})\right\} = \partial_x^2 v \partial_y^2 (-\Delta)^{-1}(Y*\{v,v\})+ \\
&+\partial_y^2 v \partial_x^2 (-\Delta)^{-1}(Y*\{v,v\}) - 2\partial_{xy} v \partial_{xy}(-\Delta)^{-1}(Y*\{v,v\}) \\
&= R_x^2 D^2 v R_y^2 (Y*\{v,v\}) + R_y^2 D^2 v R_x^2 (Y*\{v,v\}) - \\
&- 2R_x R_y D^2 v R_x R_y (Y*\{v,v\})
\end{aligned}
\tag{30}
$$

We can study this norm with the generalized Hölder inequality obtaining (in the case of the first term, the others are the same)

$$
\begin{aligned}
\left\|R_x^2 D^2 v R_y^2 (Y*\{v,v\})\right\|_{\mathcal{L}^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}} &\le \left\|R_x^2 D^2 v\right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\|R_y^2 (Y*\{v,v\})\right\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}} \\
&\lesssim \left\|D^2 v\right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\|(Y*\{v,v\})\right\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}}
\end{aligned}
$$

with $\dfrac{1}{\tilde{q}'} = \dfrac{1}{q_1} + \dfrac{1}{q_2}$ and $\dfrac{1}{\tilde{r}'} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$.

Now we can apply Young's inequality obtaining

$$
\|(Y*\{v,v\})\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}} \le \|Y\|_{\mathcal{L}^{\infty}\mathcal{L}^{1}} \cdot \|\{v,v\}\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}}
$$

but we proved in (28) that $\|Y\|_{\mathcal{L}^{\infty}\mathcal{L}^{1}} \le C_Y$ constant. Notice also that, for (16),

$$
\|\{v,v\}\|_{\mathcal{L}^{q_2}\mathcal{L}^{r_2}} \le \left\|D^2 v\right\|_{\mathcal{L}^{q_3}\mathcal{L}^{r_3}} \left\|D^2 v\right\|_{\mathcal{L}^{q_4}\mathcal{L}^{r_4}}
$$

with $\dfrac{1}{q_2} = \dfrac{1}{q_3} + \dfrac{1}{q_4}$, $\dfrac{1}{r_2} = \dfrac{1}{r_3} + \dfrac{1}{r_4}$, and so

$$
\left\|R_x^2 D^2 v R_y^2 (Y*\{v,v\})\right\|_{\mathcal{L}^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}} \lesssim C_Y \left\|D^2 v\right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\|D^2 v\right\|_{\mathcal{L}^{q_3}\mathcal{L}^{r_3}} \left\|D^2 v\right\|_{\mathcal{L}^{q_4}\mathcal{L}^{r_4}}
$$

like before we have to control that the norms found on the right side are still admissible

$$\begin{cases} \frac{1}{\tilde{q}'} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q_3} + \frac{1}{q_4} \\ \frac{1}{\tilde{r}'} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_4} \\ \frac{1}{\tilde{q}'} + \frac{1}{\tilde{r}'} = \frac{3}{2} \\ \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{2} \\ \frac{1}{q_3} + \frac{1}{r_3} = \frac{1}{2} \\ \frac{1}{q_4} + \frac{1}{r_4} = \frac{1}{2} \end{cases}$$

but if we choose $r_3 = r_4$; $q_3 = q_4$ this is exactly the same as the condition in (20) and we have already proved that exists a quadruple that solve this problem.

For subadditivity of norm and because all the terms in (30) have the same structure we obtain again

$$\|F\|_{\mathcal{L}^{\tilde{q}'}\mathcal{L}^{\tilde{r}'}} \lesssim \left\|D^2 v\right\|_{\mathcal{L}^{q_1}\mathcal{L}^{r_1}} \left\|D^2 v\right\|^2_{\mathcal{L}^{q_3}\mathcal{L}^{r_3}}$$

With this upper bound for $\|F\|$ we can finish the proof in the same way as the Theorem 2.3. □

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