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REMARK ON THE GLOBAL NON-EXISTENCE OF SEMIRELATIVISTIC EQUATIONS WITH NON-GAUGE INVARIANT POWER TYPE NONLINEARITY WITH MASS

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The non-existence of global solutions for semirelativistic equations with non-gauge invariant power type nonlinearity with mass is studied in the frame work of weighted L^1. In particular, a priori control of weighted integral of solutions is obtained by introducing a pointwise estimate of fractional derivative of some weight functions. Especially, small data blowup with small mass is obtained.

1. Introduction

We consider the Cauchy problem for the following semirelativistic equations with non-gauge invariant power type nonlinearity:

$$\begin{cases} i\partial_t u + (m^2 - \Delta)^{1/2} u = \lambda |u|^p, & t \in [0,T), \quad x \in \mathbb{R}^n, \\ u(0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $m \geq 0$, $\lambda \in \mathbb{C}\backslash\{0\}$, where $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}^n$. Here $(m^2-\Delta)^{1/2}$ is realized as a Fourier multiplier with symbol $(m^2+|\xi|^2)^{1/2}$: $(m^2 - \Delta)^{1/2} = \mathfrak{F}^{-1}(m^2+|\xi|^2)^{1/2}\mathfrak{F}$, where $\mathfrak{F}$ is the Fourier transform defined by

$$(\mathfrak{F}u)(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx.$$

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We remark that the Cauchy problem such as (1) arises in various physical settings and accordingly, semirelativistic equations are also called half-wave equations, fractional Schrödinger equations, and so on, see [3, 15, 16] and reference therein.

The local existence for (1) in the $H^s(\mathbb{R}^n)$ framework is easily seen if $s > n/2$, where $H^s(\mathbb{R}^n)$ is the usual Sobolev space defined by $(1-\Delta)^{-s}L^2(\mathbb{R}^n)$. Here the local existence in the $H^s(\mathbb{R}^n)$ framework means that for any $H^s(\mathbb{R}^n)$ initial data, there is a positive time T such that there is a solution for the corresponding integral equation,

$$u(t) = e^{it(m^2-\Delta)^{1/2}}u_0 - i\lambda \int_0^t e^{i(t-t')(m^2-\Delta)^{1/2}}|u(t')|^p dt', \tag{2}$$

in $C([0,T); H^s(\mathbb{R}^n))$. We remark that for $s > n/2$, local solution for (2) may be constructed by a standard contraction argument with the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ which holds if and only if $s > n/2$. We also remark that in the one dimensional case, $s > 1/2$ is also the necessary condition for the local existence in the $H^s(\mathbb{R})$ framework because the non-existence of local weak solutions to (1) with some $H^{1/2}(\mathbb{R})$ data is shown in [8]. In general setting, the necessary condition is still open and partial results are discussed in [2, 9, 13]. We also remark that in massless case, (1) is scaling invariant. Namely, when u is a solution to (1) with initial data u_0, then for any $\rho > 0$, the pair,

$$u_\rho(t,x) = \rho^{1/(p-1)}u(\rho t, \rho x), \quad u_{0,\rho} = \rho^{1/(p-1)}u_0(\rho x)$$

also satisfies (1). Then the case where (s,q) satisfies that for $u_0 \in H^s_q \backslash \{0\}$,

$$\|(-\Delta)^{s/2}u_{0,\rho}\|_{L^q(\mathbb{R}^n)} \to \infty \quad \text{as} \quad \rho \to \infty \iff s - \frac{n}{q} + \frac{1}{p-1} > 0$$

is called $H^s_q(\mathbb{R}^n)$ scaling subcritical case, where $H^s_q(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^q(\mathbb{R}^n)$. Moreover, if $s = n/q + 1/(p-1)$, we call the case as $H^s_q(\mathbb{R}^n)$ scaling critical case. In the $H^s(\mathbb{R}^n)$ scaling subcritical case, in general, the local existence in $H^s(\mathbb{R}^n)$ framework is expected but this is not our case because the case where $n = 1$ and $s = 1/2$ is $H^s(\mathbb{R})$ scaling subcritical with any $p > 1$.

In the present paper, we revisit the global non-existence of (1). In order to go back to prior works, we define weak solutions for (1) and its lifespan.

Definition 1. *Let $u_0 \in L^2(\mathbb{R}^n)$. We say that u is a weak solution to* (1) *on $[0,T)$, if u belongs to $L^1_{\mathrm{loc}}(0,T; L^2(\mathbb{R}^n)) \cap L^1_{\mathrm{loc}}(0,T; L^p(\mathbb{R}^n))$ and the following* *identity*

$$\int_0^\infty \big(u(t)\big|i\partial_t\psi(t) + (m^2-\Delta)^{1/2}\psi(t)\big)dt = i(u_0|\psi(0)) + \lambda\int_0^\infty \big(|u(t)|^p\big|\psi(t)\big)dt$$

holds for any $\psi \in C([0,\infty); H^1(\mathbb{R}^n)) \cap C^1([0,\infty); L^2(\mathbb{R}^n))$ *satisfying*

$$\operatorname{supp}\psi \subset [0,T]\times\mathbb{R},$$

where $(\cdot \mid \cdot)$ *is the usual* $L^2(\mathbb{R}^n)$ *inner product defined by*

$$(f \mid g) = \int_{\mathbb{R}^n} \overline{f(x)}g(x)dx.$$

Moreover we define T_w *as*

$$T_w = \inf\{T>0\ ;\ \textit{There is no weak solutions for } (1) \textit{ on } [0,T).\}.$$

At first, in $L^1(\mathbb{R})$ scaling critical and subcritical massless cases, the global non-existence is shown in [10].

Proposition 1. ([10, Theorem 1.3]) *If* $n=1$, $m=0$, $1<p\le 2$, *and* $u_0 \in (L^1\cap L^2)(\mathbb{R})$ *satisfying that*

$$\operatorname{Re}(\overline{\lambda}u_0)=0, \quad -\operatorname{Im}\Big(\int_{\mathbb{R}}\overline{\lambda}u_0(x)dx\Big)>0, \tag{3}$$

then there is no global weak solution, namely, if T *is big enough, there is no weak solution on* $[0,T)$.

Here we remark that the case when $p=2$ is $L^1(\mathbb{R})$ scaling critical.

Later, Inui [15] obtained the following global non-existence in $H^s(\mathbb{R}^n)$ scaling critical and subcritical cases for large data with $0\le s<n/2$ and in $L^2(\mathbb{R}^n)$ scaling subcritical massless case for small data:

Proposition 2. ([15, Theorem 1.2]) *Let* $s\ge 0$ *and* $m\ge 0$. *We assume that* $1<p\le 1+2/(n-2s)$. *Let* $f\in H^s(\mathbb{R}^n)$ *satisfy*

$$\operatorname{Re}(\overline{\lambda}f)=0, \quad -\operatorname{Im}(\overline{\lambda}f) \ge \begin{cases} |x|^{-k}, & \text{if} \quad |x|\le 1,\\ 0, & \text{if} \quad |x|>1,\end{cases} \tag{4}$$

with $k < n/2 - s(\leq 1/(p-1))$. If initial value u_0 is given by μf with positive constant μ, then there exists μ_0 such that there is no global weak solution for $\mu > \mu_0$. Moreover, for any $\mu \in [\mu_0, \infty)$, T_w is estimate by

$$T_w \leq C\mu^{-\frac{1}{\frac{1}{p-1}-k}}.$$

with positive constant C which is independent of μ.

Proposition 3. ([15, Theorem 1.4]) *We assume that $1 < p < 1+2/n$, $m = 0$. Let $f \in L^2(\mathbb{R}^n)$ satisfy*

$$\mathrm{Re}(\overline{\lambda} f) = 0, \quad -\mathrm{Im}(\overline{\lambda} f) \geq \begin{cases} 0, & \text{if} \quad |x| \leq 1, \\ |x|^{-k}, & \text{if} \quad |x| > 1, \end{cases} \tag{5}$$

with $n/2 < k < 1/(p-1)$. If initial value $u_0(x)$ is given by $\mu f(x)$ with $\mu > 0$, then there is no global weak solution. Moreover, there exist $\varepsilon > 0$ and a positive constant $C > 0$ such that

$$T_w \leq \begin{cases} C\mu^{-\frac{1}{\frac{1}{p-1}-k}}, & \text{if} \quad 0 < \mu < \varepsilon, \\ 2, & \text{if} \quad \mu > \varepsilon. \end{cases}$$

We remark that for $0 < s < n/2$, there are $H^s(\mathbb{R}^n)$ functions satisfying (4). For details, see [14, Example 5.1].

In [10, 15], the non-existence of weak solutions are shown by a test function method introduced by Baras-Pierre [1] and Zhang [17, 18]. In the classical test function argument, the classical Leibniz rule plays a critical role. On the other hand, the fractional derivative $(m^2 - \Delta)^{1/2}$ of compact supported functions is not controlled pointwisely like classical derivative. Indeed, since $(m^2 - \Delta)^{1/2}$ is non-local, $\mathrm{supp}\,(m^2 - \Delta)^{1/2}\phi$ is bigger than $\mathrm{supp}\,\phi$ for $\phi \in C_c^\infty(\mathbb{R}^n)$ in general, where $C_c^\infty(\mathbb{R}^n)$ denotes the collection of smooth compactly supported functions. Therefore, it is impossible to have the following pointwise estimate: There exists a positive constant C such that for any $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$|((m^2 - \Delta)^{1/2}\phi^\ell)(x)| \leq C|\phi^{\ell-1}(x)((m^2 - \Delta)^{1/2}\phi)(x)|, \quad \forall x \in \mathbb{R}^n \tag{6}$$

with $\ell > 1$. In order to avoid from the difficulty of nonlocality, in [10, 15], (1) is transformed into

$$\partial_t^2 v + m^2 v - \Delta v = -|\lambda|^2 \partial_t |u|^p, \tag{7}$$

where $v = \mathrm{Im}(\overline{\lambda} u)$. (7) may be obtained by applying $-\mathrm{Im}(\overline{\lambda}(i\partial t - (m^2 - \Delta)^{1/2}))$ to both sides of (1). Propositions above were obtained by applying test function method to (7) with some special test functions. Here we remark that test function method is relatively indirect method. Especially, it is impossible to see the behavior of blowup solution with test function method because the lifespan is obtained by comparison between initial data and scaling parameter.

On the other hand, in [7], the global nonexistence of (1) was studied in a more direct manner.

Proposition 4. ([7, Proposition 4]) *Let $m = 0$. Let*

$$X(T) = C([0,T); L^2(\mathbb{R}^n)) \cap C^1([0,T), H^{-1}(\mathbb{R}^n)) \cap L^\infty(0,T; L^p(\mathbb{R}^n)).$$

Let $u_0 \in L^2(\mathbb{R}^n)$ satisfy that

$$M_R(0) > C_{n,p,\alpha} R^{n-1/(p-1)}, \tag{8}$$

with some $R > 0$ and $\alpha \in \mathbb{C}$ satisfying that

$$\mathrm{Re}(\alpha\lambda) > 0. \tag{9}$$

Here $M_R(0)$ and $C_{n,p,\alpha}$ is given by

$$M_R(0) = -\mathrm{Im}\left(\alpha \int_{\mathbb{R}^n} u_0(x) \langle x/R\rangle^{-n-1} dx\right),$$
$$C^p_{n,p,\alpha} = 2^{1+p'/p} p^{-p'/p} p'^{-1} \mathrm{Re}(\alpha\lambda)^{-p'} |\alpha|^{p+p'} A^{p'}_{n,n+1} \left(\int_{\mathbb{R}^n} \langle x\rangle^{-n-1} dx\right)^p$$

and constant $A_{n,n+1}$ is determined below. Then there is no solution for (1) in $X(T)$ with $u(0) = u_0$ and $T > T_{n,p,\lambda,\alpha,R}$, where

$$T_{n,p,\lambda,\alpha,R} = (p-1)^{-1} D^{-1}_{n,p,\lambda,\alpha} R^{n(p-1)} (M_R(0) - C_{n,p,\alpha} R^{n-1/(p-1)})^{-p+1},$$
$$D_{n,p,\lambda,\alpha} = 2^{-1} \mathrm{Re}(\alpha\lambda) |\alpha|^{-p} \left(\int_{\mathbb{R}^n} \langle x\rangle^{-n-1} dx\right)^{-p+1}.$$

We remark that in the subcritical massless case, Propositions 1, 2, and 3 may be obtained as corollaries of Proposition 3. Especially, by (9), conditions (3), (4), and (5) may be relaxed. For details, see Corollaries 1, 2, and 3 in [7] and also Corollaries 1, 2, and 3 below.

Proposition 4 may be obtained by a modification of test function method of [11]. Particularly, one can show that, for solution u to (1),

$$M_R(t) = -\mathrm{Im}\left(\alpha \int_{\mathbb{R}^n} u(t,x)\langle x/R\rangle^{-n-1} dx\right)$$

satisfies the ordinary differential inequality,

$$\frac{d}{dt}(M_R(t) - C_1) \geq C_2(M_R(t) - C_1)^p \tag{10}$$

with some positive constants C_1 and C_2. Since a priori weight L^1 control of blowup solutions (10) is given, the approach of [11] may be regarded as relatively direct comparing to test function methods of [10, 15]. In order to show (10), again, pointwise control of wight functions like (6) is required. Since (6) fails for general compactly supported functions, we consider the estimate of weight functions decaying polynomially and obtain the following:

Lemma 1. *Let* $\langle x\rangle = (1+|x|^2)^{1/2}$*. For* $q > 0$*, there exists a positive constant* $A_{n,q}$ *depending only on* n *and* q *such that for any* $x \in \mathbb{R}^n$*,*

$$|((-\Delta)^{1/2}\langle\cdot\rangle^{-q})(x)| \leq \begin{cases} A_{n,q}\langle x\rangle^{-q-1}, & \text{if} \quad 0 < q < n, \\ A_{n,q}\langle x\rangle^{-n-1}(1+\log(1+|x|)), & \text{if} \quad q = n, \\ A_{n,q}\langle x\rangle^{-n-1}, & \text{if} \quad q > n. \end{cases}$$

Lemma 1 may be shown by a direct computation with the following representation:

$$((-\Delta)^{1/2} f)(x) = B_{n,s} \lim_{\varepsilon \searrow 0} \int_{|y|\geq\varepsilon} \frac{f(x) - f(x+y)}{|y|^{n+1}} dy, \tag{11}$$

where

$$B_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi_1)}{|\xi|^{n+1}} d\xi\right)^{-1}.$$

For details of this representation, for example, we refer the reader [6]. If one regards $(-\Delta)^{1/2}$ as ∇, Lemma 1 seems natural at least for $0 < q < n$. When $q \geq n$, the decay rate of fractional derivative is worse than the expectation form the classical first derivative but it is sufficient to prove Proposition 4 and actually sharp. For details, see Remarks 1 and 2 in Section 2 of [7].

We also remark that Córdoba and Córdoba [4] showed that

$$(-\Delta)^{s/2}(\phi^2)(x) \leq 2\phi(x)((-\Delta)^{s/2}\phi)(x) \tag{12}$$

for any $0 \leq s \leq 2$, $\phi \in \mathcal{S}(\mathbb{R}^2)$, and $x \in \mathbb{R}^2$, where $\mathcal{S}$ denotes the collection of rapidly decreasing functions. In general, $\phi \geq 0$ does not imply $(-\Delta)^{s/2}\phi \geq 0$, and therefore (12) does not imply (6) even with positive ϕ. We also remark that they also used the integral representation of $(-\Delta)^{s/2}$, which is (11) when $s = 1$. By generalizing (12), D'Abbicco and Reissig [5] studied global non-existence for structural damped wave equation possessing fractional derivative. For the study of structural damped wave equation, (12) works well because we have non-negative solutions([5, Lemma 1]), which we cannot expect for (1).

The aim of this paper is to generalize Proposition 4 by introducing the following pointwise estimate:

$$|((m^2 - \Delta)^{1/2}\langle\cdot\rangle^{-n-1})(x)| \leq C\langle x\rangle^{-n-1} \tag{13}$$

for any $x \in \mathbb{R}^n$ with some positive constant C.

The difficulty to study (13) is the non-existence of integral representation of $(m^2 - \Delta)^{1/2}$ like (11). Therefore, we divide our operator into two parts as follows:

$$(m^2 - \Delta)^{1/2} = (-\Delta)^{1/2} + \mathcal{R},$$

where $\mathcal{R}$ is a Fourier multiplier with the following symbol:

$$(m^2 + |\xi|^2)^{1/2} - |\xi| = \int_0^m (\theta^2 + |\xi|^2)^{-1/2}\theta d\theta.$$

Thanks to Lemma 1, it is sufficient to show the pointwise control of $\mathcal{R}$. Fortunately, $\mathcal{R}$ consists of Bessel potential and the Bessel potential $(1-\Delta)^{-1/2}$ has an integral kernel K. In particular, we have the following:

Proposition 5. ([12, Proposition 1.2.5]) *Let K be a mesurable function satisfying*

$$(1 - \Delta)^{-1/2}\phi = K * \phi,$$

for $\phi \in \mathcal{S}$, where $*$ denotes the convolution. Then K is strictly positive and $\|K\|_{L^1(\mathbb{R}^n)} = 1$. Moreover there is a positive constant $\widetilde{B}_n$ depending only on n and satisfying that

$$K(x) \leq \widetilde{B}_n e^{-|x|/2}, \qquad \text{if} \quad |x| > 2,$$

$$K(x) \leq \widetilde{B}_n \begin{cases} \log(\dfrac{2}{|x|}) + 1 + O(|x|^2), & \text{if} \quad n = 1, \\ 1 + |x|^{1-n}, & \text{if} \quad n > 1, \end{cases} \qquad \text{if} \quad |x| < 2.$$

Since K has only integrable singularity at the origin and decays exponentially, nonlinear estimate $\mathcal{R}$ may be obtained by a direct computation. The next estimate is essential in this paper.

Proposition 6. *For $q > n/2$ and $x \in \mathbb{R}^n$,*

$$(14)\quad |((m^2-\Delta)^{1/2}\langle\cdot\rangle^{-q})(x)| \le |((-\Delta)^{1/2}\langle\cdot\rangle^{-q})(x)| + 2^{q/2}\|\langle\cdot\rangle^q K\|_{L^1(\mathbb{R}^n)}\langle m\rangle^{q+1}\langle x\rangle^{-q}.$$

Especially,

$$(15)\quad |((m^2-\Delta)^{1/2}\langle\cdot/R\rangle^{-n-1})(x)| \le R^{-1}\widetilde{A}_n\langle Rm\rangle^{n+2}\langle x/R\rangle^{-n-1},$$

where $\widetilde{A}_n = A_{n,n+1} + 2^{q/2}\|\langle\cdot\rangle^q K\|_{L^1(\mathbb{R}^n)}$.

Here the condition of q is given to consider the domain of $\mathcal{R}$ as $L^2(\mathbb{R}^n)$. Then by replacing Lemma 1 by Lemma 6, we can generalize Proposition 4 in case with mass.

Proposition 7. *Let $m \in \mathbb{R}$. Let $u_0 \in L^2(\mathbb{R}^n)$ satisfy that*

$$(16)\quad M_R(0) > \widetilde{C}_{n,p,\alpha}\langle Rm\rangle^{(n+2)/(p-1)}R^{n-1/(p-1)},$$

with some $R > 0$ and $\alpha \in \mathbb{C}$ satisfying (9), where $\widetilde{C}_{n,p,\alpha}$ is given by

$$\widetilde{C}^p_{n,p,\alpha} = 2^{1+p'/p}p^{-p'/p}p'^{-1}\mathrm{Re}(\alpha\lambda)^{-p'}|\alpha|^{p+p'}\widetilde{A}_n^{p'}\left(\int_{\mathbb{R}^n}\langle x\rangle^{-n-1}dx\right)^p.$$

Then there is no solution for (1) in $X(T)$ with $u(0) = u_0$ and $T > \widetilde{T}_{n,p,m,\lambda,\alpha,R}$, where

$$\widetilde{T}_{n,p,m,\lambda,\alpha,R} = (p-1)^{-1}D^{-1}_{n,p,\lambda,\alpha}R^{n(p-1)}(M_R(0) - \langle Rm\rangle^{(n+2)/(p-1)}\widetilde{C}_{n,p,\alpha}R^{n-1/(p-1)})^{-p+1}.$$

Now, in the subcritical case, Propositions 1, 2 and 3 may be obtained as corollaries of Proposition 7. Here, we remark that since the Cauchy problem (1) is not scaling invariant essentially, Propositions 1 and 3 seem difficult to be extended in case of general mass. However, if mass is sufficiently small, solutions of (1) are shown to be estimated similarly to solutions of (1) without mass.

Corollary 1. *Let* $1 < p < 1+1/n$. *Let* $\alpha \in \mathbb{C}$ *and* $u_0 \in (L^1 \cap L^2)(\mathbb{R}^n)$ *satisfy* (9) *and*

$$-\mathrm{Im}\Big(\alpha \int_{\mathbb{R}^n} u_0(x)dx\Big) > 0. \tag{17}$$

Then, for sufficiently small m, *there exists no solution in* $X(T)$ *for sufficiently large* T.

Corollary 2. *Let* $m \in \mathbb{R}$. *Let* $u_0(x) = \mu f(x)$ *where* $\mu \gg 1$ *and* f *satisfies*

$$-\mathrm{Im}(\alpha f(x)) \geq \begin{cases} |x|^{-k}, & \text{if} \quad |x| \leq 1, \\ 0, & \text{if} \quad |x| > 1, \end{cases} \tag{18}$$

with some $k < \min(n/2, 1/(p-1))$ *and* α *satisfying* (9). *Then there exists some* $R_1 > 0$ *satisfying* (16) *and*

$$\widetilde{T}_{n,p,m,\lambda,\alpha,R_1} \leq C\mu^{-\frac{1}{1/(p-1)-k}}.$$

Corollary 3. *Let* $u_0(x) = \mu f(x)$ *where* $0 < \mu \ll 1$ *and* f *satisfies*

$$-\mathrm{Im}(\alpha f(x)) \geq \begin{cases} 0, & \text{if} \quad |x| \leq 1, \\ |x|^{-k}, & \text{if} \quad |x| > 1, \end{cases} \tag{19}$$

with some $n/2 < k < 1/(p-1)$ *and* α *satisfying* (9). *Then, for sufficiently small* m, *there exists some* $R_2 > 0$ *satisfying* (16) *and*

$$\widetilde{T}_{n,p,m,\lambda,\alpha,R_2} \leq C\mu^{-\frac{1}{1/(p-1)-\min(n,k)}}.$$

We remark that Corollaries 1, 2, and 3 correspond to Propositions 1, 2, and 3, respectively

In the next section, we show Proposition 6. In Section 3, we show the proof of Proposition 7 and Corollaries 1, 2, and 3.

2. Proof of Proposition 6

In order to show (14), it is sufficient to show for any $q > n/2$,

$$|\mathcal{R}\langle\cdot\rangle^{-n-1}(x)| \leq 2^{q/2}\|\langle\cdot\rangle^q K\|_{L^1(\mathbb{R}^n)}\langle m\rangle^{q+1}\langle x\rangle^{-q}. \tag{20}$$

For $\theta > 0$,

$$\begin{aligned}(\theta^2 - \Delta)^{-1/2}\theta f &= \theta\mathfrak{F}^{-1}((\theta^2 + |\cdot|^2)^{-1/2}\hat{f}) \\ &= \theta^n\mathfrak{F}^{-1}((1 + |\cdot|^2)^{-1/2}(\hat{f})_{1/\theta}) \\ &= (1-\Delta)^{-1/2} f_\theta.\end{aligned}$$

Therefore,

$$\theta(\theta^2 - \Delta)^{-1/2}\langle\cdot\rangle^{-q} = (1-\Delta)^{-1/2}\langle\cdot/\theta\rangle^{-q} \le \langle\theta\rangle^q K * \langle\cdot\rangle^{-q}, \tag{21}$$

where we have used the fact that K is positive and for any $x \in \mathbb{R}^n$,

$$\langle x\rangle \le \langle x/\theta\rangle\langle\theta\rangle.$$

Then by (21),

$$\begin{aligned}\mathcal{R}\langle\cdot\rangle^{-q}(x) &\le \int_0^m \langle\theta\rangle^q d\theta \cdot K * \langle\cdot\rangle^{-q}(x) \\ &\le 2^{q/2}\langle m\rangle^{q+1}\int_{\mathbb{R}^n} K(y)\langle y\rangle^q dy \cdot \langle x\rangle^{-q},\end{aligned}$$

where we have used the fact that, for any real numbers x and y,

$$\langle x\rangle \le \sqrt{2}\langle x-y\rangle\langle y\rangle.$$

This implies (20). (15) is shown by the following direct computation:

$$|(m^2-\Delta)^{1/2}\langle\cdot/R\rangle^{-n-1}| = R^{-1}|((R^2m^2-\Delta)^{1/2}\langle\cdot\rangle^{-n-1})_R|.$$

3. Proof of nonexistence results

3.1. Proof of Proposition 7

Proposition 7 is shown by the proof of Proposition 4 with replacing $A_{n,n+1}$ by $\widetilde{A}_n\langle Rm\rangle^{n+2}$. So we omit the detail.

3.2. Proof of Corollary 1

Let R_0 be a positive number satisfying that for any $R > R_0$,

$$M_R(0) > -\frac{1}{2}\mathrm{Im}\Big(\alpha\int_{\mathbb{R}^n} u_0(x)dx\Big).$$

We remark that such R_0 exists because of (17) and the Lebesgue dominant theorem. Moreover, let $R \geq R_0$ be a positive number satisfying that

$$-\frac{1}{2}\mathrm{Im}\left(\alpha \int_{\mathbb{R}^n} u_0(x)dx\right) > \widetilde{C}_{n,p,\alpha} 2^{(n+2)/(p-1)} R^{n-1/(p-1)}. \tag{22}$$

If $m < R^{-1}$, then (22) implies (16) and therefore Proposition 7 implies Corollary 1.

Proof of Corollary 2. For $0 < R < 1$, by (18),

$$\begin{aligned} M_R(0) &\geq \mu \int_{|x|\leq 1} |x|^{-k} \langle x/R \rangle^{-n-1} dx \\ &\geq 2^{-n-1}\mu \int_{|x|\leq R} |x|^{-k} dx \\ &= (n-k)^{-1} 2^{-n-1} \omega_n \mu R^{n-k}, \end{aligned}$$

where ω_n is the volume of S_{n-1}. Let $I_1 = (n-k)^{-1}2^{-n-1}\omega_n$ and

$$R_1 = \left(\frac{\mu I_1}{2^{(n+p+1)/(p-1)}\widetilde{C}_{n,p,\alpha}}\right)^{\frac{1}{k-1/(p-1)}}.$$

We put $\mu \gg 1$ so that $R_1 < 1/\max(1, m)$. Then

$$\begin{aligned} &M_{R_1}(0) - \widetilde{C}_{n,p,\alpha} \langle R_1 m \rangle^{(n+2)/(p-1)} R_1^{n-1/(p-1)} \\ &\geq R_1^{n-k}(\mu I_1 - 2^{(n+2)/(p-1)} \widetilde{C}_{n,p,\alpha} R_1^{k-1/(p-1)}) \\ &\geq 2^{-1} R_1^{n-k} \mu I_1 > 0 \end{aligned}$$

and therefore (16) is satisfied. Moreover,

$$\begin{aligned} &\widetilde{T}_{n,p,m,\lambda,\alpha,R_1} \\ &\leq (p-1)^{-1} 2^{p-1} D^{-1}_{n,p,\lambda,\alpha} \left(\frac{\mu I_1}{2^{(n+p+1)/(p-1)}\widetilde{C}_{n,p,\alpha}}\right)^{\frac{k(p-1)}{k-1/(p-1)}} (\mu I_1)^{-p+1} \\ &= (p-1)^{-1} 2^{p-1} D^{-1}_{n,p,\lambda,\alpha} (2^{(n+p+1)/(p-1)} \widetilde{C}_{n,p,\alpha})^{-\frac{k(p-1)}{k-1/(p-1)}} (\mu I_1)^{-\frac{1}{1/(p-1)-k}}. \end{aligned}$$

□

P r o o f o f C o r o l l a r y 3. For $R \gg 1$, by (19),

$$
\begin{aligned}
M_R(0) &\geq \mu \int_{|x|\geq 1} |x|^{-k} \langle x/R\rangle^{-n-1} dx \\
&\geq 2^{-n-1}\mu \int_{1\leq |x|\leq R} |x|^{-k} dx \\
&\geq 2^{-n-1}\omega_n \mu \int_1^R r^{n-k-1} dr, \\
&\geq 2^{-n-1}\omega_n\mu \begin{cases} (n-k)^{-1}(R^{n-k}-1), & \text{if} \quad k<n, \\ \int_1^2 r^{n-k-1}dr, & \text{if} \quad k\geq n, \end{cases} \\
&\geq I_2 \mu R^{(n-k)_+},
\end{aligned}
$$

where $(n-k)_+ = \max(n-k,0)$ and

$$
I_2 = \begin{cases} 2^{-n-2}\omega_n (n-k)^{-1}, & \text{if} \quad k<n, \\ 2^{-n-1}\omega_n \int_1^2 r^{n-k-1}dr, & \text{if} \quad k\geq n. \end{cases}
$$

Let

$$
R_2 = \left(\frac{\mu I_2}{2^{(n+p+1)/(p-1)} \widetilde{C}_{n,p,\alpha}} \right)^{\frac{1}{\min(n,k)-1/(p-1)}},
$$

where $R_2 \gg 1$ if $\mu \ll 1$. Then, by choosing m so that $m \leq 1/R_2$,

$$
\begin{aligned}
&M_{R_2}(0) - \widetilde{C}_{n,p,\alpha}\langle R_2 m\rangle^{(n+2)/(p-1)} R_2^{n-1/(p-1)} \\
&\geq R_2^{(n-k)_+}(\mu I_2 - 2^{(n+2)/(p-1)}\widetilde{C}_{n,p,\alpha} R_2^{\min(n,k)-1/(p-1)}) \\
&\geq 2^{-1} R_2^{(n-k)_+}\mu I_2 > 0
\end{aligned}
$$

and therefore (16) is satisfied. Moreover,

$$
\begin{aligned}
&\widetilde{T}_{n,p,m,\lambda,\alpha,R_2} \\
&\leq (p-1)^{-1} D^{-1}_{n,p,\lambda,\alpha} R_2^{n(p-1)-(n-k)_+(p-1)} (\mu I_2 - \widetilde{C}_{n,p,\alpha} R_2^{\min(n,k)-1/(p-1)})^{-p+1} \\
&\leq (p-1)^{-1} 2^{p-1} D^{-1}_{n,p,\lambda,\alpha} (2^{(n+p+1)/(p-1)}\widetilde{C}_{n,p,\alpha})^{-\frac{\min(n,k)(p-1)}{\min(n,k)-1/(p-1)}} (\mu I_2)^{-\frac{1}{1/(p-1)-\min(n,k)}}.
\end{aligned}
$$

□

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