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THE $\vec{G}/\vec{M}/r/$SIRO MACHINE INTERFERENCE MODEL WITH STATE-DEPENDENT SPEEDS

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This article treats a version of the multiple machine interference problem with r operatives under Service In Random Order (SIRO) repair discipline. The running times of the machine i are supposed to be identically and arbitrarily distributed random variables with a density function $f_i(x)$. The repair times of the machine i have negative exponential distribution with mean $1/\mu_i$, $i=1, \ldots, n$. The method of integro-differential equations is applied to obtain the main steady-state characteristics of the system through allowing state-dependent speeds.

Introduction. One of the oldest industrial engineering problems due to the phenomenon known as machine interference has been considered by a number of authors. They have used a variety of approaches and have made different assumptions about the statistical distribution of running and repair times. For references on the basic finite-source models, one may see Jaiswal [1], Kopocinska [2], Stecke and Aronson [3] or Takács [4]. The major problem when considering different types of machines is the necessity to trace where each individual machine is in the system. The machine interference problem with different types of machines has been discussed, for example, by Chandra [5], Elsayed and Norton [6], Gross and Ince [7], Kameda [8], Sztrik [9-12].

Recently the finite-source queueing models have been effectively used, for example, for the mathematical description of computer systems, cf. Bunday and Khorram [13], Gelenbe and Mitrani [14], Kleinrock [15], Lauchard and Latouche [16], Takagi [17-18].

This article treats a version of the machine interference problem which can be formulated as follows. Let us consider a set of n ($n \geq 2$) heterogeneous machines which are serviced by one of r ($r \leq n$) operatives. The machines work continuously and independently. However, at any time each of them may break down and may need service. The required running times are supposed to be random variables having an arbitrary distribution function $F_i(x)$ with a density function $f_i(x)$ for machine i, $i=1, \ldots, n$. If a machine breaks down it will be serviced immediately unless all the repairmen are busy. If so a waiting line is formed. The repairs are carried out in random order (SIRO), i. e., whenever a stop or a repair completion occur each machine has the same probability of being selected for service. The required repair times of the machine i are supposed to be exponentially distributed with mean $1/\mu_i$, $i=1, \ldots, n$. Furthermore, we assume that in any time interval when k machines are in working order the passed running time of each operating machine increases with speed $a(k)$, and the remaining repair time of each machine under service decreases with speed $b(k)$, $a(k)>0$, $k=1, \ldots, n$, $b(k)>0$, $k=0, \ldots, n-1$. The running and repair times are assumed to be independent of each other.

This paper extends further the work of Bunday and Khorram [13] where the speeds are equal to one. However, it is not difficult to find situations in practice where the speeds are state-dependent. For example, let us consider a factory where the working machines require a large amount of energy (or power). Clearly, the greater the number of running machines is, the greater the energy they need. Furthermore, let us suppose that the greater the number of failed machines is, the greater the intensity of repair will be.

The aim of the present paper is to provide the main steady-state operational characteristics of the system, such as the utilization of machines, the operative efficiency, the mean number of busy repairmen, the average number of running machines, and the expected "down" time of machines. It is shown that the steady-state distribution of the number of operating machines, and hence the other important practical measures depend on the running time distribution only through the means of these distributions, i. e., the insensitivity property holds.

The mathematical model. For any t $(t \geqq 0)$ define the following random variables.
$\nu(t)$: the number of running machines at time t;
$\alpha_1(t), \ldots, \alpha_{\nu(t)}(t)$: their indices;
$\zeta_{\alpha_1(t)}, \ldots, \zeta_{\alpha_{\nu(t)}(t)}$: the attained operating times in progress at time t, respectively.
It is easy to see that the process

$$X(t) = [\nu(t); \quad \alpha_1(t), \ldots, \alpha_{\nu(t)}(t); \quad \zeta_{\alpha_1(t)}, \ldots, \zeta_{\alpha_{\nu(t)}(t)}] \tag{1}$$

belongs to the class of piecewise-linear Markov processes subject to discontinuous changes. The states of (1) are

$$\{(k;\ i_1, \ldots, i_k;\ x_1, \ldots, x_k), \quad x_i > 0, \quad i = 1, \quad k, \quad (i_1, \ldots, i_k) \in C_n^k\}$$

which means that

$$(\nu(t) = k; \quad \alpha_1(t) = i_1, \ldots, \alpha_k = i_k; \quad \zeta_{i_1} \leqq x_1, \ldots, \zeta_{i_k} \leqq x_k: \quad x_s > 0, \quad s = 1, \ldots, k)$$
$$\text{for} \quad k = 1, \ldots, n,$$

where
C_n^k denotes the collection of all combinations of order k of integers $1, \ldots, n$. Furthermore, denote by $\{0\}$ the state when there is no running machine. To calculate the stationary distribution of (1) consider the following functions.

$$Q_0 = \lim_{t \to \infty} P(\nu(t) = 0), \tag{2}$$

$$Q_{i_1, \ldots, i_k}(x_1, \ldots, x_k) = \lim_{t \to \infty} P(\nu(t) = k; \quad \alpha_s(t) = i_s; \quad \zeta_{i_s} \leqq x_s, \quad s = 1, \ldots, k).$$

Clearly, this steady-state distribution exists and will be unique if

$$\alpha(k+1) > 0,\ b(k) > 0,\ k = 0, \ldots, n-1,\ 0 < \mu_i < \infty,\ 1/\lambda_i = \int_0^\infty x f_i(x)\, dx < \infty, \quad i = 1, \ldots, n,$$

(c.f. Gnedenko and Kowalenko [19] or Schassberger [20]). Furthermore,

$$q_{i_1, \ldots, i_k}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k$$
$$= \lim_{t \to \infty} P(\nu(t) = k; \quad \alpha_s(t) = i_s; \quad x_s \leqq \zeta_{i_s} < x_s + dx_s; \quad s = 1, \ldots, k);$$

(see Gnedenko and Kowalenko [19]).

In order to formulate the following theorem let us introduce the notation

$$q^*_{i_1, \ldots, i_k}(x_1, \ldots, x_k) = q_{i_1, \ldots, i_k}(x_1, \ldots, x_k)/((1 - F_{i_1}(x_1)) \ldots (1 - F_{i_k}(x_k))) \tag{3}$$

assuming that $1 - F_i(x_i) > 0$ for all $x_i > 0$, $i = 1, \ldots, k$, $k = 1, \ldots, n$. This is the so-called normed density function.

Then we have

Theorem 1. *The normed density functions* (3) *satisfy the following system of integro-differential equations* (4.1), (4.2), (5.1), (5.2) *with boundary conditions* (4.3), (5.3)

$$Q_0 \frac{rb(0)}{n} \sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \int_0^\infty a(1)\, q_j^*(y) f_j(y)\, dy, \tag{4.1}$$

$$\left[\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_k}\right]^* q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_k) \tag{4.2}$$

$$= - \sum_{j \neq i_1, \ldots, i_k} b(k) \frac{r}{n-k} \mu_j q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_k)$$

$$+ \sum_{j \neq i_1, \ldots, i_k} \int_0^\infty a(k+1)\, q^*_{i_1,\ldots,i_k,j}(x_1, \ldots, x_k, y) f_j(y)\, dy,$$

$$q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_{k-1}, 0)\, a(k) = b(k-1) \frac{r}{n-k+1} \mu_j q^*_{i_1,\ldots,i_{k-1}}(x_1, \ldots, x_{k-1}), \tag{4.3}$$

$$\text{for } j \neq i_1, \ldots, i_{k-1}, \quad k = 1, \ldots, n-r,$$

$$\left[\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_k}\right]^* q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_k) \tag{5.1}$$

$$= - \sum_{j \neq i_1, \ldots, i_k} b(k) \mu_j q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_k)$$

$$+ \sum_{j \neq i_1, \ldots, i_k} \int_0^\infty a(k+1)\, q^*_{i_1,\ldots,i_k,j}(x_1, \ldots, x_k, y) f_j(y)\, dy,$$

$$\left[\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\right]^* q^*_{i_1,\ldots,i_n}(x_1, \ldots, x_n) = 0, \tag{5.2}$$

$$q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_{k-1}, 0)\, a(k) = b(k-1) \mu_j q^*_{i_1,\ldots,i_{k-1}}(x_1, \ldots, x_{k-1}), \tag{5.3}$$

$$\text{for } j \neq i_1, \ldots, i_{k-1}, \quad k = n-r, \ldots, n.$$

Also they have to satisfy the normalizing condition

$$Q_0 + \sum_{k=1}^{n} \sum_{(i_1,\ldots,i_k)} \int_0^\infty \cdots \int_0^\infty q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_k) \tag{6}$$

$$\times (1 - F_{i_1}(x_1)) \cdots (1 - F_{i_k}(x_k))\, dx_1 \cdots dx_k = 1.$$

The usual notation for partial differential quotients has been applied to the left-hand side of (4.2), (5.1), (5.2) to denote the limit on the right-hand side. But that is not allowed since the existence of the individual partial differential quotients is not confirmed. That is why the operator is denoted by $[\ ]^*$. Actually, this is a $(a(k), \ldots, a(k)) \in R_k$ directional derivative (see Cohen [21]).

Proof. Since the process (1) is Markovian, its density should satisfy the Chapman — Kolmogorov equation. The derivation is based on studying the sample path of the process during an infinitesimal interval of width h. The following relations hold

$$Q_0 = Q_0 \left[1 - \frac{rb(0)}{n} \sum_{j=1}^{n} \mu_j\right] + \sum_{j=1}^{n} \int_0^\infty a(1)\, q_j(y) f_j(y)/(1 - F_j(y))\, dy + o(h), \tag{7.1}$$

$$q_{i_1,\ldots,i_k}(x_1 + a(k)\, h, \ldots, x_k + a(k)\, h) \tag{7.2}$$

$$=q_{i_1,\ldots,i_k}(x_1,\ldots,x_k)\,(1-\sum_{j\neq i_1,\ldots,i_k} b(k)\,\frac{r}{n-k}\,\mu_j h)\prod_{s=1}^{k}\frac{(1-F_{i_s}(x_s+a(k)h))}{(1-F_{i_s}(x_s))}$$

$$+\sum_{j\neq i_1,\ldots,i_k}\int_0^\infty a(k+1)\,q_{i_1,\ldots,i_k,j}(x_1,\ldots,x_k,y)\,f_j(y)/(1-F_j(y))\,dy$$

$$\times\prod_{s=1}^{k}\frac{(1-F_{i_s}(x_s+a(k)h))}{(1-F_{i_s}(x_s))}+o(h),$$

(7.3) $$q_{i_1,\ldots,i_{k-1}}(x_1+a(k)h,\ldots,x_{k-1}+a(k)h,\;0)\,a(k)h$$

$$=b(k-1)\,\frac{r}{n-k+1}\,\mu_j h q_{i_1,\ldots,i_{k-1}}(x_1,\ldots,x_{k-1})\prod_{s=1}^{k-1}\frac{(1-F_{i_s}(x_s+a(k-1)h))}{(1-F_{i_s}(x_s))}+o(h),$$

$$\text{for } j\neq i_1,\ldots,i_{k-1},\quad k=1,\ldots,n-r.$$

Similarly,

(8.1) $$q_{i_1,\ldots,i_k}(x_1+a(k)h,\ldots,x_k+a(k)h)$$

$$q_{i_1,\ldots,i_k}(x_1,\ldots,x_k)\,(1-\sum_{j\neq i_1,\ldots,i_k} b(k)\mu_j h)\prod_{s=1}^{k}\frac{(1-F_{i_s}(x_s+a(k)h))}{(1-F_{i_s}(x_s))}$$

$$+\sum_{j\neq i_1,\ldots,i_k}\int_0^\infty a(k+1)\,q_{i_1,\ldots,i_k,j}(x_1,\ldots,x_k,y)\,f_j(y)/(1-F_j(y))\,dy$$

$$\times\prod_{s=1}^{k}\frac{(1-F_{i_s}(x_s+a(k)h))}{(1-F_{i_s}(x_s))}+o(h),$$

(8.2) $$q_{1,\ldots,n}(x_1+a(n)h,\ldots,x_n+a(n)h)$$

$$=\prod_{s=1}^{n}\frac{(1-F_{i_s}(x_s+a(k)h)}{(1-F_{i_s}(x_s)))}\,q_{1,\ldots,n}(x_1,\ldots,x_n)+o(h),$$

(8.3) $$q_{i_1,\ldots,i_{k-1}}(x_1+a(k)h,\ldots,x_{k-1}+a(k)h,\;0)\,a(k)h$$

$$=b(h-1)\mu_j h q_{i_1,\ldots,i_{k-1}}(x_1,\ldots,x_{k-1})\prod_{s=1}^{k-1}\frac{(1-F_{i_s}(x_s+a(k-1)h))}{(1-F_{i_s}(x_s))}+o(h),$$

$$\text{for } j\neq i_1,\ldots,i_{k-1},\quad k=n-r,\ldots,n.$$

Hence the derivation of equations (4.1), (4.2), (5.1), (5.2) and boundary conditions (4.3), (5.3) is quite simple. Indeed, dividing both sides by factor $\prod_{s=1}^{k}(1-F_{i_s}(x_s+a(k)h))$, taking into account (3) and passing to the limit as $h\to 0$, we get the desired result. Next we solve the equations subject to the boundary conditions. If we set

$$Q_0=\frac{n!}{r!\,r^{n-r}}\;\frac{a(1)\ldots a(n)}{b(0)\ldots b(n-1)}\;\frac{1}{\prod_{j=1}^{n}\mu_j}\,c,$$

$$q^*_{i_1,\ldots,i_k}(x_1,\ldots,x_k)=\frac{(n-k)!}{r!\,r^{n-r-k}}\;\frac{a(k+1)\ldots a(n+k)}{b(k)\ldots b(n+k-1)}\;\frac{1}{\prod_{s\neq i_1,\ldots,i_k}\mu_s}\,c,$$

$$\text{for}\quad k=1,\ldots,n-r,$$

$$q^*_{i_1,\ldots,i_k}(x_1, \ldots, x_k) = \frac{a(k+1)\ldots a(n+k)}{b(k)\ldots b(n+k-1)} \frac{1}{\prod\limits_{s\neq i_1,\ldots,i_k} \mu_s} c,$$

$$\text{for} \quad k=n-r, \ldots, n$$

where $b(m)=a(m+1)=1$, if $m \geq n$, then they satisfy the corresponding equations and boundary conditions.

Let us introduce the following probabilities

$$Q(i_1, \ldots, i_k) = \lim_{x_i \to \infty,\ i=1,\ldots,k} Q_{i_1,\ldots,i_k}(x_1, \ldots, x_k)$$

$$\text{for } (i_1, \ldots, i_k) \in C_n^k, \quad k=1, \ldots, n,$$

$$\widehat{Q}_k = \lim_{t\to\infty} P(\nu(t)=k), \quad \text{for } k=0, \ldots, n.$$

Then it can easily be verified that we have

$$Q(i_1, \ldots, i_k) = \frac{(n-k)!}{r!\, r^{n-r-k}} \frac{a(k+1)\ldots a(n+k)}{b(k)\ldots b(n+k-1)} \frac{1}{\prod\limits_{s\neq i_1,\ldots,i_k} \mu_s} \frac{1}{\prod\limits_{j=1}^{k} \lambda_{i_j}} c,$$

$$\text{for} \quad k=1, \ldots, n-r,$$

$$Q(i_1, \ldots, i_k) = \frac{a(k+1)\ldots a(n+k)}{b(k)\ldots b(n+k-1)} \frac{1}{\prod\limits_{s\neq i_1,\ldots,i_k} \mu_s} \frac{1}{\prod\limits_{j=1}^{k} \lambda_{i_j}} c,$$

$$\text{for } k=n-r, \ldots, n.$$

Since

$$\widehat{Q}_n = Q(1, \ldots, n), \quad c=\lambda_1 \ldots \lambda_n \widehat{Q}_n,$$

$$\widehat{Q}_0 = Q_0, \quad \widehat{Q}_k = \sum_{(i_1,\ldots,i_k)\in C_n^k} Q(i_1, \ldots, i_k),$$

the whole system depends on $\widehat{Q}_n$ which can be obtained by means of the normalizing condition (6).

Comments. 1. In the case $a(k+1)=b(k)=1$, $k=0, \ldots, n-1$, i. e. when the speeds are independent of the number of running machines, we obtain

$$Q(i_1, \ldots, i_k) = \frac{(n-k)!}{r!\, r^{n-r-k}} \frac{1}{\prod\limits_{s\neq i_1,\ldots,i_k} \mu_s} \frac{1}{\prod\limits_{j=1}^{k} \lambda_{i_j}} c,$$

$$\text{for} \quad k=0, \ldots, n-r,$$

$$Q(i_1, \ldots, i_k) = \frac{1}{\prod\limits_{s\neq i_1,\ldots,i_k} \mu_s} \frac{1}{\prod\limits_{j=1}^{k} \lambda_{i_j}} c,$$

$$\text{for} \quad k=n-r, \ldots, n,$$

which coincides with the result of Bunday and Khorram [13]. By definition $\left(\prod\limits_{s=1}^{0} \cdot = 1\right)$

2. In case of homogeneous service, i. e. $\mu_i=\mu$, $i=1, \ldots, n$, we get

$$Q(i_1, \ldots, i_k)=\frac{(n-k)!}{r!\, r^{n-r-k}} \frac{a(k+1)\ldots a(n+k)}{b(k)\ldots b(n+k-1)} \frac{1}{\mu^{n-k}} \frac{1}{\prod_{j=1}^{k} \lambda_{i_j}} c,$$

for $k=1, \ldots, n-r$,

$$Q(i_1, \ldots, i_k)=\frac{a(k+1)\ldots a(n+k)}{b(k)\ldots b(n+k-1)} \frac{1}{\mu^{n-k}} \frac{1}{\prod_{j=1}^{k} \lambda_{i_j}} c,$$

for $k=n-r, \ldots, n$,

which agrees with the result derived by Sztrik [11] under FIFO service rule. Furthermore, it should be noted that the same formulas are valid when the breakdowns do not stop the repair and the random selection occurs only at the service completions (see Sztrik [12]).

Since the main steady-state characteristics of the system, such as the machine utilization, the average number of running machines, the operative efficiency, the expected "down" time of machines are calculated in the same way as in Bunday and Scraton [22], Bunday and Khorram [13], Sztrik [9-12], their derivation is omitted.

Concluding remarks. The method of integro-differential equations is applied to get the stationary distribution of the underlying Markov process. It should be noted, however, that the same result can be obtained by progressive methods of queueing theory as represented in Franken et al. [23], Gnedenko and König [24], Schassberger [25]. Sometimes, however, it is more difficult to derive results for a specific system from the general theory than to treat a given problem only, since we have to get acquainted with the notations, the whole development, etc.

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