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CONTINUITY AND NON-MULTIVALUEDNESS PROPERTIES OF METRIC PROJECTIONS AND ANTIPROJECTIONS

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Single-valuedness and continuity properties of some classes of multivalued metric projections and antiprojections in strictly convex Banach spaces are concerned. A more general topological result about upper semicontinuous multivalued mappings is applied for the establishment of analogs of Stechkin's projections results for antiprojections. A different approach based on Gâteaux differentiability of metric antiprojection functions yields that any antiprojection in a strictly convex and weakly differentiable space is non-multivalued over a second Baire category subset of the space.

Let M be a subset of the metric space (X, d). The multivalued mapping $P: X \to M$, which maps every point $x \in X$ into the set $Px = \{y \in M : d(x, y) = \inf_{z \in M} d(x, z)\}$ is called a metric projection. A Lipschitzian function is associated in a natural way with every projection mapping: $p(x) = \inf_{z \in M} d(x, z)$. Analogously, any bounded subset $M \subset X$ generates a metric antiprojection $Q: X \to M$ according to the formula: $Qx = \{y \in M : d(x, y) = \sup_{z \in M} d(x, z)\}$. The corresponding function $q(x) = \sup_{z \in M} d(x, z)$ is Lipschitzian too and, if an addition X is normed, then q as a supremum of convex functions is also convex.

In 1963 Stechkin [16] showed (under certain conditions) that in a strictly convex Banach space X the metric projection $P: X \to M$ has one-point or empty images for the „majority" of the points of X. The "majority" here should be understood as a set of points B such that its complement $X \setminus B$ is a countable union of sets A_i, $i = 1, 2, \ldots$, whose closures $\bar{A}_i$ have empty interiors: $\operatorname{int} \bar{A}_i = \emptyset$. Sometimes, sets like B are called residual in X, and according to the classical Baire theorem they are dense in the space. Stechkin indicated two different ways of proving results of this type. In the first case he based his argument on special properties of M (e. g. bounded compactness of M) and in the second case he took the special shape of the unit ball of X (e. g. locally uniform convexity) as a starting point. Later Kenderov [5-7] used certain continuity properties of the multivalued mappings as approach to the same kind of problems but his way was appropriate only for Stechkin's first type of results.

In the paper we show that both types of results as well as analogous results for antiprojections can be deduced from a general result (Theorem 1.1) concerning points of single-valuedness of a given multivalued mapping. Without any additional conditions this theorem enables us to get more information about the points of continuity of metric projections and antiprojections.

In the second part of the paper only metric antiprojections are considered. The main idea here is contained in a proposition, which states that if the convex function $q(x)=\sup_{z\in M}\|x-z\|$ is Gâteaux differentiable at some point $x_0\in X$, then the set Qx_0 has not more than one element. This makes it possible to formulate the problem into another way: Is the convex function q Gâteaux differentiable at the points of a residual subset of X? If it is, then the metric antiprojection is single-valued at the points of some residual subset of X.

The simplest way to ensure that q is Gâteaux differentiable at the points of a residual subset of X is to suppose that every convex function defined in X is Gâteaux differentiable on a residual subset of its domain of continuity. Banach spaces satisfying this condition were studied by Asplund [2] who gave them the name Weakly Differentiable Spaces (WDS). Asplund proves that every weakly compactly generated (in particular every separable or reflexive) Banach space is WDS. Therefore, at least when M is a bounded subset of the strictly convex separable or reflexive Banach space, we can assert that $Q: X\to M$ has one-point or empty images for the "majority" of points of X (Theorem 2.1) Related problems have been discussed in the papers of Konyagin [8], [9] and Zajiček [17] in the projections case and also Panda and Kapoor [14] in the antiprojections case.

This paper carries out the proofs of results announced in [18].

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1. Let X and Y be two topological spaces. The multivalued mapping $F: X\to Y$ is upper-semicontinuous (u. s. c.) at $x_0\in X$ iff for every neighbourhood $W\supset Fx_0$, there exists a neighbourhood $V\ni x_0$, such that $Fz\subset W$ whenever $z\in V$. F is u. s. c. iff it is u. s. c. at any $x\in X$. It is possible that $Fx=\emptyset$ for some $x\in X$. If it is the case, then the definition of u. s. c. implies that the set of all such points is open. If F is single-valued at x, i. e. $Fx=\{y\}$ and if F is u. s. c. at x we call F continuous at x. The following notations will be used in the sequel:

$$D(F)=\{x\in X:\ Fx=\emptyset\};$$

$S(F)=\{x\in X: Fx=\{y\}\}$, the set of points of single-valuedness of F,

$C(F)=\{x\in X: Fx=\{y\}\ \&\ F$ is u. s. c. at $x\}$, i. e. these are the points of continuity of F;

$ES(F)=\{x\in X: Fx=\emptyset$ or $Fx=\{y\}\}$, i. e. the points of non-multivaluedness of F;

$$U(F)=\{x\in X: (F \text{ is u. s. c. at } x)\ \&\ (Fx=\emptyset \text{ or } Fx=\{y\})\};$$

$$EC(F)=\{x\in X: Fx=\emptyset \text{ or } (F \text{ is u. s. c. at } x\ \&\ Fx=\{y\})\}.$$

Let $F: X\to Y$ and $F^*: X\to Y$ be two multivalued mappings. F^* is said to be an extension of F iff $Fx\subset F^*x$, $\forall x\in X$, $F|_{C(F)}\equiv F^*|_{C(F)}$ and $C(F)\subset C(F^*)$.

Theorem 1.1 *Let $F: X\to Y$ be a multivalued mapping from the topological space (X,τ) into the metric space (Y,d).*

i. *If $D(F)$ is a $\mathfrak{G}_\delta$ subset of X, then $C(F)$ and $U(F)$ are also $\mathfrak{G}_\delta$ subsets of X.*

ii. *If $C(F)$ is dense in $D(F)$ i. e. $\overline{C(F)}\supset D(F)$, then $ES(F)$ and $EC(F)$ are residual in X.*

iii. *If every closed subset of* X *is* $\mathfrak{G}_\delta$ *in* X *and if* (Y, d) *is a complete metric space, then there exists an extention* F^* *of* F *such that* $C(F^*)$ *and* $U(F^*)$ *are* $\mathfrak{G}_\delta$ *sets.*

Before giving the proof we note that in the particular case when F is a single-valued mapping, iii. is an assertion of Lavrentiev-theorem-type concerning extention of a continuous map to a $\mathfrak{G}_\delta$ subset containing its domain (see Kuratowski [10], vol. 1, p. 432]). Our method of proof is an adaptation of Kuratowski's one.

Proof. i. Let $V \in \tau(X)$. Denote $FV = \cup\{Fz : z \in V\}$ and $r(x) = \inf\{\operatorname{diam} FV : V \in \tau(x)\}$. Consider the open sets $U_n = \cup\{V \in \tau(X) : \operatorname{diam} FV < n^{-1}\} = \{x \in X : r(x) < n^{-1}\}$ and the set $U = \cap_{n=1}^{\infty} U_n = \{x \in X : r(x) = 0\}$. The latter is a $\mathfrak{G}_\delta$ set and so is $D(F) \cap U$. It is a routine matter to verify that $C(F) = D(F) \cap U$ and that $U(F)$ is $\mathfrak{G}_\delta$ whenever $C(F)$ is so.

iii. Consider now the map $F^*x = \cap\{\overline{FV} : V \in \tau(x)\}$. It is clear that $Fx \subset F^*x$, $\forall x \in X$ and $F/_{C(F)} = F^*/_{C(F)}$. Let $x \in \overline{D(F)} \cap U$. There exists a sequence of neighbourhoods of x (V_n) such that $V_i \supset V_j$, $i \leq j$; $\operatorname{diam} FV_n < n^{-1}$ and $V_n \cap D(F) \neq \emptyset$, $n = 1, 2, \ldots$ According to Cantor theorem $\cap_{n=1}^{\infty} \overline{FV_n} = \{y\}$ and obviously $F^*x = \{y\}$. For any $\varepsilon > 0$ there is $V \in \tau(x)$, $\operatorname{diam} FV < \varepsilon$, and for $z \in V$ then $V \in \tau(z)$ and since $y \in \overline{FV}$, then $F^*z \subset \overline{FV} \subset B(y, \varepsilon)$. Thus F^* is continuous at $x \in \overline{D(F)} \cap U$. If F^* is continuous at x then $r^*(x) = \inf\{\operatorname{diam} F^*V : V \in \tau(x)\} = 0$, but $r(x) \leq r^*(x)$ because $Fx \subset F^*x$ and therefore $x \in \overline{D(F)} \cap U$. Since X has the property that every closed subset is $\mathfrak{G}_\delta$ then $\overline{D(F)} \cap U = C(F^*)$ and $U(F^*)$ are $\mathfrak{G}_\delta$ sets too.

To prove ii apply i to the mapping $F : D(F) \to Y$, the topology in $D(F)$ being induced by $\tau(X)$. Having in mind the fact that F is u. s. c. at $x \in D(F)$ iff F is u. s. c. at x relative to the induced topology we see that $C(F)$ is $\mathfrak{G}_\delta$ and dense in $D(F)$. Then $D(F) \setminus C(F) = \cup_{n=1}^{\infty} T_n$ are nowhere dense in $D(F)$ and so they are in $\tau(X)$. The theorem is proved.

Corollary 1.2. *Suppose* $F : X \to Y$ *is a multivalued mapping with non-empty images from the topological space* X *into the metric space* Y, *which is single-valued and u. s. c. at the points of some dense subset of* X. *Then* F *is single-valued and u. s. c. at the points of a dense* $\mathfrak{G}_\delta$ *subset of* X.

Proof. Immediate from Theorem 1.1 i.

In order to underline the analogy between metric projections and antiprojections (as far as these kinds of problems are concerned) and for the sake of brevity, we formulate and prove both the sorts of results simultaneously.

Theorem 1.3 *Let* X *be a complete metric space,* $M \subset X$ *be a closed (respectively bounded and closed) subset. Then the sets* $C(P)$, $U(P)$ *(resp.* $C(Q)$, $U(Q)$ *are* $\mathfrak{G}_\delta$ *sets. If* X *is a Banach space and* $M \cap \operatorname{bd}\overline{M}$ *is closed then* $C(P)$ *and* $U(P)$ *are* $\mathfrak{G}_\delta$ *in* $X \setminus \overline{M}$ *(Respectively if* X *is a Banach space and* M *is bounded and* $M \cap \operatorname{bd}\overline{M}$ *is closed then* $C(Q)$ *and* $U(Q)$ *are* $\mathfrak{G}_\delta$ *sets in* X).

The proof is based on the argument that under the above conditions every projection (resp. antiprojection) coincide with its extension constructed in Theorem 1.1 iii., i. e. $Px = P^*x = \cap\{\overline{PV} : V \in \tau(x)\}$, $\forall x \in X$ (resp. $Qx = Q^*x = \cap\{\overline{QV} : V \in \tau(x)\}$, $\forall x \in X$).

Indeed let $y \in P^*x = \bigcap_{n=1}^{\infty} \overline{PB(x, n^{-1})}$ (resp. $y \in Q^*x = \bigcap_{n=1}^{\infty} \overline{QB(x, n^{-1})}$). For any positive integer $B(y, n^{-1}) \cap PB(x, n^{-1}) \neq \emptyset$ (resp. $B(y, n^{-1}) \cap QB(x, n^{-1}) \neq \emptyset$), which implies the existence of sequences (x_n) and (y_n) such that $y_n \in Px_n$ (resp. $y_n \in Qx_n$), $d(x, x_n) < n^{-1}$ and $d(y, y_n) < n^{-1}$, $n = 1, 2, \ldots$ Therefore $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) < 2n^{-1} + p(x_n)$. After transition at infinity $d(x, y) \leq px$), $\lim_{n\to\infty} y_n = y \in M$. If X is normed and $x \notin \overline{M}$, then also $y \in \operatorname{bd} \overline{M}$ (For antiprojections the inequality $q(x_n) = d(x_n, y_n) < 2n^{-1} + d(x, y)$ holds). The proof is completed.

The space X is strictly convex if $\|x+y\| = \|x\| + \|y\|$ implies $\|x\| y = \|y\| x$ whenever $x, y \in X$.

Theorem 1.4. *Let M be a subset (resp. a bounded subset) of the strictly convex Banach space X and let the metric projection $P: X \to M$ (resp. the metric antiprojection $Q: X \to M$) be u. s. c. at every point $x \in X$ where the image of P (resp. Q) is a singleton. Then $EC(P)$ (resp. $EC(Q)$) is a residual subset of X. Moreover, if M is closed or $M \cap \operatorname{bd} \overline{M}$ is closed then $U(P)$ (resp. $U(Q)$) is $\mathfrak{G}_\delta$ dense, where in the latter case $U(P)$ is $\mathfrak{G}_\delta$ dense in $X \setminus \overline{M}$.*

Proof. If $y_0 \in Px_0$ (resp. $y_0 \in Qx_0$), it is not difficult to see because of the strict convexity of X, as Stechkin indicated for the projection case, that at every point x from the set $\{x_t = (1-t)x_0 + ty_0 : 0 < t \leq 1\}$ (resp. $\{x_t = x_0 + t(x_0 - y_0) : t > 0\}$) the metric projection P (resp. the antiprojection Q) is single-valued and according to our assumption continuous. Let $t \to 0$ then x_t approaches x_0 arbitrarily close. It remains to apply Theorem 1.1 ii. in order to get the first part of the theorem. For the second one we observe that $U(P)$ (resp. $U(Q)$) is dense in X and apply Theorem 1.3.

A set $M \subset X$ is approximatively compact [4] (resp. Λ-compact [3]) if every minimizing (resp. every maximizing) sequence has a subsequence convergent in M.

Corollary 1.5. *Let M be an approximatively compact (resp. Λ-compact) subset of the strictly convex space X. Then $C(P)$ (resp. $C(Q)$) is dense $\mathfrak{G}_\delta$ set.*

Proof. According to a result of Singer [15] (resp. Blatter [3]) the projection mapping $P: X \to M$ (resp. the antiprojection $Q: X \to M$) is upper-semicontinuous. Since M is proximinal, i. e. $Px \neq \emptyset$ for every $x \in X$, and hence closed; and since $U(P) = C(P)$ the proof is an application of Theorem 1.4 (If M is Λ-compact it is not necessarily closed but $D(Q) = X$ and as follows from Theorem 1.4 $C(Q)$ is dense. Apply Theorem 1.1 i.).

Remarks 1.6. The metric projection part of Corollary 1.5 belongs to Kenderov [5]. If M is boundedly compact, i. e. the intersection of M with every closed ball is a compact set, it is approximatively compact and the result of Stechkin [16], that in this situation $P: X \to M$ is single-valued on a $\mathfrak{G}_\delta$ dense subset is a corollary of Theorem 1.4.

Let now the space X be locally uniformly convex, i. e. for every sequence $(x_n)_{n=0}^{\infty}$ in X with $\|x_n\| = 1$, $n = 0, 1, 2, \ldots$, and $\lim_{n\to\infty} \|x_0 + x_n\| = 2$ it follows $\lim_{n\to\infty} x_n = x_0$ (def. i.). A slightly modified version of the above definition is the following one: For every sequence $(x_n)_{n=0}^{\infty}$ with $\lim_{n\to\infty} \|x_0 + x_n\| = \lim_{n\to\infty} \|x_n\| + \|x_0\|$ it follows $\|x_0\| \lim_{n\to\infty} x_n = (\lim_{n\to\infty} \|x_n\|) x_0$ (def. ii.).

To establish the equivalence between these two definitions. Only the implication (def. i.) $\Rightarrow$ (def. ii.) has to be proved.

Consider the sequence $(\|(\|x_0\|^{-1})x_0+(\|x_n\|^{-1})x_n\|)_{n=1}^{\infty}$. If every subsequence of it has a convergent to 2 sub-subsequence then the sequence itself converges to 2. The proof is from the contrary and for convenience the indexation is not changed. Then for some $\delta>0$ the inequality $\|(\|x_0\|^{-1})x_0+(\|x_n\|^{-1})x_n\|\leq 2-\delta$ holds. Assume additionally $\|x_0\|>\lim_{n\to\infty}\|x_n\|$ (Later the other possibilities will be discussed).

Hence

$$(1)\qquad (\|x_n\|^{-1}\|x_0+x_n\|\leq\|(\|x_0\|^{-1})x_0+(\|x_n\|^{-1})x_n\|+(\|x_n\|^{-1}-\|x_0\|^{-1})\|x_0\|$$
$$\leq 1-\delta+(\|x_n\|^{-1})\|x_0\|=\|x_n\|^{-1}.(\|x_0\|+\|x_n\|)-\delta.$$

Thus

$$(2)\qquad \|x_0+x_n\|\leq\|x_0\|+\|x_n\|-\delta\|x_n\|,$$

which contradicts (def. ii.) assertion.

If $\|x_0\|<\lim_{n\to\infty}\|x_n\|$, one replaces everywhere in (1) x_0 with x_n and derives an inequality similar to (2) which contradicts the assumption again.

The case $\|x_0\|=\lim_{n\to\infty}\|x_n\|$ is trivial.

It only remains to take into account (def. i) assertion in order to get (def. ii.).

Lemma 1.7. *Let M be a subset (resp. a bounded subset) of the locally uniformly convex Banach space X and let $y_0\in Px_0$ (resp. $y_0\in Qx_0$). Then at every point of the set $\{x_t=(1-t)x_0+ty_0:0<t\leq 1\}$ (resp. $\{x_t=x_0+t(x_0-y_0):t>0\}$ the metric projection $P:X\to M$ (resp. the antiprojection $Q:X\to M$) is both single-valued and upper-semicontinuous.*

Even a stronger conclusion is true, i. e. every minimizing for x_t sequence (resp. every maximizing for x_t sequence) converges in norm topology to y_0, thus an approximative compactness (resp. Λ-compactness) at x_t is obtained and after Singer [15] (resp. Blatter [3]) this means that P (resp. Q) is u. s. c. at x_t.

Proof of Lemma 1.7. Considerations are for projections only, since the antiprojection part of the proof goes over the same pattern.

Let (y_n) be a minimizing for x_t sequence, i. e. $\lim_{n\to\infty}\|x_t-y_n\|=p(x_t)$. Then (y_n) is minimizing for x_0 too. Indeed

$$(3)\qquad \|x_0-y_0\|\leq\|x_0-y_n\|\leq\|x_0-x_t\|+\|x_t-y_n\|.$$

Having in mind that $Px_t=\{y_0\}$ from (3) after transition at infinity we yield:

$$(4)\qquad \lim_{n\to\infty}\|x_0-y_n\|=\lim_{n\to\infty}\|x_t-y_n\|+\|x_0-x_t\|.$$

Combine (4) and (def. ii.)

$$(5)\qquad \lim_{n\to\infty}(x_t-y_n)/\lim_{n\to\infty}\|x_t-y_n\|=(x_0-x_t)/\|x_0-x_t\|.$$

Assume for simplicity $x_0=0$, then $x_t=ty_0$ and therefore from (5) $[ty_0-\lim_{n\to\infty}y_n]/(1-t)\|y_0\|=-y_0/\|y_0\|$, that means $\lim_{n\to\infty}y_n=y_0$. The proof is completed.

Theorem 1.8. *Let M be a subset (resp. a bounded subset) of the locally uniformly convex Banach space X. Then the set $EC(P)$ (resp. $EC(Q)$)*

is residual in X. If in addition M is closed then $U(P)$ (resp. $U(Q)$ is $\mathfrak{G}_\delta$ dense in X and if $M \cap \operatorname{bd} \overline{M}$ is closed then $U(P)$ is $\mathfrak{G}_\delta$ dense in $X \setminus \overline{M}$ (resp. $U(Q)$ is $\mathfrak{G}_\delta$ dense in X).

Proof. Having in mind Lemma 1.7 it is more or less line in line the same as the proof of Theorem 1.4.

Remark 1.9 The metric projection part of this theorem belongs to Stechkin [16].

Recently Lau [12] has proved that any projection generated by a closed subset M of a reflexive locally uniformly convex Banach space X is single-valued on a residual part of X. In [1] Asplund has proved that in analogous situation the antiprojection $Q: X \to M$ has non-void images for the points from some $\mathfrak{G}_\delta$ dense subset of X. Their results can be presized involving continuity properties of P (resp. Q). Thus the following nice conclusion is true:

Theorem 1.10 (Lau, Asplund). *Let $P: X \to M$ be a projection mapping (resp. $Q: X \to M$ be an antiprojection mapping) generated by the closed (resp. bounded and closed) subset M of the reflexive locally uniformly convex Banach space X. Then the set of points $x \in X$ at which simultaneously*

i. *Px (resp. Qx) is a singleton,*

ii. *P (resp. Q) is upper-semicontinuous at x is a $\mathfrak{G}_\delta$ and dense subset of X, i. e. $C(P)$ (resp. $C(Q)$) is $\mathfrak{G}_\delta$ dense.*

The proof is a combination of Lau's [12] (resp. Asplund's [1]) result with Lemma 1.7 and Theorem 1.3.

2. Let X be a strictly convex normed space. Denote by $n(t)$ the restriction of the very norm function $\|.\|$ upon some straight line $L=\{x_t=x_0+th : t \in \mathbf{R}, \|h\|=1\}$ not passing through θ. Since n is convex, for each couple $t_1, t_2 \in \mathbf{R}$, $t_1<t_2$, the inequalities hold:

$$dn(t_1)/dt_+ \leq (n(t_2)-n(t_1))/(t_2-t_1) \leq dn(t_2)/dt_- . \tag{6}$$

From $\theta \notin L$ and strict triangle inequality it follows

$$-1<(n(t_2)-n(t_1))/(t_2-t_1)<1. \tag{7}$$

Then (6) and (7) imply

$$-1<dn(t_0)/dt_- \leq dn(t_0)/dt_+<1, \quad \text{for every} \quad t_0 \in \mathbf{R}. \tag{8}$$

That simple fact enables us to give a sufficient condition for a projection (resp. antiprojection) to be non-multivalued at $x \in X$ if there exists $h \in X$, $\|h\|=1$ such that $\overline{\lim}_{t\to 0_+}(p(x+th)-p(x))/t=1$ (resp. $\lim_{t\to 0_+}(q(x+th)-q(x))/t=-1$). Indeed if $\theta \in Px$ (resp. $\theta \in Qx$) which is no loss of generality,

$$\overline{\lim_{t\to 0_+}} \frac{p(x+th)-p(x)}{t} \leq \lim_{t\to 0_+} \frac{\|x+th\|-\|x\|}{t}=1 \tag{9}$$

but (9) compared with (8) requires x and h to be colinear and then $Px \subset \{x-p(x)h\}$, so an impossible task is to find two different points that belong to Px. In the antiprojection situation

$$\lim_{t\to 0_+} q(x+th)-q(x))/t \geq \lim_{t\to 0_+} (\|x+th\|-\|x\|)/t, \quad \text{which}$$

means that $\lim s \to 0_-(\|x+s(-h)\|-\|x\|)/s=1$ and then $Qx \subset \{x+q(x)h\}$.

Thus if p (resp. q) is Gâteaux differentiable at some point $x \in X$ one cannot expect more than single-valuedness of P (resp. Q) at x.

This method proved to be reliable when dealing with antiprojections. There is a large class of spaces, namely Weakly Differentiable spaces for which every convex function is Gâteaux differentiable at the points of a residual part of the space (Asplund [2]). In particular every separable [13] and every reflexive [2] Banach space is WDS.

Theorem 2.1 *Suppose the strictly convex Banach space X is WDS and M is a bounded subset of it. Then the set $ES(Q)$, where $Q: X \to M$ is the antiprojection generated by M, is residual in X.*

Definition 2.2. (Rockafellar). *Let X be a normed space and $\varphi: X \to \mathrm{R}$ be a convex function. The set $\partial\varphi(x) = \{x^* \in X^* : \langle x^*, y-x\rangle \leq \varphi(y) - \varphi(x),\ \forall y \in X\}$ is called the subdifferential of φ at x.*

Theorem 2.3. *Let X be a strictly convex Banach space and M weakly compact. Then the antiprojection $Q: X \to M$ is single-valued on a residual subset of X, i.e. the set $S(Q)$ is residual.*

Proof. The existance part of the proof belongs to Lau [11] and even it does not depend on strict convexity assumptions. He considered the set

$$D = \{x \in X : \sup_{z \in M} \langle x^*, x-z\rangle = q(x), \quad \forall x^* \in \partial q(x)\}$$

and proved that it is $\mathfrak{G}_\delta$ dense. We shall demonstrate that in the strictly convex case $D \subset S(Q)$. Suppose the contrary. For some $x_0 \in D$ and $z_1, z_2 \in Qx_0$ it is true that $\|x_0 - z_i\| = q(x_0)$, $i=1, 2$. There exist x_1^* and $x_2^* \in X$ such that $\langle x_i^*, x_0 - z_i\rangle = \|x_0 - z_i\|$, $\|x_i^*\| = 1$, $i=1, 2$. But $x_i^* \in \partial q(x_0)$, $i=1, 2$. Indeed, for arbitrary $z' \in M$ and $y \in X$

$$\langle x_i^*, y-x_0\rangle = \langle x_i^*, y-z'\rangle - \langle x_i^*, x_0 - z'\rangle$$

$$\leq \sup_{z \in M} \sup_{\|x^*\| \leq 1} \langle x^*, y-z\rangle - \langle x_i^*, x_0 - z'\rangle = q(y) - \langle x_i^*, x_0 - z'\rangle, \quad i=1, 2.$$

Hence $\sup_{z \in M} \langle x_i^*, x_0 - z\rangle \leq q(y) - \langle x_i^*, y - x_0\rangle$ and then $q(x_0) - q(y) = \langle x_i^*, x_0 - y\rangle$, $\forall y \in X$, $i=1, 2$. Denote $x^* = x_1^*/2 + x_2^*/2$. As $\partial q(x_0)$ is a convex set $x^* \in \partial q(x_0)$. Since $x_0 \in D$ there exists a sequene $(z_n)_{n=3}^{\infty}$, such that $q(x_0) = \lim_{n\to\infty} \langle x^*, x_0 - z_n\rangle = \langle x^*, x_0 - z_0\rangle$ for some $z_0 \in M$ and $z_n \xrightarrow{w} z_0$. The last equalities also imply that $\langle x_i^*, x_0 - z_0\rangle = q(x_0)$, $i=1, 2$ and therefore x_i^* attains its supremum on the closed ball $\overline{B(x_0, q(x_0))}$ at z_0 and z_i. Having in mind the strict convexity of the norm we conclude: $z_1 = z_0 = z_2$. The theorem is proved.

The analogous result for projections is also true and it belongs to Kenderov [7].

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