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MICROLOCAL PARAMETRIX FOR THE CAUCHY PROBLEM FOR HYPERBOLIC SYMMETRIC SYSTEMS WITH SYMPLECTIC MULTIPLE CHARACTERISTICS I

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In the present paper a hyperbolic system P with a principal symbol of a diagonal form is considered. Its determinant is $(\tau-t\xi_1)^{n_1}(\tau+t\xi_1)^{n_2}$. A microlocal parametrix for the Cauchy problem for P at the point $(x^0, 0, \xi^0, 0)$, where $\xi_1^0>0$, is constructed.

1. Introduction. The construction of microlocal parametrices and the propagation of singularities for hyperbolic operators with characteristics of variable multiplicity is an active area of current research. In these problems the geometry of the characteristic set

$$\Sigma=\{(x, \xi)\in T^*(X)\setminus 0 : \det p(x, \xi)=0\},$$

where $p(x, \xi)$ is the principal symbol of the pseudo-differential operator (ψDO). is essential.

The set of double characteristics is defined by $\Sigma_2=\{(x, \xi)\in\Sigma: d_{x,\xi}(\det p(x, \xi))=0\}$. When $\det p(x, \xi)=p_1(x, \xi)\,p_2(x, \xi)$ with dp_1, dp_2 linearly independent we have $\Sigma_2=\{p_1(x, \xi)=p_2(x, \xi)=0\}$.

In local coordinates (x, ξ) we define the Poisson bracket $\{f, g\}$ by

$$\{f, g\}=\sum_{j=1}^{n}\Big(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial\xi_j}\Big).$$

The involutive case $\{p_1, p_2\}=0$ on Σ_2 has been considered by Uhlmann [1] for a scalar operator. In [2] Uhlmann studies operators with multiple involutive characteristics, i. e. with a principal symbol $p_1^k p_2^l$, $k\geq 1$, $l\geq 1$, $\{p_1, p_2\}=0$, when $p_1=p_2=0$.

Nosmas [3] studies the Cauchy problem for a class of uniformly symmetrizable hyperbolic systems with involutive characteristics of not necessarily constant multiplicity.

The symplectic case, when $\{p_1, p_2\}\neq 0$ on Σ_2, has been investigated by Melrose [4], Alinhac [5], Ivrii [6], [7], Petkov [8], etc.

In a more general context Hörmander [9] considers the case, when the symplectic form has a constant rank on Σ_2. In contrast with this Lascar [11] studies a case, when the rank of the symplectic form is not constant on Σ_2.

In the present paper our attention is restricted to a $(n_1+n_2)\times(n_1+n_2)$ system which has the following form

$$P(x, t, D_x, D_t)=\begin{pmatrix}(D_t-tD_{x_1})I_{n_1} & 0\\ 0 & (D_t+tD_{x_1})I_{n_2}\end{pmatrix}+\begin{pmatrix}0 & B_{12}\\ B_{21} & 0\end{pmatrix}(x, t, D_x)$$

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in a conic neighbourhood $\tilde{\Gamma}$ of $(x^0, 0, \xi^0, 0) \in T^*(\mathbb{R}^{n+1})\setminus 0$, $\xi_1^0 > 0$. Here (x, ξ) denotes a point in $T^*(\mathbb{R}^n)$, $t \in \mathbb{R}^1$. $I_{n_\varkappa}$ is the identity $(n_\varkappa \times n_\varkappa)$ matrix, $\varkappa = 1, 2$. B_{12}, B_{21} are classical ψDo of order 0 in the tangential variables depending smoothly on t near $t=0$. We have $\{\tau - t\xi_1, \tau + t\xi_1\} = 2\xi_1 > 0$ if $\tilde{\Gamma}$ is small enough.

Let Γ be a closed cone in $T^*(\mathbb{R}^n)\setminus 0$. Denote by $\mathcal{D}'_\Gamma(\mathbb{R}^n)$ the space of distributions with wave front sets contained in Γ.

Our main result is the following:

Theorem 1. *There exists a conic neighbourhood Γ of (x^0, ξ^0) and an operator $E : \mathcal{D}'_\Gamma(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n+1})$ such that for every $f \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ we have $PEf \in C^\infty(\mathbb{R}^{n+1})$, $\gamma_0 Ef - f \in C^\infty(\mathbb{R}^n)$, where γ_0 denotes the trace on $t=0$.*

In the forthcoming paper [12] we shall reduce a system L whose principal part has the form

$$\begin{pmatrix} (D_t - \lambda_1(x, t, D_x)) I_{n_1} & 0 \\ 0 & (D_t - \lambda_2(x, t, D_x)) I_{n_2} \end{pmatrix}$$

with $\{\tau - \lambda_1, \tau - \lambda_2\} \neq 0$ in a point of multiple characteristics to the simple microlocal form of P mentioned above. This enables us to construct a microlocal parametrix for L and obtain results concerning the propagation of singularities.

The plan of this paper is as follows. In section 2 we carry out, in a formal way, the construction of the parametrix E. Our approach was suggested by the method of Gautesen and Ludwig [13], Kucherenko [15], for construction of asymptotic solutions. The reader may also see Ludwig and Granoff [14]. In section 3 we carry out the asymptotic summation of the obtained symbols. In section 4 we obtain some estimates which we have used in section 3. We study the asymptotic behaviour of the integrals

$$\int_{\Delta(\lambda, m)} \exp\{\pm i(s_1^2 - s_2^2 + \dots + (-1)^{m-1} s_m^2)\} ds_1 \dots ds_m$$

with $\Delta(\lambda, m) = \{(s_1, s_2, \dots, s_m) \in \mathbb{R}^m : 0 \leq s_m \leq s_{m-1} \leq \dots \leq s_1 \leq \lambda\}$ which is of independent interest.

2. Construction of the Parametrix. Consider the operator

$$P = \begin{pmatrix} (D_t - tD_{x_1}) I_{n_1} & B_{12}(x, t, D_x) \\ B_{21}(x, t, D_x) & (D_t + tD_{x_1}) I_{n_2} \end{pmatrix}$$

acting on distributions whose wave front sets are contained in a small conic neighbourhood $\tilde{\Gamma}$ of $(x^0, 0, \xi^0, 0)$. Let $\Gamma = X \times \Gamma_1$, $X \ni x^0$, $\Gamma_1 \ni \xi^0$, be a conic neighbourhood of (x^0, ξ^0), $x \subset\subset \pi_x \tilde{\Gamma}$ (the projection of $\tilde{\Gamma}$ on $\mathbb{R}^n_x$), $\Gamma_1 \subset \pi_\xi \tilde{\Gamma}$. Remember that $\xi_1^0 > 0$. We suppose that Γ_1 is so small that $\inf_{\xi \in \Gamma_1} \xi_1 > 0$.

Our aim is to determine an operator $E : D'_\Gamma \mathcal{D}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n+1})$ such that for every $f \in \mathcal{D}'_\Gamma(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ we have

$$PEf \in C^\infty(\mathbb{R}^{n+1}), \quad \gamma_0 Ef - f \in C^\infty(\mathbb{R}^n). \tag{1}$$

Consider a sequence of phase functions

$$\varphi_1(x, t, \xi),\ \varphi_2(x, t, \xi),\ \varphi_1^{(1)}(x, t, \xi; \tau_1),\ \varphi_2^{(1)}(x, t, \xi; \tau_1), \dots,$$

$$\varphi_1^{(m)}(x, t, \xi; \tau_1, \dots, \tau_m),\ \varphi_2^{(m)}(x, t, \xi; \tau_1, \dots, \tau_m), \dots$$

which are solutions of the following Cauchy problems:

$$\begin{cases}(D_t - tD_{x_1})\varphi_1 = 0, \\ \varphi_1{}_{|t=0} = \langle x, \xi\rangle,\end{cases} \qquad \begin{cases}(D_t + tD_{x_1})\varphi_2 = 0 \\ \varphi_2{}_{|t=0} = \langle x, \xi\rangle,\end{cases}$$

$$\begin{cases}(D_t - tD_{x_1})\varphi_1^{(1)} = 0 \\ \varphi^{(1)}_{1\,|\,t=\tau_1} = \varphi_2(x, \tau_1, \xi),\end{cases} \qquad \begin{cases}(D_t + tD_{x_1})\varphi_2^{(1)} = 0 \\ \varphi^{(1)}_{2\,|\,t=\tau_1} = \varphi_1(x, \tau_1, \xi),\end{cases}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{cases}(D_t - tD_{x_1})\varphi_1^{(m)} = 0 \\ \varphi^{(m)}_{1\,|\,t=\tau_1} = \varphi_2^{(m-1)}(x, \tau_1, \xi; \tau_2, \dots, \tau_m),\end{cases}$$

$$\begin{cases}(D_t + tD_{x_1})\varphi_2^{(m)} = 0 \\ \varphi^{(m)}_{2\,|\,t=\tau_1} = \varphi_1^{(m-1)}(x, \tau_1, \xi; \tau_2, \dots, \tau_m),\end{cases}$$

$$\dots\dots\dots\dots\dots\dots$$

We can easily see that the solutions of these Cauchy problems are

$$\varphi_\varkappa = (-1)^{\varkappa+1}\frac{t^2}{2}\xi_1 + \langle x, \xi\rangle,$$

$$\varphi_\varkappa^{(m)} = (-1)^\varkappa \xi_1\Big(-\frac{t^2}{2} + \tau_1^2 - \tau_2^2 + \dots + (-1)^{m-1}\tau_m^2\Big) + \langle x, \xi\rangle,\ \varkappa = 1, 2;\ m = 1, 2, 3, \dots$$

Suppose that we have already constructed an operator $\tilde{E}: \mathcal{D}'_\Gamma(\mathbf{R}^n) \cap \mathcal{E}'(\mathbf{R}^n) \to \mathcal{D}'(\mathbf{R}^{n+1})$ such that for every $f \in \mathcal{D}'_\Gamma(\mathbf{R}^n) \cap \mathcal{E}'(\mathbf{R}^n)$ we have

$$(2) \qquad P\tilde{E}f \in C^\infty(\mathbf{R}^{n+1}), \quad \gamma_0\tilde{E}f - f = Rf,$$

where R is a classical ψDO of order -1. This will be denoted further as $R \in CL^{-1}(\mathbf{R}^n)$. Then the operator $Id + R$ is elliptic. Denoting by $(Id+R)^{-1}$ a properly supported parametrix for $Id + R$, the operator $E = \tilde{E}(Id+R)^{-1}$ will satisfy (1).

We try to find the operator $\tilde{E}$, satisfying (2), as a sum of two Fourier integral operators (FIO): $\tilde{E} = \tilde{E}_1 + \tilde{E}_2$,

$$\tilde{E}_\varkappa f(x, t) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\varphi_\varkappa(x, t, \xi)} e_\varkappa(x, t, \xi)\hat{f}(\xi)d\xi, \quad \varkappa = 1, 2,$$

with

$$e_\varkappa(x, t, \xi) \sim \sum_{j=0}^\infty c_\varkappa^{-j}(x, t, \xi) + \sum_{j=0}^\infty \int_0^t e^{i(-1)^\varkappa \tau_1^2 \xi_1} c^{-j}_{(1)\varkappa}(x, t, \xi; \tau_1)d\tau_1 + \dots$$

$$+ \sum_{j=0}^\infty \int_{\Delta(t, m)} \exp\{i(-1)^\varkappa \xi_1(\tau_1^2 - \tau_2^2 + \dots + (-1)^{m-1}\tau_m^2)\} c^{-j}_{(m)\varkappa}(x, t, \xi; \tau^{(m)})d\tau^{(m)} + \dots$$

Here by $\tau^{(m)}$ we have denoted $(\tau_1, \tau_2, \dots, \tau_m)$, $d\tau^{(m)} = d\tau_1\, d\tau_2 \dots d\tau_m$, $c_\varkappa^{-j}$, $c^{-j}_{(m)\varkappa}$ are $(n_1+n_2)\times(n_1+n_2)$-matrices depending smoothly on x, t, $\tau^{(m)}$ and homogeneous of degree $-j$ in ξ. Sometimes we shall write

$$c_{\varkappa}^{-j}=\begin{pmatrix} c_{\varkappa;11}^{-j} & c_{\varkappa;12}^{-j} \\ c_{\varkappa;21}^{-j} & c_{\varkappa;22}^{-j} \end{pmatrix},$$

where $c_{\varkappa;11}^{-j}$, $c_{\varkappa;12}^{-j}$, $c_{\varkappa;21}^{-j}$, $c_{\varkappa;22}^{-j}$ are matrices of dimension $(n_1\times n_1)$, $(n_1\times n_2)$, $(n_2\times n_1)$, $(n_2\times n_2)$, and similarly for $c_{(m)\varkappa}^{-j}$. Up to now it is not clear whether these asymptotic sums have any sense. The form of the parametrix was suggested by [13; 15].

Applying the operator P to an arbitrary member of these sums we obtain

$$P\Big(\int_{\mathbf{R}^n}\int_{\Delta(t,m)} e^{i\varphi_{\varkappa}^{(m)}(x,t,\xi;\tau^{(m)})} c_{(m)\varkappa}^{-j}(x,t,\xi;\tau^{(m)})d\tau^{(m)}\,.\,\hat{f}(\xi)d\xi\Big)$$

$$=-i\int_{\mathbf{R}^n}\int_{\Delta(t,m-1)} e^{i\varphi_{3-\varkappa}^{(m-1)}(x,t,\xi;\tau^{(m-1)})} c_{(m)\varkappa}^{-j}(x,t,\xi;t,\tau^{(m-1)})d\tau^{(m-1)}\,.\,\hat{f}(\xi)d\xi$$

$$+\int_{\mathbf{R}^n}\int_{\Delta(t,m)} e^{i\varphi_{\varkappa}^{(m)}(x,t,\xi;\tau^{(m)})}(\tilde{A}_{\varkappa}+\tilde{P})c_{(m)\varkappa}^{-j}(x,t,\xi;\tau^{(m)})d\tau^{(m)}\,.\,\hat{f}(\xi)d\xi$$

with

$$\tilde{A}_1=\begin{pmatrix} 0 & 0 \\ 0 & 2t\xi_1 I_{n_2}\end{pmatrix},\quad \tilde{A}_2=\begin{pmatrix} -2t\xi_1 I_{n_1} & 0 \\ 0 & 0\end{pmatrix},\quad \tilde{P}=\begin{pmatrix} (D_t-tD_{x_1})I_{n_1} & 0 \\ 0 & (D_t+tD_{x_1})I_{n_2}\end{pmatrix}+\tilde{B}$$

and $\tilde{B}$ is defined by

$$\tilde{B}c_{(m)\varkappa}^{-j}(x,t,\xi;\tau^{(m)})=e^{-i\langle x,\xi\rangle}\begin{pmatrix} 0 & B_{12} \\ B_{12} & 0\end{pmatrix}\{e^{i\langle x,\xi\rangle}c_{(m)\varkappa}^{-j}(x,t,\xi;\tau^{(m)})\}$$

$$\sim\sum_{\nu=0}^{\infty}\Big(\sum_{0\le|\alpha|\le\nu}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\begin{pmatrix} 0 & B_{12}^{-\nu+|\alpha|} \\ B_{21}^{-\nu+|\alpha|} & 0\end{pmatrix}(x,t,\xi)\,.\,D_x^{\alpha}c_{(m)\varkappa}^{-j}(x,t,\xi;\tau^{(m)})\Big).$$

Therefore

$$P\tilde{E}f(x,t)=(2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\varphi_1(x,t,\xi)}g_1(x,t,\xi)\hat{f}(\xi)d\xi$$

$$+(2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\varphi_2(x,t,\xi)}g_2(x,t,\xi)\hat{f}(\xi)d\xi$$

with

$$g_{\varkappa}(x,t,\xi)\sim\sum_{j=0}^{\infty}\{(\tilde{A}_{\varkappa}+\tilde{P})c_{\varkappa}^{-j}(x,t,\xi)-ic_{(1)3-\varkappa}^{-j}(x,t,\xi;t)\}$$

$$+\sum_{j=0}^{\infty}\int_0^t e^{i(-1)^{\varkappa}\tau_1^2\xi_1}\{(\tilde{A}_{\varkappa}+\tilde{P})c_{(1)\varkappa}^{-j}(x,t,\xi;\tau_1)-ic_{(2)3-\varkappa}^{-j}(x,t,\xi;t,\tau_1)\}d\tau_1$$

$$+\ldots+\sum_{j=0}^{\infty}\int_{\Delta(t,m)}\exp\{i(-1)^{\varkappa}\xi_1(\tau_1^2-\ldots+(-1)^{m-1}\tau_m^2\}\{(\tilde{A}_{\varkappa}+\tilde{P})c_{(m)\varkappa}^{-j}(x,t,\xi;\tau^{(m)})$$

$$-ic_{(m+1)3-\varkappa}^{-j}(x,t,\xi;t,\tau^{(m)})\}d\tau^{(m)}+\ldots$$

Moreover $\tilde{E}f(x,0)=(2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}(e_1(x,0,\xi)+e_2(x,0,\xi))\hat{f}(\xi)d\xi$. Since $e_1(x, 0,\xi)+e_2(x,0,\xi)\sim\sum_{j=0}^{\infty}(c_1^{-j}(x,0,\xi)+c_2^{-j}(x,0,\xi))$, we must have

$$c_1^0(x,0,\xi)+c_2^0(x,0,\xi)=I_{n_1+n_2}, \tag{3}$$

which implies the second condition of (2).

In order to determine c_1^{-j}, c_2^{-j}, $c_{(m)1}^{-j}$, $c_{(m)2}^{-j}$ we equate to 0 the coefficients at like powers of $|\xi|$ in the asymptotic expansion of each integral in g_1 and g_2:

i) We start with

$$\begin{pmatrix} 0 & 0 \\ 0 & 2t\xi_1 I_{n_2}\end{pmatrix} c_1^0=0.$$

Therefore c_1^0 is of the form

$$\begin{pmatrix} c_{1;11}^0 & c_{1;12}^0 \\ 0 & 0\end{pmatrix}$$

and similarly

$$c_2^0=\begin{pmatrix} 0 & 0 \\ c_{2;21}^0 & c_{2;22}^0\end{pmatrix},\quad c_{(m)1}^0=\begin{pmatrix} c_{(m)1;11}^0 & c_{(m)1;12}^0 \\ 0 & 0\end{pmatrix},\quad c_{(m)2}^0=\begin{pmatrix} 0 & 0 \\ c_{(m)2;21}^0 & c_{(m)2;22}^0\end{pmatrix},\quad m=1,2,3,\ldots$$

ii) The terms in g_1 which do not contain the integration over τ_1 yield

$$\begin{pmatrix} 0 & 0 \\ 0 & 2t\xi_1 I_{n_2}\end{pmatrix} c_1^{-1}+\begin{pmatrix} (D_t-tD_{x_1})I_{n_1} & B_{12}^0(x,t,\xi) \\ B_{21}^0(x,t,\xi) & (D_t+tD_{x_1})I_{n_2}\end{pmatrix}\begin{pmatrix} c_{1;11}^0 & c_{1;12}^0 \\ 0 & 0\end{pmatrix}$$
$$-i\begin{pmatrix} 0 & 0 \\ c_{(1)2;21}^0 & c_{(1)2;22}^0\end{pmatrix}(x,t,\xi;t)=0.$$

This equality and (3) lead to

$$\begin{cases}(D_t-tD_{x_1})c_{1;11}^0=0\\ c_{1;11}^0(x,0,\xi)=I_{n_1},\end{cases}\qquad \begin{cases}(D_t-tD_{x_1})c_{1;12}^0=0\\ c_{1;12}^0(x,0,\xi)=0\end{cases}$$

and we conclude that $c_1^0=\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0\end{pmatrix}$. For $t=0$ we get the following equalities: $B_{21}^0(x,0,\xi)-ic_{(1)2;21}^0(x,0,\xi;\ 0)=0$, $c_{(1)2;22}^0(x,0,\xi;0)=0$. In this way we obtain initial conditions for $c_{(1)2;21}^0$, $c_{(1)2;22}^0$, which enables us to define the homogeneous of degree -1 symbols

$$c_{1;21}^{-1}=[-B_{21}^0(x,t,\xi)+ic_{(1)2;21}^0(x,t,\xi;t)]/2t\xi_1,\quad c_{1;22}^{-1}=ic_{(1)2;22}^0(x,t,\xi;\ t)/2t\xi_1.$$

The terms in g_2 similar to those discussed above imply $c_2^0=\begin{pmatrix} 0 & 0 \\ 0 & I_{n_2}\end{pmatrix}$. For $t=0$ we get the equalities

$$c_{(1)1;11}^0(x,0,\xi;0)=0,\qquad B_{12}^0(x,0,\xi)-ic_{(1)1;12}^0(x,0,\xi;0)=0;$$
$$c_{2;11}^{-1}=ic_{(1)1;11}^0(x,t,\xi;t)/-2t\xi_1,\qquad c_{2;12}^{-1}=[-B_{12}^0(x,t,\xi)+ic_{(1)1;12}^0(x,t,\xi:t)]/-2t\xi_1.$$

The terms in g_2 which are under the first integral yield

$$\begin{pmatrix} -2t\xi_1 I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} c^{-1}_{(1)2} + \begin{pmatrix} (D_t - tD_{x_1}) I_{n_1} & B^0_{12}(x, t, \xi) \\ B^0_{21}(x, t, \xi) & (D_t + tD_{x_1}) I_{n_2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c^0_{(1)2;21} & c^0_{(1)2;22} \end{pmatrix}$$

$$-i\begin{pmatrix} c^0_{(2)1;11} & c^0_{(2)1;12} \\ 0 & 0 \end{pmatrix}(x, t, \xi ; t, \tau_1) = 0.$$

From the equations

$$\begin{cases} (D_t + tD_{x_1}) c^0_{(1)2;21} = 0 \\ c^0_{(1)2;21}(x, 0, \xi ; 0) = -iB^0_{21}(x, 0, \xi), \end{cases} \qquad \begin{cases} (D_t + tD_{x_1}) c^0_{(1)2;22} = 0 \\ c^0_{(1)2;22}(x, 0, \xi ; 0) = 0 \end{cases}$$

we find $c^0_{(1)2;22} \equiv 0$ and $c^0_{(1)2;21}$ which does not depend on τ_1. Moreover, we have obviously $c^{-1}_{1;22} \equiv 0$.

For $t=0$ we get

$$B^0_{12}(x, 0, \xi) c^0_{(1)2;21}(x, 0, \xi) - ic^0_{(2)1;11}(x, 0, \xi ; 0, \tau_1) = 0, \quad c^0_{(2)1;12}(x, 0, \xi ; 0, \tau_1) = 0.$$

Since $(D_t - tD_{x_1}) c^0_{(2)1;1l} = 0$, $l = 1, 2$, we find that $c^0_{(2)1;12} \equiv 0$, $c^0_{(2)1;12}$ does not depend on τ_1, τ_2; etc.

iii) Equating to 0 the coefficients of $|\xi|^{-j}$, we determine $c^{-j}_{(m)\varkappa;\varkappa l}$, $l = 1, 2$, as solutions of the following Cauchy problems:

$$\begin{cases} (D_t) + (-1)^\varkappa tD_{x_1}) c^{-j}_{(m)\varkappa;\varkappa l} = a^{-j}(x, t, \xi) \\ c^{-j}_{(m)\varkappa;\varkappa l}(x, 0, \xi ; 0, \tau^{(m-1)}) = d^{-j}(x, \xi), \end{cases}$$

where a^{-j}, d^{-j} are known functions, homogeneous of degree $-j$ in ξ. The solutions $c^{-j}_{(m)\varkappa;\varkappa l}$ are homogeneous of degree $-j$ in ξ and they do not depend on $\tau^{(m)}$. We determine $c^{-j-1}_{(m)1;2l}$, $c^{-j-1}_{(m)2;1l}$, as before, dividing by $\pm 2t\xi_1$. These symbols are homogeneous of degree $-j-1$ in ξ and do not depend on $\tau^{(m)}$.

Thus for $\varkappa = 1, 2$

$$e_\varkappa(x, t, \xi) \sim \sum_{j=0}^\infty c^{-j}_\varkappa(x, t, \xi) + \sum_{j=0}^\infty c^{-j}_{(1)\varkappa}(x, t, \xi) \int_0^t e^{i(-1)^\varkappa \tau_1^2 \xi_1} d\tau_1 + \dots$$

$$+ \sum_{j=0}^\infty c^{-j}_{(m)\varkappa}(x, t, \xi) \int_{A(t,m)} \exp\{i(-1)^\varkappa \xi_1(\tau_1^2 - \tau_2^2 + \dots + (-1)^{m-1}\tau_m^2)\} d\tau^{(m)} + \dots,$$

$$c^0_1 = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad c^0_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}.$$

4. Asymptotic Summation. Remember that $\Gamma = X \times \Gamma_1$ is a conic neighbourhood of (x^0, ξ^0) such that $\inf_{\xi \in \Gamma_1 \cap S^{n-1}} \xi_1 = \delta > 0$. Let $\xi \in \Gamma_1$. Then $\xi/|\xi| \in \Gamma_1 \cap S^{n-1}$ implies $\delta|\xi| \le \xi_1 \le |\xi|$ on Γ_1.

Definition. *Let* $a(x, t, \xi) \in C^\infty(X \times I \times \Gamma_1)$, *where* I *is an open interval in* $\mathbf{R}^1_t$ *containing* 0. *We say that* $a \in S^m_{\varrho_1, \varrho'; \delta}(X \times I \times \Gamma_1)$ *if for every pair of multiindices* $\alpha = (\alpha_1, \alpha') = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, *for every nonnegative integer* l, *for every closed interval* J: $0 \in J \subset I$ *and for every compact subset* K *of* X *there exists a constant* $C_{l,\alpha,\beta,J,K} > 0$ *such that*

$$|D_t^l D_\xi^\alpha D_x^\beta a(x, t, \xi)| \leqq C_{l,\alpha,\beta,J,K}(1+|\xi|)^{m-\varrho_1\alpha_1-\varrho'|\alpha'|+\delta l}$$

for $x \in K$, $t \in J$, $\xi \in \Gamma_1$.

We are going to consider the product

$$c_{(m)\varkappa}^{-j}(x, t, \xi) \int_{\Delta(t,m)} \exp\{i(-1)^\varkappa \xi_1(\tau_1^2-\tau_2^2+\ldots+(-1)^{m-1}\tau_m^2)\}d\tau^{(m)}$$

which we denote by $I_{(m)\varkappa}^{-j}(x, t, \xi)$. In the above integral we carry out the following change of the variables: $\tau_\mu = s_\mu \xi_1^{-1/2}$, $\mu = 1, 2, \ldots, m$. Thus we get

$$I_{(m)\varkappa}^{-j}(x, t, \xi) = c_{(m)\varkappa}^{-j}(x, t, \xi)\xi_1^{-m/2} \int_{\Delta(t\sqrt{\xi_1},\, m)} \exp\{i(-1)^\varkappa(s_1^2-s_2^2+\ldots+(-1)^{m-1}s_m^2)\}ds^{(m)}$$

Introduce the notation

$$Q_{(m)\varkappa}(\lambda) = \int_{\Delta(\lambda,\, m)} \exp\{i(-1)^\varkappa(s_1^2-s_2^2+\ldots+(-1)^{m-1}s_m^2)\}ds^{(m)}.$$

Sometimes, for the sake of simplicity, we shall write $Q_{(m)}^-$ instead of $Q_{(m)1}$ and $Q_{(m)}^+$ instead of $Q_{(m)2}$. Now we have

$$I_{(m)\varkappa}^{-1}(x, t, \xi) = c_{(m)\varkappa}^{-j}(x, t, \xi)\xi_1^{-m/2}Q_{(m)\varkappa}(t\sqrt{\xi_1}).$$

The following two propositions will play an essential role in our construction.

Proposition 1. *For every* $r>0$, *for every pair of nonnegative integers* l, k *and for* $\varkappa=1, 2$ *there exists a constant* $C_{r,l,2,\varkappa}>0$ *such that*

$$\sup_{|t|\leqq r\xi_1^{-1/2}} |D_t^l D_{\xi_1}^k Q_{(m)\varkappa}(t\sqrt{\xi_1})| \leqq C_{r,l,k,\varkappa}\xi_1^{l/2-k}.$$

Proposition 2. *Let* $|\lambda|>1$. *Then* $Q_{(m)\varkappa}(\lambda)$ *has the representation* $Q_{(m)\varkappa}(\lambda) = p_{(m)\varkappa}(\lambda) + e^{i(-1)^\varkappa\lambda^2}q_{(m)\varkappa}(\lambda)$, *where* $p_{(m)\varkappa}(\lambda)$ *and* $q_{(m)\varkappa}(\lambda)$ *admit the following asymptotic expansions*:

$$p_{(m)\varkappa}(\lambda) \sim a_{(m)\varkappa}\ln^{[m/2]}|\lambda| + \sum_{\mu=0}^{[m/2]-1} \ln^\mu|\lambda| \sum_{\nu=0}^{\infty} b_{(m)\varkappa;\mu\nu}\lambda^{-2\nu},$$

$$q_{(m)\varkappa}(\lambda) \sim \sum_{\mu=0}^{[(m-1)/2]} \ln^\mu|\lambda| \sum_{\nu=1}^{\infty} c_{(m)\varkappa;\mu\nu}\lambda^{-2\nu+1}, \qquad \lambda \to \pm\infty.$$

Assuming that Propositions 1 and 2 have been already proved we shall finish the asymptotic summation. First we shall need some estimates. Let $\chi(\lambda) \in C_0^\infty(\mathrm{R}^1)$. $0 \leqq \chi(\lambda) \leqq 1$. $\chi(\lambda)=1$ for $|\lambda| \leqq 3/2$, $\chi(\lambda)=0$ for $|\lambda| \geqq 2$. Introduce the sum

$$I_{(m)\varkappa}^{-j}(x, t, \xi) = \chi(t\sqrt{\xi_1})I_{(m)\varkappa}^{-j}(x, t, \xi) + (1-\chi(t\sqrt{\xi_1}))I_{(m)\varkappa}^{-j}(x, t, \xi).$$

We begin with the term $\chi(t\sqrt{\xi_1})I_{(m)\varkappa}^{-j}(x, t, \xi)$. Note that $D_t^l(\chi(t\sqrt{\xi_1})) = (-i)^l\xi_1^{l/2}\chi^{(l)}(t\sqrt{\xi_1})$. The chain rule yields

$$D_t^l D_{\xi_1}^{\alpha_1}(\chi(t\sqrt{\xi_1})) = (-i)^{l+\alpha_1} \sum_{k=0}^{\alpha_1} c_{k,\alpha_1}\xi_1^{l/2-\alpha_1+k}\partial_{\xi_1}^k(\chi^{(l)}(t\sqrt{\xi_1})).$$

We have $\partial_{\xi_1}(\chi^{(l)}(t\sqrt{\xi_1}))=\frac{1}{2}t\xi_1^{-1/2}\chi^{(l+1)}(t\sqrt{\xi_1})$. Suppose that we have already proved

$$\partial_{\xi_1}^k(\chi^{(l)}(t\sqrt{\xi_1}))=\sum_{\nu=1}^{k}\tilde{c}_{k,\nu}t^\nu\xi_1^{\nu/2-k}\chi^{(l+\nu)}(t\sqrt{\xi_1}).$$

Then we have

$$\partial_{\xi_1}^{k+1}(\chi^{(l)}(t\sqrt{\xi_1}))=\sum_{\nu=1}^{k}\tilde{c}_{k,\nu}\{(\frac{\nu}{2}-k)t^\nu\xi_1^{\nu/2-k-1}\chi^{(l+\nu)}(t\sqrt{\xi_1})$$
$$+\frac{1}{2}t^{\nu+1}\xi_1^{\nu/2-k-1/2}\chi^{(l+\nu+1)}(t\sqrt{\xi_1})\}$$
$$=\tilde{c}_{k,1}(1/2-k)t\xi_1^{1/2-(k+1)}\chi^{(l+1)}(t\sqrt{\xi_1})+\sum_{\nu=2}^{k}\{\tilde{c}_{k,\nu}(\frac{\nu}{2}-k)$$
$$+\frac{1}{2}\tilde{c}_{k,\nu-1}\}t^\nu\xi_1^{\nu/2-(k+1)}\chi^{(l+\nu)}(t\sqrt{\xi_1})+\frac{1}{2}\tilde{c}_{k,k}t^{k+1}\xi_1^{-(k+1)/2}\chi^{(l+k+1)}(t\sqrt{\xi_1})$$

which is a sum of the desired type with $k+1$ instead of k. Thus we have

$$\partial_{\xi_1}^k(\chi^{(l)}(t\sqrt{\xi_1}))=\sum_{\nu=1}^{k}\tilde{c}_{k,\nu}t^\nu\xi_1^{\nu/2-k}\chi^{(\nu+l)}(t\sqrt{\xi_1})=\xi_1^{-k}\sum_{\nu=1}^{k}\tilde{c}_{k,\nu}(t\sqrt{\xi_1})^\nu\chi^{(l+\nu)}(t\sqrt{\xi_1}).$$

Since for $|t\sqrt{\xi_1}|\geqq 2$ we have $\chi(t\sqrt{\xi_1})=0$, it follows that $|\partial_{\xi_1}^k(\chi^{(l)}(t\sqrt{\xi_1}))|\leqq C_{k,l}\xi_1^{-k}$ and $|D_t^lD_{\xi_1}^{\alpha_1}(\chi(t\sqrt{\xi_1}))|\leqq C_{l,\alpha_1}\xi_1^{l/2-\alpha_1}$. According to Proposition 1, for $|t\sqrt{\xi_1}|\leqq 2$ the derivatives of $Q_{(m)\varkappa}(t\sqrt{\xi_1})$ satisfy estimates of the same type, hence

$$|D_t^lD_{\xi_1}^{\alpha_1}(\chi(t\sqrt{\xi_1})Q_{(m)\varkappa}(t\sqrt{\xi_1}))|\leqq C_{l,\alpha_1}\xi_1^{l/2-\alpha_1}.$$

Finally, using the fact that $\delta|\xi|\leqq\xi_1\leqq|\xi|$ in Γ_1, we obtain

$$|D_t^lD_\xi^\alpha D_x^\beta(\chi(t\sqrt{\xi_1})I_{(m)\varkappa}^{-j}(x,t,\xi))|\leqq C(1+|\xi|)^{-j-m/2-|\alpha|+l/2},$$

i. e. $\chi(t\sqrt{\xi_1})I_{(m)\varkappa}^{-j}(x,t,\xi)\in S_{1,1;1/2}^{-j-m/2}(X\times I\times\Gamma_1)$. An application of Proposition 2 implies

$$\exp[i(-1)^{\varkappa+1}\frac{t^2}{2}\xi_1](1-\chi(t\sqrt{\xi_1}))I_{(m)\varkappa}^{-j}(x,t,\xi)$$
$$=\exp[i(-1)^{\varkappa+1}\frac{t^2}{2}\xi_1]c_{(m)\varkappa}^{-j}(x,t,\xi)\xi_1^{-m/2}(1-\chi(t\sqrt{\xi_1}))p_{(m)\varkappa}(t\sqrt{\xi_1})$$
$$+\exp[i-1)^{\varkappa}\frac{t^2}{2}\xi_1]c_{(m)\varkappa}^{-j}(x,t,\xi)\xi_1^{-m/2}(1-\chi(t\sqrt{\xi_1}))q_{(m)\varkappa}(t\sqrt{\xi_1}).$$

Introduce $h(\lambda)=(1-\chi(\lambda))p_{(m)\varkappa}(\lambda)$. Then we have

$$\frac{d^kh(\lambda)}{d\lambda^k}=\sum_{\mu=0}^{k}c_{k,\mu}\frac{d^\mu}{d\lambda^\mu}(1-\chi)(\lambda)\frac{d^{k-\mu}}{d\lambda^{k-\mu}}p_{(m)\lambda}(\lambda).$$

Let $m \geq 2$. Then $|p_{(m)\varkappa}(\lambda)| \leq C \ln^{[m/2]} |\lambda|$,

$$\left|\frac{d^{k-\mu}}{d\lambda^{k-\mu}} p_{(m)\varkappa}(\lambda)\right| \leq C \ln^{[m/2]-1} |\lambda| . |\lambda|^{-k+\mu}$$

for $\mu < k$, hence $\left|\frac{d^k h}{d\lambda^k}(\lambda)\right| \leq C \ln^{[m/2]} |\lambda| \leq C |\lambda|^{\varepsilon}$, where $0 < \varepsilon < 1$. For $m = 1$, $\left|\frac{d^k h(\lambda)}{d\lambda^k}\right| \leq C$. As before we have

$$D_t^l D_{\xi_1}^{\alpha_1}(h(t\sqrt{\xi_1}))$$

$$= c_{\alpha_1,0}\xi_1^{l/2-\alpha_1} h^{(l)}(t\sqrt{\xi_1}) + \sum_{k=1}^{\alpha_1} \sum_{\nu=1}^{k} c_{\alpha_1,k,\nu}\xi^{l/2-\alpha_1+\nu/2} t^\nu h^{(l+\nu)}(t\sqrt{\xi_1})$$

$$= t^{\alpha_1}\xi^{(l-\alpha_1)/2} \sum_{\nu=0}^{\alpha_1} \tilde{c}_{\alpha_1,\nu}(t\sqrt{\xi_1})^{\nu-\alpha_1} h^{(l+\nu)}(t\sqrt{\xi_1}).$$

Then for $m = 1$ we have

$$|D_t^l D_{\xi_1}^{\alpha_1}(h(t\sqrt{\xi_1}))| \leq |t|^{\alpha_1}\xi_1^{(l-\alpha_1)/2} \sum_{\nu=0}^{\alpha_1} c_{\alpha_1,\nu} (t\sqrt{\xi_1})^{\nu-\alpha_1}$$

while for $m \geq 2$ we obtain

$$|D_t^l D_{\xi_1}^{\alpha_1}(h(t\sqrt{\xi_1}))| \leq |t|^{\alpha_1+\varepsilon}\xi_1^{(l-\alpha_1+\varepsilon)/2} \sum_{\nu=0}^{a} c_{\alpha_1,\nu}(t\sqrt{\xi_1})^{\nu-\alpha_1}.$$

Furthermore, since $|t| \leq T$ and $|t\sqrt{\xi_1}| \geq 3/2$, we obtain the estimates

$$|D_t^l D_{\xi_1}^{\alpha_1}[(1-\chi(t\sqrt{\xi_1}))\, p_{(1)\varkappa}(t\sqrt{\xi_1})]| \leq C\xi_1^{(l-\alpha_1)/2},$$

$$|D_t^l D_{\xi_1}^{\alpha_1}[(1-\chi(t\sqrt{\xi_1}))\, p_{(m)\varkappa}(t\sqrt{\xi_1})]| \leq C\xi_1^{(l-\alpha_1+\varepsilon)/2} \quad \text{for} \quad m \geq 2$$

and in a similar way we get

$$|D_t^l D_{\xi_1}^{\alpha_1}[1-\chi(t\sqrt{\xi_1}))q_{(1)\varkappa}(t\sqrt{\xi_1})| \leq C\xi_1^{(l-\alpha_1-1)/2},$$

$$|D_t^l D_{\xi_1}^{\alpha_1}[(1-\chi(t\sqrt{\xi_1}))q_{(m)\varkappa}(t\sqrt{\xi_1})]| \leq C\xi_1^{(l-\alpha_1-1+\varepsilon)/2} \quad \text{for} \quad m \geq 2.$$

Finally, we have

$$\exp[i(-1)^{\varkappa+1}\frac{t^2}{2}\xi_1](1-\chi(t\sqrt{\xi_1}))I_{(m)\varkappa}^{-j}(x, t, \xi)$$

$$= \exp[i(-1)^{\varkappa+1}\frac{t^2}{2}\xi_1]f_{(m)\varkappa}^{-j}(x, t, \xi) + \exp[i(-1)^{\varkappa}\frac{t^2}{2}\xi_1]g_{(m)\varkappa}^{-j}(x, t, \xi)$$

with

$$f_{(1)\varkappa}^{-j}(x, t, \xi) \in S_{1/2,1;1/2}^{-j-1/2}(X \times I \times \Gamma_1),$$

$$f_{(m)\varkappa}^{-j}(x, t, \xi) \in S_{1/2,1;1/2}^{-j-(m-\varepsilon)/2}(X \times I \times \Gamma_1) \quad \text{for} \quad m \geq 2,$$

$$g_{(1)\varkappa}^{-j}(x, t, \xi) \in S_{1/2,1;1/2}^{-j-1}(X \times I \times \Gamma_1),$$

$$g_{(m)\varkappa}^{-j}(x, t, \xi) \in S_{1/2,1;1/2}^{-j-(m+1-\varepsilon)/2}(X\times I\times \Gamma_1) \quad \text{for} \quad m\geqq 2,\ 0<\varepsilon<1.$$

After this preparatory work we shall carry out the asymptotic summation. First we find a classical symbol $c_{(m)\varkappa}(x, t, \xi)$ of order 0 for each positive integer m and for $\varkappa=1, 2$ in such manner that

$$c_{(m)\varkappa}(x, t, \xi) \sim \sum_{j=0}^{\infty} c_{(m)\varkappa}^{-j}(x, t, \xi).$$

For the moment we claim that $\tilde{E}$ is defined by

$$\tilde{E}f(x, t) = (2\pi)^{-n} \int_{\mathbf{R}^n} \exp\left[i\left(\frac{t^2}{2}\xi_1 + \langle x, \xi\rangle\right)\right] e_1(x, t, \xi)\hat{f}(\xi)d\xi$$

$$+(2\pi)^{-n} \int_{\mathbf{R}^n} \exp\left[i\left(-\frac{t^2}{2}\xi_1 + \langle x, \xi\rangle\right)\right] e_2(x, t, \xi)\hat{f}(\xi)d\xi, \tag{4}$$

where in a formal sense we take

$$e_\varkappa(x, t, \xi) \sim c_\varkappa(x, t, \xi) + \sum_{m=1}^{\infty} c_{(m)\varkappa}(x, t, \xi)\xi_1^{-m/2}Q_{(m)\varkappa}(t\sqrt{\xi_1}), \quad \varkappa=1, 2.$$

As before we have

$$e^{it^2\xi_1/2}c_{(m)1}(x, t, \xi)\xi_1^{-m/2}Q_{(m)1}(t\sqrt{\xi_1}) + e^{-it^2\xi_1/2}c_{(m)2}(x, t, \xi)\xi_1^{-m/2}Q_{(m)2}(t\sqrt{\xi_1})$$

$$= e^{it^2\xi_1/2}(\tilde{f}_{(m)1}(x, t, \xi) + f_{(m)1}(x, t, \xi) + g_{(m)2}(x, t, \xi))$$

$$+ e^{-it^2\xi_1/2}(\tilde{f}_{(m)2}(x, t, \xi) + f_{(m)2}(x, t, \xi) + g_{(m)1}(x, t, \xi))$$

with

$$\tilde{f}_{(m)\varkappa}(x, t, \xi) = c_{(m)\varkappa}(x, t, \xi)\xi_1^{-m/2}\chi(t\sqrt{\xi_1})\, Q_{(m)\varkappa}(t\sqrt{\xi_1}),$$

$$f_{(m)\varkappa}(x, t, \xi) = c_{(m)\varkappa}(x, t, \xi)\xi_1^{-m/2}(1-\chi(t\sqrt{\xi_1}))\, p_{(m)\varkappa}(t\sqrt{\xi_1}),$$

$$g_{(m)\varkappa}(x, t, \xi) = c_{(m)\varkappa}(x, t, \xi)\xi_1^{-m/2}(1-\chi(t\sqrt{\xi_1}))q_{(m)\varkappa}(t\sqrt{\xi_1}),$$

$$\tilde{f}_{(m)\varkappa} \in S_{1,1;1/2}^{-m/2}(X\times I\times \Gamma_1), \quad m\geqq 1,$$

$$f_{(1)\varkappa} \in S_{1/2,1;1/2}^{-1/2}(X\times I\times \Gamma_1),\ f_{(m)\varkappa} \in S_{1/2,1;1/2}^{-(m-\varepsilon)/2}(X\times I\times \Gamma_1) \quad \text{for} \quad m\geqq 2,$$

$$g_{(1)\varkappa} \in S_{1/2,1;1/2}^{-1}(X\times I\times \Gamma_1),\ g_{(m)\varkappa} \in S_{1/2,1;1/2}^{-(m+1-\varepsilon)/2}(X\times I\times \Gamma_1) \quad \text{for} \quad m\geqq 2.$$

The asymptotic summation of symbols can be carried out for arbitrary ϱ, δ, $0\leq\varrho\leq 1$, $0\leq\delta\leq 1$, hence we can find symbols

$$\tilde{l}_\varkappa \in S_{1,1;1/2}^{-1/2}(X\times I\times \Gamma_1), \quad l_\varkappa \in S_{1/2,1;1/2}^{-1/2}(X\times I\times \Gamma_1), \ \varkappa=1, 2$$

where

$$\tilde{l}_\varkappa(x, t, \xi) \sim \sum_{m=1}^{\infty} \tilde{f}_{(m)\varkappa}(x, t, \xi),$$

$$I_\varkappa(x,t,\xi)\sim f_{(1)\varkappa}(x,t,\xi)-\sum_{m=2}^{\infty}(f_{(m)\varkappa}(x,t,\xi)+g_{(m-1)_s-\varkappa}(x,t,\lambda)).$$

Now define the operator $\tilde{E}$ by (4), where

$$e_\varkappa(x,t,\xi)=c_\varkappa(x,t,\xi)+\tilde{I}_\varkappa(x,t,\xi)+I_\varkappa(x,t,\xi),\quad \varkappa=1,\ 2.$$

For every function $f\in C_0^\infty(\mathbf{R}^n)$ we write $\tilde{E}f$ as a sum of two oscillatory integrals:

$$\tilde{E}f(x,t)=(2\pi)^{-n}\iint \exp[i(\frac{t^2}{2}\xi_1+\langle x-y,\xi\rangle)]\,e_1(x,t,\xi)f(y)dyd\xi$$

$$\times(2\pi)^{-n}\iint \exp[i(-\frac{t^2}{2}\xi_1+\langle x-y,\xi\rangle)]\,e_2(x,t,\xi)f(y)dyd\xi.$$

Here the amplitudes $e_\varkappa(x,t,\xi)$, $\varkappa=1,2$, belong to the class $S^0_{1/2}((X\times I)\times\Gamma_1)$ which is not covered by the calculus of Hörmander [10]. However, as it was mentioned by Melin and Sjöstrand [16], the spaces $I^m_\varrho(X,\Lambda)$ of Lagrangean distributions can be defined for $\varrho=1/2$. Moreover, the calculus of FIO with real phase functions can be extended to the case when $\varrho=1/2$. (The only difference which appears is that there is no simple definition of the principal symbol). In particular, the results concerning the composition of FIO remain true in this case.

Proposition 3. *For every $f\in\mathcal{D}'_\Gamma(\mathbf{R}^n)\cap\mathcal{E}'(\mathbf{R}^n)$ the operator $\tilde{E}$ satisfies the conditions*

(2) $$P\tilde{E}f\in C^\infty(\mathbf{R}^{n+1}),\quad \gamma_0\tilde{E}f=f+Rf,\quad R\in CL^{-1}(\mathbf{R}^n).$$

Proof. We have

$$\tilde{E}f(x,0)=f(x)+(2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}c(x,\xi)\hat{f}(\xi)d\xi$$

with

$$c(x,\xi)\sim\sum_{j=1}^{\infty}(c_1^{-j}(x,0,\xi)+c_2^{-j}(x,0,\xi)).$$

From the representation

$$\exp[i\frac{t^2}{2}\xi_1]e_1(x,t,\xi)+\exp[-i\frac{t^2}{2}\xi_1]e_2(x,t,\xi)$$

$$=\exp[i\frac{t^2}{2}\xi_1]\{\sum_{j=1}^{N}c_1^{-j}(x,t,\xi)+\sum_{j=0}^{N}\sum_{m=1}^{2N}c^{-j}_{(m)1}(x,t,\xi)$$

$$\times\int_{\Delta(t,m)}\exp[-i\xi_1(\tau_1^2\dots+(-1)^{m-1}\tau_m^2)]d\tau^{(m)}+\tilde{p}_N(x,t,\xi)\}.$$

$$+\exp[-i\frac{t^2}{2}\xi_1]\{\sum_{j=0}^{N}c_2^{-j}(x,t,\xi)+\sum_{j=0}^{N}\sum_{m=1}^{2N}c^{-j}_{(m)2}(x,t,\xi)$$

$$\times\int_{\Delta(t,m)}\exp[i\xi_1(\tau_1^2-\dots+(-1)^{m-1}\tau_m^2)]d\tau^{(m)}+\tilde{q}_N(x,t,\xi)\}$$

with $\tilde{p}_N, \tilde{q}_N \in S^{-N-1}_{1/2,1;1/2}(X\times I\times\Gamma_1)$, N being an arbitrary positive integer, as well as from the transport equations for $c_{\varkappa}^{-j}$, $c_{(m)\varkappa}^{-j}$ we find that the first condition holds.

As we remarked before, the modified operator $E=\tilde{E}(Id+R)^{-1}$ will be a parametrix for the Cauchy problem for P.

Remark. Let $t\geq t_0>0$. Then $\tilde{f}_{(m)\varkappa}\in S^{-\infty}(X\times\tilde{I}\times\Gamma_1)$ as for $|\xi|\geq 4/\delta t_0$ the corresponding terms are negligeable. Here $\tilde{I}$ is a semi-open interval of the form $t_0\leq t<T$. Furthermore, we have

$$f_{(1)\varkappa}\in CS^{-1/2}(X\times\tilde{I}\times\Gamma_1),$$

$$f_{(m)\varkappa}+g_{(m-1)_s-\varkappa}=\xi_1^{-m/2}\sum_{\mu=0}^{[m/2]}\log^{\mu}\xi_1\cdot p_{(m)\mu\varkappa}(x,t,\xi)$$

with $p_{(m)\mu\varkappa}(x,t,\xi)\in CS^0(X\times\tilde{I}\times\Gamma_1)$. After a rearrangement of the symbols we get

$$e_{\varkappa}(x,t,\xi)\sim\sum_{\mu=0}^{\infty}\xi_1^{-\mu}\cdot\log^{\mu}\xi_1(a_{(\mu)\varkappa}(x,t,\xi)+\xi_1^{-1/2}b_{(\mu)\varkappa}(x,t,\xi)),$$

$$a_{(\mu)\varkappa},\ b_{(\mu)\varkappa}\in CS^0(X\times\tilde{I}\times\Gamma_1),\quad \mu=0,1,2,\ldots;\ \varkappa=1,2.$$

4. Proofs of the Propositions 1 and 2. Proof of Proposition 1 For the sake of convenience we shall use the notations $Q^{\pm}_{(0)}=1$, $Q^{\pm}_{(m)}=0$ for $m<0$. Since

$$Q^{\pm}_{(m)}(t\sqrt{\xi_1})=\int_0^{t\sqrt{\xi_1}}e^{\pm is_1^2}Q^{\mp}_{(m-1)}(s_1)ds_1,$$

we have $D_{\xi_1}Q^{\pm}_{(m)}(t\sqrt{\xi_1})=-i/2e^{\pm it^2\xi_1}Q^{\mp}_{(m-1)}(t\sqrt{\xi_1})$. Suppose we have already proved that

$$D^k_{\xi_1}Q^{\pm}_{(m)}(t\sqrt{\xi_1})$$

$$=e^{\pm it^2\xi_1}\sum_{\nu=1}^{[(k+1)/2]}Q^{\mp}_{(m-2\nu+1)}(t\sqrt{\xi_1})\sum_{\alpha=0}^{k-\nu}c_{k,\alpha}t^{2\alpha+1}\xi_1^{\alpha+1/2-k}$$

$$+\sum_{\nu=1}^{[k/2]}Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1})\sum_{\alpha=1}^{k-\nu}d_{k,\alpha}t^{2\alpha}\xi_1^{\alpha-k}.$$

Then we have

$$D^{k+1}_{\xi_1}Q^{\pm}_{(m)}(t\sqrt{\xi_1})$$

$$=e^{\pm it^2\xi_1}\sum_{\nu=1}^{[(k+1)/2]}Q^{\mp}_{(m-2\nu+1)}(t\sqrt{\xi_1})\sum_{\alpha=1}^{k-\nu}c_{k,\alpha}\{\pm t^{2\alpha+3}\xi_1^{\alpha+1/2-k}$$

$$-i(\alpha+1/2-k)t^{2\alpha+1}\xi_1^{\alpha+1/2-k-1}\}$$

$$+\sum_{\nu=1}^{[(k+1)/2]}Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1})\sum_{\alpha=0}^{k-\nu}(-ic_{k,\alpha})/2t^{2\alpha+2}\xi_1^{\alpha+1/2-k-1/2}$$

$$+\sum_{\nu=1}^{[k/2]} Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1}) \sum_{\alpha=1}^{k-\nu} id_{k,\alpha}(k-\alpha)t^{2\alpha}\xi_1^{\alpha-k-1}$$

$$+e^{\pm it^2\xi_1}\sum_{\nu=1}^{[k/2]} Q^{\mp}_{(m-2\nu-1)}(t\sqrt{\xi_1}) \sum_{\alpha=1}^{k-\nu} (-id_{k,\alpha})/2\,.\,t^{2\alpha+1}\xi_1^{\alpha-1/2-k}$$

$$=e^{\pm it^2\xi_1}\{\sum_{\nu=1}^{[(k-1)/2]} Q^{\mp}_{(m-2\nu+1)}(t\sqrt{\xi_1})[\pm \sum_{\alpha=1}^{k+1-\nu} c_{k,\alpha-1}t^{2\alpha+1}\xi_1^{\alpha+1/2-(k+1)}$$

$$-i\sum_{\alpha=0}^{k-\nu} c_{k,\alpha}(\alpha+1/2-k)t^{2\alpha+1}\xi^{\alpha+1/2-(k+1)}]$$

$$+\sum_{\nu=2}^{[(k+2)/2]} Q^{\mp}_{(m-2\nu+1)}(t\sqrt{\xi_1}) \sum_{\alpha=1}^{k+1-\nu}(-id_{k,\alpha})/2\,.\,t^{2\alpha+1}\xi_1^{\alpha+1/2-(k+1)}\}$$

$$+\sum_{\nu=1}^{[(k+1)/2]} Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1}) \sum_{\alpha=1}^{k+1-\nu} (-ic_{k,\alpha-1})/2\,.\,t^{2\alpha}\xi_1^{\alpha-(k+1)}$$

$$-i\sum_{\nu=1}^{[k/2]} Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1}) \sum_{\alpha=1}^{k-\nu} d_{k,\alpha}(\alpha-k)t^{2\alpha}\xi_1^{\alpha-(k+1)}$$

which is a sum of the desired type with $k+1$ instead of k.

Next we shall prove that

$$D_t^l D_{\xi_1}^k Q^{\pm}_{(m)}(t\sqrt{\xi_1})$$

(5) $$=e^{\pm it^2\xi_1}\sum_{\nu=1}^{[(k+l+1)/2]} Q^{\mp}_{(m-2\nu+1)}(t\sqrt{\xi_1}) \sum_{\alpha\geqq 0} c_{k,l,\alpha}t^{\alpha}\xi_1^{(l+\alpha)/2-k}$$

$$+\sum_{\nu=1}^{[(k+l)/2]} Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1}) \sum_{\alpha\geqq 0} d_{k,l,\alpha}t^{\alpha}\xi_1^{(l+\alpha)/2-k}.$$

The sums over α are finite, of course, we are not interested in the exact way α changes. The equality (5) is proved for $l=0$. Suppose that (5) holds for some l. Then we obtain

$$D_t^{l+1}D_{\xi_1}^k Q^{\pm}_{(m)}(t\sqrt{\xi_1})$$

$$=e^{\pm it^2\xi_1}\sum_{\nu=1}^{[(k+l+1)/2]} Q^{\mp}_{(m-2\nu+1)}(t\sqrt{\xi_1})[\pm \sum_{\alpha\geqq 0} 2c_{k,l,\alpha}t^{\alpha+1}\xi_1^{(\alpha+l)/2-k+1}$$

$$-i\sum_{\alpha\geqq 1} \alpha c_{k,l,\alpha}t^{\alpha-1}\xi^{(\alpha+l)/2-k}]$$

$$-i\sum_{\nu=1}^{[(k+l+1)/2]} Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1}) \sum_{\alpha\geqq 0} c_{k,l,\alpha}t^{\alpha}\xi_1^{(\alpha+l+1)/2-k}$$

$$-i\sum_{\nu=1}^{[(k+l)/2]} Q^{\pm}_{(m-2\nu)}(t\sqrt{\xi_1}) \sum_{\alpha\geqq 1} \alpha d_{k,l,\alpha}t^{\alpha-1}\xi_1^{(\alpha+l)/2-k}$$

$$-ie^{\pm it^2\xi_1}\sum_{\nu=1}^{[(k+l)/2]} Q^{\mp}_{(m-2\nu-1)}(t\sqrt{\xi_1}) \sum_{\alpha\geqq 0} d_{k,l,\alpha}t^{\alpha}\xi_1^{(\alpha+l+1)/2-k}$$

which is a sum of the desired type. Thus (5) is proved for each positive integer k and each nonnegative integer l. Moreover, (5) can easily be proved in the case $k=0$ as well.

Thus we have

$$D_t^l D_{\xi_1}^k Q_{(m)}^{\pm}(t\sqrt{\xi_1})$$
$$=\xi_1^{-k+l/2}\{e^{\pm i(t\sqrt{\xi_1})^2}\sum_{\nu=1}^{[(k+l+1)/2]} Q_{(m-2\nu+1)}^{\mp}(t\sqrt{\xi_1})\sum_{\alpha\geqq 0} c_{k,l,\alpha}(t\sqrt{\xi_1})^{\alpha}$$
$$+\sum_{\nu=1}^{[(k+l)/2]} Q_{(m-2\nu)}^{\pm}(t\sqrt{\xi_1})\sum_{\alpha\geqq 0} d_{k,l,\alpha}(t\sqrt{\xi_2})^{\alpha}\}=\xi_1^{-k+l/2}f_{l,k,\varkappa}(t\sqrt{\xi_1}),$$

where, obviously, $f_{l,k,\varkappa}(\lambda)$ are continuous functions of the variable λ. Denoting $C_{r,l,k,\varkappa}=\sup_{|\lambda|\leqq r}|f_{l,k,\varkappa}(\lambda)|$, we obtain the inequality

$$\sup_{|t|\leqq r\xi_1^{-1/2}}|D_t^l D_{\xi_1}^k Q_{(m)\varkappa}(t\sqrt{\xi_1})|\leqq C_{r,l,k,\varkappa}\xi_1^{-k+l/2}$$

and the proof of Proposition 1 is complete.

Proof of Proposition 2. We shall carry out the proof by induction. Suppose, for the sake of definiteness, that λ is positive. In order to obtain the desired asymptotic expansions we integrate by parts in the interval $[\lambda, \infty)$. If the integrals under consideration are not convergent when $\lambda\to\infty$, we integrate by parts in the interval $[1, \lambda]$ until we obtain absolutely convergent integrals.

First remark the well-known fact about the integrals of Fresnel $\int_0^\infty e^{\pm is^2}ds=\sqrt{\pi/8}(1\pm i)$. Write

$$Q_{(1)}^{\pm}(\lambda)=\int_0^\lambda e^{\pm is^2}ds=\int_0^\infty e^{\pm is^2}dt-\int_\lambda^\infty e^{\pm is^2}ds.$$

Put

$$p_{(1)}^{\pm}(\lambda)=\int_0^\infty e^{\pm is^2}ds=\sqrt{\pi/8}(1\pm i),\quad q_{(1)}^{\pm}(\lambda)=-e^{\mp i\lambda^2}\int_\lambda^\infty e^{\pm is^2}ds.$$

In order to obtain the asymptotic expansion of $q_{(1)}^{\pm}(\lambda)$, we use an integration by parts:

$$q_{(1)}^{\pm}(\lambda)=\mp\frac{e^{\mp i\lambda^2}}{2i}\int_\lambda^\infty\frac{de^{\pm is^2}}{s}=\pm\frac{1}{2i\lambda}\mp\frac{e^{\mp i\lambda^2}}{2i}\int_\lambda^\infty\frac{e^{\pm is^2}}{s^2}ds$$
$$=\pm\frac{1}{2i\lambda}-\frac{e^{\mp i\lambda^2}}{(2i)^2}\int_\lambda^\infty\frac{de^{\pm is^2}}{s^3}=\pm\frac{1}{2i\lambda}+\frac{1}{(2i)^2\lambda^3}-\frac{3e^{\mp i\lambda^2}}{(2i)^2}\int_\lambda^\infty\frac{e^{\pm is^2}}{s^4}ds,$$

etc. Thus

$$q_{(1)}^{\pm}(\lambda)\sim\sum_{\nu=1}^{\infty}c_{(1);0\nu}^{\pm}\lambda^{-2\nu+1}$$

with $c_{(1);01}^{\pm}=\pm 1/2i$, $c_{(1);0\nu}^{\pm}=(\pm 1/2i)^{\nu}(2\nu-1)!!$ when $\nu\geqq 2$ and Proposition 2 is proved for $m=1$.

Next we have

$$Q^{\pm}_{(2)}(\lambda)=\int_0^{\lambda} e^{\pm is_1^2}Q^{\mp}_{(1)}(s_1)ds_1=Q^{\pm}_{(2)}(1)+\int_1^{\lambda} e^{\pm is_1^2}Q^{\mp}_{(1)}(s_1)ds_1.$$

We integrate by parts in the last integral and obtain

$$\int_1^{\lambda} e^{\pm is_1^2}Q^{\mp}_{(1)}(s_1)ds_1=\pm\frac{1}{2i}\int_1^{\lambda}\frac{Q^{\mp}_{(1)}(s_1)}{s_1}\,de^{\pm is_1^2}$$

$$=\pm\frac{1}{2i}\,\frac{e^{\pm i\lambda^2}}{\lambda}\,Q^{\mp}_{(1)}(\lambda)\mp\frac{1}{2i}\,e^{\pm i}Q^{\mp}_{(1)}(1)\pm\frac{1}{2i}\int_1^{\lambda}\frac{e^{\pm is_1^2}}{s_1^2}\,Q^{\mp}_{(1)}(s_1)ds_1\mp\frac{1}{2i}\int_1^{\lambda}\frac{ds_1}{s_1}.$$

In the following we consider separately each of the four terms given above. We start with

$$\pm\frac{e^{\pm i\lambda^2}}{2i\lambda}\,Q^{\mp}_{(1)}(\lambda)=\pm\frac{e^{\pm i\lambda^2}}{2i\lambda}\,p^{\mp}_{(1)}(\lambda)\pm\frac{1}{2i\lambda}\,q^{\mp}_{(1)}(\lambda)\sim e^{\pm i\lambda^2}\sqrt{\frac{\pi}{2}}\cdot\frac{1\pm i}{4\lambda}\pm\frac{1}{2i}\sum_{\nu=1}^{\infty}c^{\mp}_{(1);0_\nu}\lambda^{-2\nu}.$$

Next $\frac{1}{2i}\,e^{\pm i}Q^{\mp}_{(1)}(1)=\text{const.}$, $\int_1^{\lambda}\frac{ds_1}{s_1}=\log\lambda.$

On the other hand, the integral $\int_1^{\infty}(e^{\pm is_1^2}/s_1^2)Q^{\mp}_{(1)}(s_1)ds_1$ is absolutely convergent, that is why the remaining term can be written down as

$$\int_1^{\lambda}\frac{e^{\pm is_1^2}}{s_1^2}\,Q^{\mp}_{(1)}(s_1)ds_1=C-\int_{\lambda}^{\infty}\frac{e^{\pm is_1^2}}{s_1^2}\,Q^{\mp}_{(1)}(s_1)ds_1.$$

Our main interest is to study the asymptotics when $\lambda\to+\infty$, and now we have $s_1\geqq\lambda$. Therefore, we can replace $Q^{\mp}_{(1)}(s_1)$ by its asymptotic expansion and then integrate each term. Thus we obtain

$$\int_{\lambda}^{\infty}\frac{e^{\pm is_1^2}}{s_1^2}\,Q^{\mp}_{(1)}(s_1)ds_1\sim a^{\mp}_{(1)}\int_{\lambda}^{\infty}\frac{e^{\pm is_1^2}}{s_1^2}\,ds_1+\sum_{\nu=1}^{\infty}c^{\mp}_{(1);0_\nu}\int_{\lambda}^{\infty}\frac{ds_1}{s_1^{2\nu+1}}$$

$$\sim a^{\mp}_{(1)}\int_{\lambda}^{\infty}\frac{e^{\pm is_1^2}}{s_1^2}\,ds_1+\sum_{\nu=1}^{\infty}\frac{c^{\mp}_{(1);0\nu}}{2\nu}\,\lambda^{-2\nu}$$

and

$$\int_{\lambda}^{\infty}\frac{e^{\pm is_1^2}}{s_1^2}\,ds_1\sim\mp e^{\pm i\lambda^2}2i\sum_{\nu=2}^{\infty}c^{\mp}_{(1);0_\nu}\lambda^{-2\nu+1}.$$

Hence Proposition 2 is proved for $m=2$.

Suppose that the desired asymptotic expansions for $Q^{\pm}_{(l)}(\lambda)$, $l\leqq m-1$, hold. We have

$$Q^{\pm}_{(m)}(\lambda)=\int_0^{\lambda} e^{\pm is_1^2}\,Q^{\mp}_{(m-1)}(s_1)ds_1=Q^{\pm}_{(l)}(1)+\int_1^{\lambda} e^{\pm is_1^2}Q^{\mp}_{(m-1)}(s_1)ds_1.$$

In the last integral we integrate by parts:

$$\int_1^\lambda e^{\pm is_1^2} Q^{\mp}_{(m-1)}(s_1)ds_1 = \pm\frac{1}{2i}\int_1^\lambda \frac{Q^{\mp}_{(m-1)}(s_1)}{s_1}\, de^{\pm is_1^2}$$

$$= \pm\frac{e^{\pm i\lambda^2}}{2i\lambda} Q^{\mp}_{(m-1)}(\lambda) \mp \frac{1}{2i} e^{\pm i} Q^{\mp}_{(m-1)}(1) \pm \int_1^\lambda \frac{e^{\pm is_1^2}}{s_1^2} Q^{\mp}_{(m-1)}(s_1)ds_1 \mp \int_1^\lambda \frac{Q^{\pm}_{(m-2)}(s_2)}{s_2}\, ds_2.$$

According to our assumption $|Q^{\mp}_{(m-1)}(s_1)| \leq C(1+\log^{[(m-1)/2]} s_1)$, hence $\int_1^\infty e^{\pm is_1^2} s_1^{-2} Q^{\mp}_{(m+1)}(s_1)ds_1$ is absolutely convergent and

$$\int_1^\lambda \frac{e^{\pm is_1^2}}{s_1^2} Q^{\mp}_{(m-1)}(s_1)ds_1 = C - \int_\lambda^\infty \frac{e^{\pm is_1^2}}{s_1^2} Q^{\mp}_{(m-1)}(s_1)ds_1.$$

Now consider

$$\int_1^\lambda \frac{Q^{\pm}_{(m-2)}(s_2)ds_2}{s_2} = \int_1^\lambda Q^{\pm}_{(m-2)}(s_2)d\log s_2 = \log\lambda\,.\, Q^{\pm}_{(m-2)}(\lambda) - \int_1^\lambda e^{\pm is_3^2}\log s_3 Q^{\mp}_{(m-3)}(s_3)ds_3.$$

In the last integral we again integrate by parts:

$$\int^\lambda e^{\pm is_3^2}\log s_3\,.\,Q^{\mp}_{(m-3)}(s_3)ds_3 = \mp i/2\int_1^\lambda \log s_3/s_3\,.\,Q^{\mp}_{(m-3)}(s_3)de^{\pm is_3^2}$$

$$= \mp i/2\,.\,\log\lambda\,.\,\lambda^{-1}Q^{\mp}_{(m-3)}(\lambda)e^{\pm i\lambda^2} \pm i/2\int_1^\lambda (1-\log s_3)s_3^{-2}Q^{\mp}_{(m-3)}(s_3)e^{\pm is_3^2}ds_3$$
$$\pm i\int_1^\lambda Q^{\pm}_{(m-4)}(s_4)d\log^2 s_4.$$

Then, as before, we write down

$$\int_1^\lambda (1-\log s_3)s_3^{-2}e^{\pm is_3^2}Q^{\mp}_{(m-3)}(s_3)ds_3 = C - \int_\lambda^\infty (1-\log s_3)s_3^{-2}e^{\pm is_3^2}Q^{\mp}_{(m-3)}(s_3)ds_3,$$

etc. Suppose that $m=2k-1$. Then

$$Q^{\pm}_{(2k-1)}(\lambda)$$
$$\sim C + c_1\log\lambda\,.\,Q^{\pm}_{(2k-3)}(\lambda) + c_2\log^2\lambda\,.\,Q^{\pm}_{(2k-3)}(\lambda) + \dots + c_{k-1}\log^{k-1}\lambda\,.\,Q^{\pm}_{(1)}(\lambda)$$
$$+ e^{\pm i\lambda^2}\lambda^{-1}(d_0 Q^{\mp}_{(2k-2)}(\lambda) + d_1\log\lambda\,.\,Q^{\mp}_{(2k-4)}(\lambda) + \dots + d_{k-1}\log^{k-1}\lambda)$$
$$+ b_0\int_1^\infty e^{\pm is_1^2}s_1^{-2}Q^{\mp}_{(2k-2)}(s_1)ds_1 + b_1\int_\lambda^\infty (1-\log s_3)s_3^{-2}e^{\pm is_3^2}Q^{\mp}_{(2k-4)}(s_3)ds_3$$
$$+ \dots + b_{k-1}\int_\lambda^\infty \{(k-1)\log^{k-2}s_{2k-1} - \log^{k-1}s_{2k-1}\}\, s_{2k-1}^{-2}e^{\pm is_{2k-1}^2}ds_{2k-1}.$$

From the inductive assumption it follows easily that

$$C + c_1\log\lambda\,.\,Q^{\pm}_{(2k-3)}(\lambda) + \dots + c_{k-1}\log^{k-1}\lambda\,.\,Q^{\pm}_{(1)}(\lambda)$$
$$+ e^{\pm i\lambda^2}\lambda^{-1}(d_0 Q^{\mp}_{(2k-2)}(\lambda) + d_1\log\lambda\,.\,Q^{\mp}_{(2k-4)}(\lambda) + \dots + d_{k-1}\log^{k-1}(\lambda)$$

$$\sim a \log^{k-1}\lambda + \sum_{\mu=0}^{k-2} \log^{\mu}\lambda \sum_{\nu=0}^{\infty} b_{\mu\nu}\lambda^{-\nu 2} + e^{\pm i\lambda^2} \sum_{\mu=0}^{k-1} \log^{\mu}\lambda \sum_{\nu=1}^{\infty} c_{\mu\nu}\lambda^{-2\nu+1}$$

and

$$b_0 \int_{\lambda}^{\infty} e^{\pm i s_1^2} s_1^{-2} Q^{\mp}_{(2k-2)}(s_1) ds_1 + \dots$$

$$+ b_{k-1} \int_{\lambda}^{\infty} \{(k-1)\log^{k-2} s_{2k-1} - \log^{k-1} s_{2k-1}\} s_{2k-1}^{-2} e^{\pm i s_{2k-1}^2} ds_{2k-1}$$

$$\sim \sum_{\mu=0}^{k-2} \sum_{\nu=1}^{\infty} a_{\mu\nu} \int_{\lambda}^{\infty} \frac{\ln^{\mu} s}{s^{2\nu+1}} ds + \sum_{\mu=0}^{k-1} \beta_{\mu} \int_{\lambda}^{\infty} \frac{\ln^{\mu} s}{s^2} e^{\pm i s^2} ds + \sum_{\mu=0}^{k-2} \sum_{\nu=2}^{\infty} \int_{\lambda}^{\infty} \frac{\ln^{\mu} s}{s^{2\nu}} e^{\pm i s^2} ds.$$

Integrating by parts we have

$$\sum_{\mu=0}^{k-2} \sum_{\nu=1}^{\infty} a_{\mu\nu} \int_{\lambda}^{\infty} \frac{\ln^{\mu} s}{s^{2\nu+1}} ds \sim \sum_{\mu=0}^{k-2} \sum_{\nu=1}^{\infty} \tilde{a}_{\mu\nu} \frac{\ln^{\mu}\lambda}{\lambda^{2\nu}}$$

and

$$\sum_{\mu=0}^{k-1} \beta_{\mu} \int_{\lambda}^{\infty} \frac{\ln^{\mu} s}{s^2} e^{\pm i s^2} ds + \sum_{\mu=0}^{k-2} \sum_{\nu=2}^{\infty} \gamma_{\mu\nu} \int_{\lambda}^{\infty} \frac{\ln^{\mu} s}{s^{2\nu}} e^{\pm i s^2} ds \sim e^{\pm i\lambda^2} \sum_{\mu=0}^{k-1} \sum_{\nu=1}^{\infty} \tilde{\gamma}_{\mu\nu} \frac{\ln^{\mu}\lambda}{\lambda^{2\nu+1}}.$$

Thus Proposition 2 is proved for $m=2k-1$. A similar argument works in the case $m=2k$. The details are left to the reader.

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