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COMMUTATIVITY RESULTS FOR ONE-SIDED s-UNITAL AND SEMI-PRIME RINGS

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ABSTRACT. Let $n \neq 1$ be a fixed non-negative integer. In this paper we prove that: If R is a left or right s-unital ring satisfying the polynomial identity $[xy-(yx)^n, y] = 0$ for all $x, y \in R$, then R is commutative. Another commutativity result is obtained for right s-unital ring satisfying the polynomial identity $[xy-(xy)^n, y] = 0$ for all $x, y \in R$. Moreover, we establish the commutativity of semi-prime ring satisfying $xy \pm (yx)^n \in Z(R)$ or $xy \pm (xy)^n \in Z(R)$

1. Introduction. In [6, Theorem 11], Jacobson proved the following interesting theorem

Theorem J. *Let R be a ring and let $n(x) > 1$ be an integer for every $x \in R$. If $x^{n(x)} = x$, then R is necessarily commutative.*

Let $Z(R)$ be the center of any ring R. In [3, Theorem 18], Herstein has generalized the above mentioned result. In fact, Herstein proved the following:

Theorem H_1. *Let $n > 1$ be a fixed positive integer. If R is a ring in which*

$$x^n - x \in Z(R) \quad \text{for all} \quad x \in R,$$

then R is commutative.

For any $x \in R$, $p_x(t)$ will denote a polynomial in the indeterminate t with rational integral coefficients, where we further suppose that these coefficients are functions of x. In an attempt to extend the above results, Herstein [4] proved

Theorem H_2. *Let R be any ring, and let $x \in R$. If there exists a polynomial $p_x(t)$ such that*

$$x - x^2 p_x(t) \in Z(R),$$

then R is commutative.

Later, it was proved [8] that if for every $x, y \in R$,

$$xy - (xy)^2 p(xy) \in Z(R),$$

then $R^2 \subseteq Z(R)$. Recently, Bell et al. [2] studied the commutativity of rings satisfying the following properties:
(P): there exists a polynomial $p(xy)$ such that

$$[xy - (xy)^2 p(xy), x] = 0 \text{ for all } x, y \in R. \tag{1}$$

(Q): there exists a polynomial $q(xy)$ such that

$$[xy - (xy)^2 q(xy), y] = 0 \text{ for all } x, y \in R. \tag{2}$$

In [2, Theorem 2] it has been proved that a ring with unity 1 satisfying the polynomial identity (1) is commutative. On the other hand, one could study the analogous situation for a ring satisfying the polynomial identity (2). For instance, if R satisfies (2), is it true that R is commutative or $R^2 \subseteq Z(R)$. In the present paper, we study the commutativity of a ring R satisfying the polynomial identity

$$[xy - (yx)^n, y] = 0 \text{ for all } x, y \in R, \tag{3}$$

where $n \neq 1$ is a fixed non-negative integer. Also, another analogous result is obtained. Further, we establish the commutativity of semi-prime ring satisfying either $xy \pm (yx)^n \in Z(R)$ or $xy \pm (xy)^n \in Z(R)$ for all $x, y \in R$.

2. Preliminary Results. Throughout the present paper, R will represent an associative ring (may be without unity 1). Let $C(R)$ be the commutator ideal of R, $N(R)$ the set of all nilpotent elements in R, and $N'(R)$ the set of all zero-divisors in R. Let $GF(q)$ denote the Galois field with q elements and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$.

Definition 1. *A ring R is called a left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. Further, R is called s-unital if it is both left as well as right s-unital, that is $x \in Rx \cap xR$ for all $x \in R$.*

Definition 2. *If R is s-unital (resp. left or right s-unital) ring, then for any subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e is called the pseudo (resp. pseudo left or pseudo right) identity of F in R.*

The following results are pertinent for developing the proofs of our results.

Lemma 1 ([7, Page 221]). *Let $x, y \in R$. If $[[x, y], y] = 0$, then*

$$[x, y^m] = my^{m-1}[x, y]$$

for any positive integer m.

Lemma 2 ([1, Theorem]). *Let* f *be a polynomial in* n *non-commuting indeterminates* $x_1, x_2, \ldots, x_n$ *with relatively prime integral coefficients. Then the following statements are equivalent:*

(1) *For every prime* p, $(GF(p))_2$ *fails to satisfy* $f = 0$.

(2) *For any ring satisfying the polynomial identity* $f = 0$, $C(R)$ *is a nil ideal.*

(3) *Every semi-prime ring satisfying* $f = 0$ *is commutative.*

Lemma 3 ([2, Lemma 1]). (*i*) : *If* R *is a ring such that each element has a power lying in* $Z(R)$, *then there is no distinction between left and right zero-divisors in* R. *Thus* $N'(R)R \subseteq N'(R)$ *and* $RN'(R) \subseteq N'(R)$.

(*ii*) : *If* R *is a subdirectly irreducible ring with heart* H, *then every central zero-divisor of* R *annihilates* H.

Lemma 4 ([9,Lemma]). *Let* R *be a left (resp. right) s-unital ring. If for each pair of elements* x *and* y *in* R *there exists a positive integer* $m = m(x, y)$ *and an element* $e = e(x, y) \in R$ *such that* $x^m e = x^m$ *and* $y^m e = y^m$ *(resp.* $ex^m = x^m$ *and* $ey^m = y^m$*), then* R *is an s-unital ring.*

3. Commutativity of one-sided s-unital rings. Now, we present a commutativity theorem for left or right s-unital rings.

Theorem 1. *Let* $n \neq 1$ *be a fixed non-negative integer. If* R *is a left or right s-unital ring satisfying the polynomial identity* (3), *then* R *is commutative.*

The following lemma shows that the ring considered in the above theorem is in fact an s-unital ring. According to Proposition 1 of [5], this enables us to reduce the proof of Theorem 1 to a ring with unity 1.

Lemma 5. *Let* m *be a fixed non-negative integer. If* R *is a left or right s-unital ring satisfying the polynomial identity* (3), *then* R *is an s-unital ring.*

Proof. Let R be a left s-unital ring and let x and y be arbitrary elements in R. Then there exists an element $e = e(x, y) \in R$ such that $ex = x$ and $ey = y$. In (3), replace x by e to get $[ey, y] = [(ye)^n, y] = 0$. Thus $(ye)^n y = y(ye)^n$. Therefore, $y^{n+1} = y^{n+1}e$. Similarly, $x^{n+1} = x^{n+1}e$. Hence, by Lemma 4, R is an s-unital ring.

Next, if R is a right s-unital, then for any $x, y \in R$, there exists an element $f = f(x, y) \in R$ such that $xf = x$ and $yf = y$. Replace x by f in the polynomial identity (3) to get $[fy - (yf)^n, y] = 0$. Therefore, $y^2 = fy^2$. Similarly, $x^2 = fx^2$. Hence, by Lemma 4, R is an s-unital ring. □

Lemma 6. *Let* n *be a fixed non-negative integer and let* R *be a ring satisfying the polynomial identity* (3). *Then* $C(R) \subseteq N(R)$.

Proof. Let $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be elements in $(GR(p))_2$ for a prime p. Then x and y fail to satisfy the polynomial identity (3). So by Lemma 2, $C(R) \subseteq N(R)$. □

Lemma 7. *Let $n \neq 1$ be a fixed non-negative integer and let R be a ring with unity 1 satisfying the polynomial identity* (3). *Then*

$$C(R) \subseteq N(R) \subseteq Z(R).$$

Proof. Let $u \in R$ be an invertible element. Replace y by u and x by $u^{-1}y$ in (3) to get

$$[u^{-1}yu, u] = [(uu^{-1}y)^n, u] \quad \text{for all} \quad y \in R$$

and hence

$$[u^{-1}yu, u] = [y^n, u] \quad \text{for all} \quad y \in R.$$

Therefore,

$$(*) \qquad [u, y]u = u[u, y^n] \quad \text{for all} \quad y \in R.$$

Thus for any positive integer t the above polynomial identity gives

$$[u, y]u^t = u[u, y^n]u^{t-1} = u^2[u, y^{n^2}]u^{t-2} = \cdots$$

Repeating the above process, we finally get

$$[u, y]u^t = u^t[u, y^{n^t}] \quad \text{for all} \quad y \in R.$$

Now, let $a \in N(R)$. Then the last polynomial identity gives

$$[u, a]u^t = u^t[u, a^{n^t}].$$

Indeed, we have $[u, a]u^t = 0$, for sufficiently large t, and any unit element u. Therefore, $[u, a] = 0$. By Lemma 6, we have $C(R) \subseteq N(R)$. Thus $[u, y] \in N(R)$, and hence $(*)$ gives $u[u, y] = [u, y]u = u[u, y^n]$ for all $y \in R$. Since u is a unit element, we have

$$(5) \qquad [u, y] = [u, y^n] \quad \text{for all} \quad y \in R.$$

Now, there exists an integer $r \geq 1$ such that

$$(6) \qquad a^q \in Z(R) \quad \text{for all} \quad q \geq r \quad \text{and} \quad r \quad \text{minimal}.$$

If $r = 1$, then $a \in Z(R)$. Let $r > 1$. Then (5) and (6) imply that

$$(7) \qquad [a^q, y^n] = [1 + a^q, y^n] = [1 + a^q, y] = [a^q, y] = 0 \quad \text{for all} \quad y \in R \quad \text{and} \quad q \geq r.$$

By (5) we have

$$[a^q, y^n - y] = 0 \text{ for all } y \in R. \tag{8}$$

Since $r > 1$, then $a^{r-1} + 1$ is an invertible element. So (5) gives

$$[a^{r-1} + 1, y^n] = [a^{r-1} + 1, y] \text{ for all } y \in R,$$

and hence

$$[a^{r-1}, y^n] = [a^{r-1}, y] \text{ for all } y \in R.$$

Thus

$$[a^{r-1}, y^n - y] = 0 \text{ for all } y \in R.$$

By (6) and (8), we have a contradiction to the minimality of r. Therefore, $r = 1$ and thus $[a, y] = 0$ for all $y \in R$ and $a \in N(R)$. So $N(R) \subseteq Z(R)$. By Lemma 5, (4) holds.

Remark. In view of (4), $C(R) \subseteq Z(R)$. Therefore, $[[x, y], y] = 0$ for all $x, y \in R$ and thus the conclusion of Lemma 1 holds. Hence in the Proof of Theorem 1, we shall therefore routinely use Lemma 1 without mentioning it explicitly.

Now, we shall prove Theorem 1.

Proof of Theorem 1. According to Lemma 5, R is an s-unital ring. So, in view of Proposition 1 of [5], it suffices to prove the theorem for R with unity 1.

If $n = 0$, then (3) yields

$$xy^2 = yxy \text{ for all } x, y \in R. \tag{9}$$

Replacing y by $y+1$ in (9) and making repeated use of (9) gives $xy = yx$ for all $x, y \in R$. Thus R is commutative.

Let $n > 1$. By (3) we have

$$[xy, y] = [(yx)^n, y] \text{ for all } x, y \in R. \tag{10}$$

Suppose that $t > 1$ is a positive integer. Replacing x by tx in (10) gives

$$t[xy, y] = t^n[(yx)^n, y] = t^n[xy, y] \text{ for all } x, y \in R.$$

Hence

$$(t^n - t)[xy, y] = 0 = (t^n - t)[x, y]y \text{ for all } x, y \in R.$$

Putting $y + 1$ for y gives

$$(t^n - t)[x, y] = 0 \text{ for all } x, y \in R.$$

Let $m = (t^n - t)$. Then $m > 1$ for $n > 1$. Thus

$$m[x, y] = 0 \text{ for all } x, y \in R. \tag{11}$$

We know that every ring R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in I$, the index set), each of which as a homomorphic image of R inherits the hypothesis placed on R. So we may assume that R itself is subdirectly irreducible satisfying the polynomial identity (3). Hence in view of (4), (11) and Lemma 1, we have

$$[x, y^m] = m[x, y]y^{m-1} = 0 \text{ for all } x, y \in R.$$

Therefore,

$$y^m \in Z(R) \text{ for each } y \in R. \tag{12}$$

By using Lemma 1 and (4), the polynomial identity (10) gives

$$\begin{aligned}
[xy, y] &= [(yx)^n, y] \\
&= n(yx)^{n-1}[yx, y] \\
&= n(yx)^{n-1}y[y, x] \\
&= n(yx)^{n-1}[y, x]y \\
&= n(yx)^{n-1}[xy, y] \\
&= n(yx)^{n-1}[(yx)^n, y] \\
&= n^2(yx)^{2(n-1)}[(yx)^n, y] \\
&= n^3(yx)^{3(n-1)}[(yx)^n, y].
\end{aligned}$$

By repeated use of the above argument, we finally get

$$[xy, y] = n^m(yx)^{m(n-1)}[(yx)^n, y] = n^m(yx)^{m(n-1)}[xy, y].$$

Hence

$$(1 - n^m(yx)^{m(n-1)})[xy, y] = 0 \text{ for all } x, y \in R. \tag{13}$$

Suppose that H ($\neq 0$) is the heart of R and $z \in N'(R)$. Then by (12) and (13) we have

$$n^m(yz)^{m(n-1)} \in Z(R) \text{ for each } y \in R.$$

By Lemma 3 (i)

$$n^m(yz)^{m(n-1)} \in N'(R) \text{ for each } y \in R.$$

Now, if $[yz, y] \neq 0$, then (13) gives $(1-n^m(yz)^{m(n-1)}) \in N'(R)$. Hence by Lemma 3 (*ii*), we get $H = H(1 - n^m(yz)^{m(n-1)} = (0)$ which is a contradiction. So $[yz, y] = 0$ and hence $y[z, y] = 0$. On replacing y by $y+1$ this gives $[z, y] = 0$ for all $y \in R$ and $z \in N'(R)$. Therefore,

$$N'(R) \subseteq Z(R). \tag{14}$$

Now, if $y \in R$, then by (12), $y^m \in Z(R)$ and $y^{mn} \in Z(R)$. Thus

$$\begin{aligned}
[x,y]y^{2m}(y - y^{m(n-1)+1}) &= [x,y]y(y^{2m} - y^{m(n+1)}) \\
&= [xy,y](y^{2m} - y^{m(n+1)}) \\
&= [xy,y]y^{2m} - [xy,y]y^{m(n+1)} \\
&= [xy,y]y^{2m} - [(yx)^n,y]y^{m(n+1)} \\
&= [xy^{m+1},y^{m+1}] - [(y^{m+1}x)^n,y^{m+1}] \\
&= [(xy^{m+1}) - (y^{m+1}x)^n,y^{m+1}].
\end{aligned}$$

Therefore,

$$[x,y]y^{2m}(y - y^{m(n-1)+1}) = 0 \quad \text{for all} \quad x, y \in R. \tag{15}$$

Now, if R is not commutative, then by Theorem H_1, there exists an element $y \in R$ such that $(y - y^{m(n-1)+1}) \notin Z(R)$. Thus $y \notin Z(R)$ and consequently neither $(y - y^{m(n-1)+1})$ nor y is a zero-divisor. So $y^{2m}(y - y^{m(n-1)+1}) \notin N'(R)$. Therefore (15) implies that $[x, y] = 0$ for all $x, y \in R$. Thus we have a contradiction. Hence R is commutative. □

Further, we prove the following result for right s-unital rings.

Theorem 2. *Let $n \neq 1$ be any fixed non-negative integer. If R is a right s-unital ring satisfying the polynomial identity*

$$[xy - (xy)^n, y] = 0 \quad \textit{for all} \quad x, y \in R, \tag{16}$$

then R is commutative.

Lemma 8. *Let $n \neq 1$ be a fixed non-negative integer, and let R be a right s-unital ring satisfying the polynomial identity* (16). *Then R is an s-unital ring.*

Proof. Let x and y be arbitrary elements in R. If R is a right s-unital ring, then there exists an element $f = f(x, y) \in R$ such that $xf = x$ and $yf = y$. If $n = 0$, then (16) gives $x = fx \in Rx$. Thus R is left s-unital.

Now, if $n > 1$, replace y by f in (16) to obtain $[xf-(xf)^n, f] = 0$ for each $x \in R$. Hence $xf - x^n f = fx - fx^n$ and hence $x = fx - fx^n + x^n = (f - fx^{n-1} + x^{n-1})x \in Rx$. Therefore, R is left s-unital. Hence R is s-unital. □

Lemma 9. *Let $n \neq 1$ be a fixed non-negative integer, and let R be a ring satisfying the polynomial identity* (16). *Then $C(R) \subseteq N(R)$.*

Proof. Let $x = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be elements in $(GF(p))_2$ for a prime p. Then x and y fail to satisfy the polynomial identity (16). By Lemma 2, $C(R) \subseteq N(R)$. □

Lemma 10. *Let $n \neq 1$ be a fixed non-negative integer, and let R be a ring with unity 1 satisfying the polynomial identity* (16). *Then*

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{17}$$

Proof. Let $v \in R$ be an invertible element. Replace x by yv^{-1} and y by v in (16) to obtain

$$[yv^{-1}v - (yv^{-1}v)^n, v] = 0 \text{ for all } y \in R.$$

Therefore,

$$[y, v] = [y^n, v] \text{ for all } y \in R.$$

Now, let $b \in N(R)$. By following the argument of the proof of Lemma 7 (see(5)), we see that $[b, y] = 0$ for all $y \in R$. Thus $N(R) \subseteq Z(R)$. By Lemma 9, (17) holds. □

Proof of Theorem 2. By Lemma 8, R is an s-unital ring. In view of Proposition 1 of [5], we prove the theorem for R with unity 1.

Let $n = 0$. Then (16) gives

$$xy^2 = yxy \text{ for all } x, y \in R. \tag{18}$$

Replace y by $y + 1$ in (18), and use (18) to get $xy = yx$ for all $x, y \in R$. Thus R is commutative.

If $n > 1$, then by (16), we have

$$[xy, y] = [(xy)^n, y] \text{ for all } x, y \in R. \tag{19}$$

Let $s > 1$ be a positive integer. Replace x by sx in (19) to get

$$s[xy, y] = s^n[(xy)^n, y] \text{ for all } x, y \in R.$$

Let $k = s^n - s$. Then $k > 1$ for $n > 1$. Following the Proof of Theorem 1 (see(11), (12) and (13)), we obtain

$$k[x, y] = 0 \text{ for all } x, y \in R,$$

and

$$[x, y^k] = k[x, y]y^{k-1} = 0 \text{ for all } x, y \in R.$$

Thus

$$(20) \qquad y^k \in Z(R) \text{ for all } y \in R,$$

and

$$(21) \qquad (1 - n^k(xy)^{k(n-1)})[xy, y] = 0 \text{ for all } x, y \in R.$$

If S ($\neq 0$) is the heart of R and $c \in N'(R)$, then we can prove that (see (14)) $[c, y] = 0$ for all $y \in R$. Therefore,

$$(22) \qquad N'(R) \subseteq Z(R).$$

Now, let $y \in R$. Then by (20), $y^k \in Z(R)$ and $y^{kn} \in Z(R)$. Again following the proof of Theorem 1, yields (see(15))

$$(23) \qquad [xy, y]y^{2k}(y - y^{k(n-1)+1}) = 0 \text{ for all } x, y \in R,$$

and

$$y^{2k}(y - y^{k(n-1)+1}) \not\subseteq N'(R).$$

So (23) implies that $[x, y] = 0$ for all $x, y \in R$. Therefore, R is commutative. □

4. Examples. The following examples demonstrate that the restriction on the hypothesis of Theorem 1 are not superfluous.

Example 1. Let

$$R = \{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} |a, b, c, d \in GF(3)\}.$$

It can be easily checked that the non-commutative ring R satisfies the polynomial identity $[xy - yx, y] = 0$ for all $x, y \in R$. Thus the Theorem is not valid for $n = 1$.

Examples 2. Suppose that D is a division ring. Let $k > 2$ be a positive integer and let

$$A_k = \{(a_{ij})|a_{ij} = 0 \text{ for } i \geq j\}.$$

Then A_3 is non-commutative nilpotent ring of index 3, which satisfies polynomial (3). This shows that we cannot disregard the condition of the unity 1 in the ring.

The following example shows that Theorem 2 is not true for left s-unital rings.

Examples 3. Let

$$R = \{a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, d = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}$$

be a subring of $(GF(2))_2$. It is easy to check that R is a left s-unital ring satisfying the polynomial identity (16) for all non-negative integers $n \neq 1$. Also, R is not a right s-unital. However, R is a non-commutative ring.

5. Commutativity of semi-prime rings.

Now, we present two commutativity theorems for semi-prime rings.

Theorem 3. *Let R be a semi-prime ring and let n be a fixed non-negative integer. If R satisfies*

$$xy \pm (yx)^n \in Z(R), \tag{24}$$

then R is commutative.

Proof. Let $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ be in $(GF(p))_2$ for a prime p. Then (24) gives

$$xy \pm (xy)^n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin Z(R).$$

Therefore, R is commutative by Lemma 2. □

Theorem 4. *Let R be a semi-prime ring and let $n \neq 1$ be a fixed non-negative integer. If R satisfies*

$$xy \pm (xy)^n \in Z(R), \tag{25}$$

then R is commutative.

Proof. If $x = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $(GF(p))_2$ for a prime p, then (25) gives

$$xy \pm (xy)^n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin Z(R).$$

By Lemma 2, R is commutative. □

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