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DECOMPOSITION OF BANACH SPACE INTO A DIRECT SUM OF SEPARABLE AND REFLEXIVE SUBSPACES AND BOREL MAPS*

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Abstract. The main results of the paper are:
Theorem 1. Let a Banach space E be decomposed into a direct sum of separable and reflexive subspaces. Then for every Hausdorff locally convex topological vector space Z and for every linear continuous bijective operator $T : E \to Z$, the inverse T^{-1} is a Borel map.
Theorem 2. Let us assume the continuum hypothesis. If a Banach space E cannot be decomposed into a direct sum of separable and reflexive subspaces, then there exists a normed space Z and a linear continuous bijective operator $T : E \to Z$ such that T^{-1} is not a Borel map.

Introduction. On a topological space X we can naturally define the σ-algebra of Borel sets Φ generated by closed (or open) sets of X. The collection of Borel subsets of a metric space X can be represented as a union of a transfinite sequence of collections: $\Phi = \bigcup\limits_{\alpha<\omega_1} F_\alpha$ (ω_1 is the first uncountable ordinal), where

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1) F_0 is the collection of closed subsets of X. 2) Elements of F_α are intersections (unions) of countable sequences of sets from $\bigcup_{\beta<\alpha} F_\beta$ when α is even (odd). We consider that the limit ordinals are even. Similarly we have $\Phi = \bigcup_{\alpha<\omega_1} G_\alpha$, where: 1) G_0 is the collection of open subsets of X. 2) Elements of G_α are intersections (unions) of countable sequences of sets from $\bigcup_{\beta<\alpha} G_\beta$, when α is odd (even).

Definition 1. *A Borel subset A of a metric space X is said to be of multiplicative class α if it belongs to F_α (G_α) for even (odd) α. It is said to be of additive class α if A belongs to G_α (F_α) for even (odd) α.*

Definition 2. *If X, Y are topological spaces, then a map $T : X \to Y$ is said to be Borel if $T^{-1}(M) \in \Phi$ for every closed subset $M \subset Y$. If X, Y are metric spaces, then a map $T : X \to Y$ is said to be of α Borel class if the set $T^{-1}(M)$ is of multiplicative class α for every closed subset $M \subset X$ (or, which is the same, the set $T^{-1}(U)$ is of additive class α for every open set $U \subset X$).*

Every map of α Baire class (analytically representable map of class α) belongs to α Borel class for finite α and $\alpha + 1$ Borel class for infinite α. In separable Banach spaces the α Baire class coincides with the α Borel class for finite α and with $\alpha + 1$ Borel class for infinite α [1], [8, §31.IX].

The main results of this paper are:

Theorem 1. *Let a Banach space E is decomposed into a direct sum of separable and reflexive subspaces. Then for every Hausdorff locally convex topological vector space Z and for every linear continuous bijective operator $T : E \to Z$, the inverse T^{-1} is a Borel map.*

Theorem 2. *Let us assume the continuum hypothesis. If a Banach space E cannot be decomposed into a direct sum of separable and reflexive subspaces, then there exists a normed space Z and a linear continuous bijective operator $T : E \to Z$ such that T^{-1} is not a Borel map.*

For separable Banach spaces E Theorem 1 is a well known result. If Z is metrizable then it follows immediately from well known Suslin theorem [8, §39.IV]. In the general separable case, Theorem 1 follows from a result of [12]. Let us note that in separable spaces the local convexity condition can be omitted [2]. For a reflexive Banach space E and a normed space Z the inverse map T^{-1} is even of the 1 Baire class; this is a result, well known in the theory of ill-posed problems (V. V. Vasin, V. P. Tanana, V. A. Vinokurov [19]–[21]). By the way, the similar fact is not valid in Frechet spaces. There exists a linear continuous injective operator from a separable reflexive Frechet space onto a normed space such that its inverse is not of the 1 Baire class [9]. For the reflexive spaces Theorem 2 was announced in [21] and proved in [22]. This result of [21, 22] is close to an Edgar's theorem which claims that the σ-algebras, generated by normed and weak topologies, coincide in a locally uniformly rotund Banach space [4, 5]. For a weakly compactly generated (WCG) space, i.e. the space which is the closed linear span of its weakly compact set, Theorem 2 (without supposing the continuum hypothesis) was given in [15]. Also, Theorem 1 was given in [15]. At the end of this paper we note a class of Banach spaces, including WCG-spaces, for which Theorem 2 is valid without the continuum hypothesis.

An important role in the proof of Theorem 2 plays a result on Borel class of inverse to a linear continuous map in separable normed spaces, which has an independent interest also. Before its formulation we recall a definition. Let X be a Banach space and F be some subset of dual space X^*. The set $F_{(1)}$ of all limits of weakly* convergent sequences in F is called the weak* sequential closure of F. By induction for an ordinal α the weak* sequential closure of order α of F is the set $F_{(\alpha)} = \underset{\beta<\alpha}{\cup} (F_{(\beta)})_{(1)}$.

Theorem 3. *Let T be a linear continuous injective operator from a separable Banach space X onto a normed space Y and α be a countable ordinal. The map T^{-1} is of α Borel class if and only if α is the first ordinal for which $F_{(\alpha)} = X^*$, where $F := T^*Y^*$.*

This theorem was proved in [18, Corollary 42, Corollary 45]. Another proof was given in [16] with some errors, it was corrected by the author. Afterwards the proof of [16] was improved by M. I. Ostrovskii. In this paper we give the Ostrovskii's proof with the kind permission of the author. We note one more known result which is used in the proof of Theorem 2. But before its formulation let us recall two definitions.

Definition 3. *Let X be a Banach space. The space X is said to be quasireflexive, if $\dim X^{**}/X < \infty$; if $\dim X^{**}/X = \infty$ then it is called non-quasireflexive. A subset $F \subset X^*$ is called total (on X) if for every $x \in X$, $x \neq 0$ there exists $f \in F$ such that $f(x) \neq 0$.*

Theorem 4 [10]**.** *Let X be a separable non-quasireflexive Banach space.*

Then for every ordinal $\alpha < \omega_1$ there exists a total on X subspace F such that $F_{(\alpha)} \neq X^*$.

By Theorems 3, 4 and [13, p. 190] for any countable ordinal α on any separable non-quasireflexive Banach space there exist linear continuous injective operators whose inverses does not belong to α Borel class. In [11] it was shown that for a large class of function non-quasireflexive spaces (f.e. for $C[0,1]$ and $L_1[0,1]$) such operators one can chose among integral operators with the infinitely differentiable kernel. Theorem 3 also finds an application in the geometry of Banach spaces [6]. We note that from the point of view of ill-posed problems the greatest interest have operators T in Banach spaces whose inverses belong to the 1 Baire class (they are called regularizable). And from the point of view of Banach valued random variables it is interesting, when the identity map from a Banach space E with the weak topology $w(E,F)$, generated by some linear, subspace $F \subset E^*$, onto the space E with the norm topology, is a Borel map (for details see [13]).

Let T be a linear continuous injective operator from a Banach space E onto a Hausdorff locally convex topological linear space Z with a topology $\mathcal{T}$ (a normed space Z with a norm $\| \, \|_Z$). Then we can introduce on E the topology $\tau = T^{-1}(\mathcal{T})$ (the norm $|||x||| = \|Tx\|_Z$), which is weaker than $\| \, \|$. Then T^{-1} is Borel if and only if the identity map $E,\ \tau(||| \, |||) \to E,\ \| \, \|$ is Borel. This fact we will take into account from now on (in particular Theorem 1, 2 and $1'$, $2'$ will be equivalent).

Recall several definitions and notations. Let X be a Banach space. We will denote by $M^\perp$ the annihilator of a subset $M \subset X$ in X^* and by $M^\top$ the annihilator of the subset $M \subset X^*$ in X. Let $[M]$ denote the norm closure of linear span of M and let $\mathrm{cl}^* F$ denote the weak* closure of a subset $F \subset X^*$. A subspace $F \subset X^*$ is called norming if the norm $\|x\|_0 = \sup\{|f(x)| : f \in F, \|f\| \leq 1\}$ is equivalent to the original norm $\| \, \|$ of X. If moreover $\|x\|_0 = \|x\|$ for every $x \in X$, then F is called 1-norming. A norm $\| \, \|$ of a Banach space is said to be locally uniformly rotund if $\|x_n\| = \|x\| = 1$ and $\|x_n + x\| \to 2$ imply $\|x_n - x\| \to 0$ as $n \to \infty$. If it is not said the opposite, by subspace we mean the closed subspace.

We refer to [8] for necessary background in topology, to [17] in locally convex spaces and to [3, 13] in Banach spaces.

Theorem 1′. *Let a Banach space $E, \| \ \|$ be decomposed into a direct sum of separable and reflexive subspaces and let τ be a locally convex topology on E weaker than $\| \ \|$. Then the identity map $I : E, \tau \to E, \| \ \|$ is Borel.*

The proof is based on some known results and the following two propositions.

Proposition 1. *Let a Banach space $E, \| \ \|$ be decomposed into a direct sum of separable and reflexive subspaces and let $|\|\ \|\|$ be a norm on E weaker than $\| \ \|$. Then the identity map $I : E, |\|\ \|\| \to E, \| \ \|$ is of α Borel class for some $\alpha < \omega_1$.*

P r o o f. Obviously, a reflexive Banach space is WCG-space. Every separable Banach space is (even) compactly generated. Since the sum of weakly compact sets is weakly compact, the sum E of separable space E_1 and reflexive space E_2 is WCG-space. By the Amir-Lindenstrauss lemma [3, p. 137], there is a linear projection $P : E \to E$ with $\|P\| = |\|P\|| = 1$, $PE \supset E_1$ and such that $X = PE$ is separable. Then $Y = \ker P$ is reflexive. By well known Suslin theorem [8, §39.IV], the restriction of I onto X is the Borel map. Since X is separable, this restriction is of α Borel class for some $\alpha < \omega_1$. And by above mentioned Vasin-Tanana-Vinokurov result, the restriction of I onto Y is of the 1 Borel class.

Let A be a $\| \ \|$-open subset of E. Since E, $\| \ \|$ is isomorphic to the topological product $X \times Y$ we can represent

$$A = \bigcup_{j \in J} (B_j + C_j)$$

where B_j are open subsets of X and C_j are open subsets of Y. By separability of X we can suppose J to be countable. The sets B_j and C_j belong to the multiplicative class α of X, $|\|\ \|\|$ and Y, $|\|\ \|\|$ respectively. The space E, $|\|\ \|\|$ is decomposed into the topological direct sum of X and Y, so it is isomorphic to the topological product $(X, |\|\ \|\|) \times (Y, |\|\ \|\|)$. Therefore, $B_j + C_j$ belong to the additive class α of E, $|\|\ \|\|$ [8, §30.III] and their countable union too.

We use in the following proposition, and later also, the following result which goes back to S. Mazur: if E is a locally convex space and E^* is the corresponding dual space then for every convex subset $V \subset E$ the closure of V in the initial topology of E coincides with its closure in the topology $w(E, E^*)$.

Proposition 2. *Let τ be a Hausdorff locally convex topology on a WCG-space E which is weaker than $\| \;\|$ and the subspace $F := (E, \tau)^* \subset E^*$ is norming. Then $I : E, \tau \to E, \| \;\|$ is Borel.*

P r o o f. Let us show first that any $\| \;\|$-closed convex neighbourhood V of zero is τ-Borel. Let K be a weak compact which generates E and U be the τ-closure of the unit ball $B(E)$ of E. We can assume, without loss of generality, $B(E) \subset V$. The set $V_n = nK \cap V$ is weakly compact, therefore is $w(E, F)$-compacts, for every n. Since the convex hull of convex compact and closed sets is closed, for every scalars a and b the set $\operatorname{conv}(aV_n, bU)$ is $w(E, F)$-closed, hence τ-closed. Since F is norming, there exists a number $r > 0$ such that $rU \subset B(E)$ [13, p. 32]. Therefore we can represent the $\| \;\|$-interior of V as

$$\operatorname{int} V = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \operatorname{conv}\{(1 - m^{-1})V_n, rU\}.$$

Since $V = \bigcap_k (1 - k^{-1}) \operatorname{int} V$, the set V is τ-Borel. To finish the proof we shall show that we can receive every open set of a WCG-space as countable unions and intersections of convex $\| \;\|$-bodies. For this purpose we use the proof of Lemma 1 [13, p. 26]. Observe that a WCG-space is isomorphic to locally uniformly rotund one [3, p. 146].

Claim. *Let W be an open subset of a locally uniformly rotund space E. There exists a sequence of convex bodies $U_n \subset E$ and a sequence of closed balls B_n, $n = \overline{1, \infty}$ such that*

$$W = \bigcup_n \Big(B_n \bigcap (E \setminus U_n) \Big).$$

P r o o f. Without loss of generality we can assume $0 \in W$. Since W is open, for every $x \in W$ there exists a closed ball $B_x \subset W$ with center in x. For $x = 0$ let B_0 has a radius a. For any $x \in W \setminus B_0$ denote by f_x a linear continuous functional such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$. Since E is locally uniformly rotund, there exists a rational number $r_x \geq \|x\|$ and positive number $\varepsilon_x < \|x\| - a$ such that

$$\sup\{f_x(y) : y \in r_x B(E) \setminus B_x\} < \|x\| - \varepsilon_x.$$

Put

$$H_x = \{y \in E : f_x(y) > \|x\| - \varepsilon_x\}.$$

Then

$$W = \bigcup_{x \in W} \left(H_x \bigcap r_x B(E) \right).$$

If we fix the numbers r_x and ε_x then we receive a decomposition of W onto classes in the following way: x and y belong to the common class $\bar{r}_x$, if $r_x = r_y$. Therefore

$$W = \bigcup_{r_x} \left(\bigcup_{x \in \bar{r}_x} \left(H_x \bigcap r_x B(E) \right) \right) \bigcup B_0 = \bigcup_{r_x} \left(r_x B(E) \bigcap \left(\bigcup_{x \in \bar{r}_x} H_x \right) \right) \bigcup B_0.$$

Evidently

$$\bigcup_{x \in \bar{r}_x} H_x = E \setminus \bigcap_{x \in \bar{r}_x} \{y \in E : f_x(y) \leq \|x\| - \varepsilon_x\}.$$

The last intersection is closed and contains B_0, hence is a body. $\square$

Remark. If E is separable then under the conditions of Proposition 2 we can represent the counter-image $I^{-1}(U)$ of any $\| \, \|$-open set U as the countable union of τ-closed sets [13, p. 186-187]. If E is non-separable, the above is not obligatory [13, p. 190]. It follows from the proof of Proposition 2 that we can receive in this proposition the counter-image $I^{-1}(U)$ of any $\| \, \|$-open set U from τ-closed sets by means of turning application of the countable union and intersection (it is sufficiently to apply these operations roughly four times).

P r o o f o f T h e o r e m $1'$. Put $F = (E, \tau)^* \subset E^*$ and introduce on E the new norm

$$\|x\|_0 = \sup\{|f(x)| \ : \ f \in F, \ \|f\| \leq 1\}.$$

The topology $w(E, F)$ is weaker than the norm $\| \, \|_0$ and we can extend them naturally onto the $\| \, \|_0$-completion $\bar{E}$ of E. Then $\bar{E}$ is WCG-space and, by the definition of $\| \, \|_0$, $F \subset (\bar{E}, \| \, \|_0)^*$ is a $\| \, \|_0$-norming subspace. Hence, by Proposition 2, the operator $I : \bar{E}, w(\bar{E}, F) \to \bar{E}, \| \, \|_0$ is Borel. Since the Borel sets of the subspace $E \subset \bar{E}$ be the intersections of Borel sets of $\bar{E}$ with E [8, §5.VI], the operator $I : E, w(E, F) \to E, \| \, \|_0$ is Borel. By Proposition 1 the operator $I : E, \| \, \|_0 \to E, \| \, \|$ is Borel. Therefore, the map $I : E, \tau \to E, \| \, \|$

is Borel as the product of $\tau \to w(E,F), w(E,F) \to \| \|_0$ and $\| \|_0 \to \| \|$ Borel maps. $\square$

To prove Theorem 3 we shall introduce another few definitions and notations and recall some known results. Let $T : X \to Y$ be a continuous linear bijective operator. Let us for every countable ordinal $\beta \geq 0$ denote by $B_\beta(X)$ the polar

$$(B(F_{(\beta)}))^\circ := \{x \in X : (\forall f \in B(F_{(\beta)})(|f(x)| \leq 1)\}$$

of the ball $B(F_{(\beta)}) = \{f \in F_{(\beta)} : \|f\| \leq 1\}$ with respect to the dual pair (X, X^*) (recall, $F := T^*Y^*$ is a total on X subspace of X^*). The gauge functional of $B_\beta(X)$ is a new norm on X. We shall denote this norm by $\| \|_\beta$ and call β-norm. The closures in this norm will be denoted by cl_β. It is clear that $B_\beta(X)$ is the unit ball of the normed space $(X, \| \|_\beta)$. The balls of this normed space will be called β-balls. In accordance with our denotes, $F_0 = F$ and $\|x\|_0 = \sup\{|f(x)| : f \in F, \|f\| \leq 1\}$. We also introduce on X the norm $\|x\|_{-1} = \|Tx\|_Y$.

For a separable space X the set $F_{(1)}$ coincides with the union of weak* closures of bounded subsets of F. Therefore, by the bipolar theorem it follows that the dual of $(X, \| \|_\beta)$ $(\beta \geq 0)$ is $F_{(\beta+1)}$ with the natural duality. For $\beta = -1$ the analogous result follows from the definition. Using the bipolar theorem once more, we obtain that $B_{\beta+1}(X)$ is the closure of $B(X) := \{x \in X : \|x\| \leq 1\}$ in the topology $w(X, F_{(\beta+1)})$. Hence, by mentioned above Mazur's result, $B_{\beta+1}(X) = \mathrm{cl}_\beta B(X)$, $(\beta \geq -1)$.

This result admits the following generalization:

Lemma 1. *For every ordinal $\beta \geq 0$ we have*

$$B_\beta(X) = \bigcap_{-1 \leq \gamma < \beta} \mathrm{cl}_\gamma B(X). \tag{1}$$

P r o o f. It is clear that the transfinite sequence in the right-hand side of (1) is decreasing. Therefore for non-limit ordinals the equality (1) is already proved. Let β be a limit ordinal and let us suppose that (1) is prove for all ordinals which are less than β. We have (polars below are taken with respect to the dual pair (X, X^*)):

$$\begin{aligned} B_\beta(X) &= (B(F_{(\beta)}))^\circ = B\left(\bigcup_{\gamma<\beta} F_{(\gamma)}\right)^\circ = \left(\bigcup_{\gamma<\beta} B(F_{(\gamma)})\right)^\circ = \bigcap_{\gamma<\beta}(B(F_{(\gamma)}))^\circ \\ &= \bigcap_{\gamma<\beta} B_\gamma(X) = \bigcap_{\gamma<\beta}\bigcap_{\tau<\gamma} \mathrm{cl}_\tau B(X) = \bigcap_{-1\le\gamma<\beta} \mathrm{cl}_\gamma B(X). \end{aligned}$$ □

Lemma 2. *For every ordinal $\beta \geq 0$ every β-closed subset of X is of multiplicative class $\beta + 1$ in (-1)-norm.*

Proof. Since $(X, \| \ \|)$ is separable, then for every ordinal β the space $(X, \| \ \|_\beta)$ is separable also. At first let us consider the case $\beta = 0$. Let $M \subset X$ be $\| \ \|_0$-closed. By $\| \ \|_0$-separability of X it follows that $X \setminus M$ can be represented as a union of countable collection of closed 0-balls. By formula (1) closed 0-balls are also (-1)-closed. Therefore, the set $X \setminus M$ is of additive class 1 in (-1)-norm, Hence M is of multiplicative class 1 in the (-1)-norm.

Let $\beta > 0$ and let us suppose that Lemma 2 is already proved for every ordinal $0 \leq \gamma < \beta$. Let $M \subset X$ be β-closed. By β-separability of X it follows that $X \setminus M$ can be represented as the union of a countable collection of closed β-balls $V_n : X \setminus M = \bigcup_{n=1}^{\infty} V_n$. By formula (1) it follows that every V_n can be represented as $V_n = \bigcap_{\gamma<\beta} V_{n,\gamma}$, where each $V_{n,\gamma}$ is γ-closed. Hence

$$M = \bigcap_{n=1}^{\infty}(X \setminus V_n) = \bigcap_{n=1}^{\infty}\bigcup_{\gamma<\beta}(X \setminus V_{n,\gamma}).$$

By the induction hypothesis each $V_{n,\gamma}$ is of multiplicative class $\gamma + 1$ in the (-1)-norm. Therefore $\bigcup_{\gamma<\beta}(X \setminus V_{n,\gamma})$ is of additive class β and M is of multiplicative class $\beta + 1$. □

Proof of necessity of Theorem 3. Let α be the first ordinal for which $F_{(\alpha)} = X^*$. As it is known, $\alpha < \omega_1$ and cannot be a limit ordinal [7]. Moreover, the subset $F_{(\alpha-1)}$ is norming [13, p. 47]. We need to prove that the identity map $(X, \| \ \|_{-1}) \to (X, \| \ \|)$ is of α Borel class. Since $F_{(\alpha-1)}$ is norming, it is sufficient to prove that the identity map $X, \| \ \|_{-1} \to X, \| \ \|_{\alpha-1}$ is of α Borel class. But this is Lemma 2 (if we put $\alpha - 1$ instead of β).

For proving the sufficiency we shall use the following result [8, §11.III]. Let M be a Borel subset of a topological space Z. Then there exists an open subset $N \subset Z$ such that the symmetrical difference $N \Delta M$ is of the first category in Z. The set N is not uniquely determined by M but the closure of N in Z is uniquely determined by M. In the case $Z = (X, \| \|)$ we shall denote this set by $F(M)$.

Lemma 3. 1) *Let M be a Borel set of additive class $\alpha \geq 1$ in (-1)-norm and the corresponding set $F(M)$ is nonempty. Then for every $\| \|$-ball D_1, contained in $F(M)$, there exists an ordinal $\beta < \alpha$ and a $\| \|$-ball $D_2 \subset D_1$ such that*

$$\bigcap_{-1 \leq \gamma < \beta} \mathrm{cl}_{\beta} D_2 \subset F(M).$$

2) *Let M be a Borel set of multiplicative class $\alpha \geq 0$ in (-1)-norm and the corresponding set $F(M)$ is nonempty. Then for every $\| \|$-ball D_1 contained in $F(M)$, there exists a $\| \|$-ball $D_2 \subset D_1$ such that*

$$\bigcap_{-1 \leq \gamma < \alpha} \mathrm{cl}_{\gamma} D_2 \subset F(M).$$

P r o o f. At first we prove the second assertion for $\alpha = 0$. Let D_1 be some $\| \|$-ball which is contained in $F(M)$. Because M is of multiplicative class 0 (i.e. closed) in (-1)-norm, it follows that M contains $\mathrm{cl}_{-1} D_1$. Therefore it is not hard to verify that $F(M) \supset \mathrm{cl}_{-1} D_1$; i.e. instead D_2 we can take D_1.

Let us now suppose, that we have already proved the second assertion of Lemma 3 for all $\beta < \alpha$, and let M be of additive class α in the (-1)-norm, i.e. $M = \bigcup_{n=1}^{\infty} M_n$, where M_n is of multiplicative class β_n for some $\beta_n < \alpha$. Let $D_1 \subset F(M)$ be some $\| \|$-ball. It is not hard to verify that for some natural n and for some $\| \|$-ball $D_1' \subset D_1$ we have $D_1' \subset F(M_n)$. By the induction hypothesis there exists a $\| \|$-ball $D_2 \subset D_1'$ such that

$$\bigcap_{-1 \leq \gamma < \beta_n} \mathrm{cl}_{\gamma} D_2 \subset F(M_n) \subset F(M).$$

From this follows the first assertion of lemma for α.

Now suppose that we have already proved the first assertion of lemma for all $\beta \leq \alpha$ and will prove the second assertion for α. Let M be of multiplicative class α in (-1)-norm and $D_1 \subset F(M)$ be some $\| \|$-ball. Take a $\| \|$-ball $D_2 \subset D_1$ which is contained in the $\| \|$-interior of $F(M)$. Put $D = \bigcap_{-1\leq\gamma<\alpha} \mathrm{cl}_\gamma D_2$. If D is not contained in $F(M)$ then it intersects with $X \setminus F(M)$. Since D is convex and contains $\| \|$-interior points, it follows that $D \setminus F(M)$ contains some $\| \|$-ball D_1'. Since $X \setminus F(M) \subset F(X \setminus M)$ it follows by the first statement that for some $\beta < \alpha$ and for some $\| \|$-ball $D_2' \subset D_1'$ we have

$$\bigcap_{-1\leq\gamma<\beta} \mathrm{cl}_\gamma D_2' \subset F(X \setminus M).$$

Therefore the center of D_2 has a β-neighbourhood which also does not intersect the interior of $F(M)$. This contradicts $D_2' \subset \mathrm{cl}_\beta D_2$. $\square$

S u f f i c i e n c e o f T h e o r e m 3. Let T^{-1} be of the α Borel class and $f \in X^*$. Then fT^{-1} is also of the α Borel class. Hence the set $M = \{x \in X : f(x) < 0\}$ is of the additive class α in (-1)-norm. It is clear that $F(M) = \{x \in X : f(x) \leq 0\}$. By Lemma 3 it follows that for some $\beta < \alpha$ and some $\| \|$-ball $D_2 \subset F(M)$ we have

$$\bigcap_{-1\leq\gamma<\beta} \mathrm{cl}_\gamma D_2 \subset F(M),$$

i.e. $F(M)$ contains some β-ball. Therefore f is β-continuous and, as it was observed in the beginning of this item, $f \in F_{(\beta+1)} \subset F_{(\alpha)}$. Because f is arbitrary it follows that $F_{(\alpha)} = X^*$. $\square$

Theorem 2′. *Let us assume the continuum hypothesis. If a Banach space E cannot be decomposed into a direct sum of separable and reflexive subspaces, then there exists a norm $||| \, |||$ on E weaker than $\| \|$ such that the identity operator $E, ||| \, ||| \to E, \| \|$ is not Borel.*

To prove this theorem we need several auxiliary lemmas.

Lemma 4. *Let Y_0 and G_0 be subspaces of E and E^* respectively, separable in the norm. Then there exist subspaces $Y_0 \subset Y \subset E$ and $G_0 \subset G \subset E^*$, separable in the norm, which norm one another, i.e.*

$$\begin{aligned} \|y\| &= \sup\{|g(y)| : g \in G, \|g\| \leq 1\} \quad \textit{for every } y \in Y \textit{ and} \\ \|g\| &= \sup\{|g(y)| : y \in Y, \|y\| \leq 1\} \quad \textit{for every } g \in G. \end{aligned}$$

This lemma was proved, in fact, in [14]. To prove we need, using the Hahn-Banach theorem, to take a norm separable subspace $G_1 \supset G_0$ which 1-norms Y_0, then $Y_1 \supset Y_0$ which 1-norms G_1, then $G_2 \supset G_1$ which 1-norms Y_1 and so on. At the end we put $Y = [Y_i : i = 1, \infty]$ and $G = [G_i : i = 1, \infty]$. $\square$

Lemma 5. *Let E, $\| \ \|$ be a Banach space, $Y \subset E$ and $G \subset E^*$ be norm closed and separable subspaces which norm one another. If the quotient space $E/G^\top$ is non-separable then in the assumption of continuum hypothesis there exists a norm $||| \ |||$ on E weaker than $\| \ \|$ such that the identity operator $E, ||| \ ||| \to E, \| \ \|$ is not Borel.*

Proof. The dual to $E/G^\top$ is the (weakly* separable) space cl^*G. And in Theorem 3 [13, p. 23] it was proved actually that under these conditions there exists a linear continuous injective operator T from $E/G^\top$ into a separable Hilbert space H such that $T^*H^* \subset G$ and $[T^*H^*] = G$. Put $U = \{T^*h : \|h\|_{H^*} \le 1\}$ and $V = \{f \in Y^\perp : \|f\| \le 1\}$. Then $U \subset G$ and both sets are weakly* compact. Therefore the set $U \oplus V$ is convex, symmetric, total on E and weakly* compact. Then $|||x||| = \sup\{|f(x)| : f \in U \oplus V\}$ is a norm on E weaker than $\| \ \|$, moreover on $G^\top$ the norms $\| \ \|$ and $||| \ |||$ are equivalent.

The completion $\bar{E}$ of the space E, $||| \ |||$ is decomposed into the direct sum $\bar{Y} \oplus G^\top$ where the $||| \ |||$-completion $\bar{Y}$ of subspace Y is isomorphic to H. Therefore, $E/G^\top$, $||| \ |||$ is isomorphic to some (unclosed) subspace of H, hence separable. Since in the assumption of continuum hypothesis separability is invariant under Borel maps [8, §31.X], the identity operator $E/G^\top, ||| \ ||| \to E/G^\top, \| \ \|$ is not Borel (recall, $E/G^\top, \| \ \|$ is non-separable). Since the norms $\| \ \|$ and $||| \ |||$ are equivalent on $G^\top$, it follows that the identity operator $E, ||| \ ||| \to E, \| \ \|$ is not Borel too. $\square$

Lemma 6. *Let there exists a quasireflexive subspace Z of a Banach space E such that E/Z is separable. Then E is decomposed into a direct sum of separable and reflexive subspaces.*

Proof. As it is well known [13, p. 73] a quasireflexive space Z contains a reflexive subspace Z' such that Z/Z' is separable. Hence we can suppose Z to be reflexive at once. It follows that E is WCG-space [3, p. 153]. Since the quotient E/Z is separable, there exists a separable subspace $X_0 \subset E$ such that $X_0 + Z = E$. Then there exist a separable subspace $X_0 \subset X \subset E$ and a projection P from E

onto X with norm one [3, p. 149]. Evidently, the kernel of this projection is reflexive. $\square$

Lemma 7. *Let E be a Banach space and X be its closed subspace. Suppose that there exists a weaker norm $|||\ |||$ on X such that the identity operator $X, |||\ ||| \to X, \|\ \|$ is not Borel. Then we can extends the norm $|||\ |||$ to a norm on whole E, weaker than $\|\ \|$, so that the identity operator $E, |||\ ||| \to E, \|\ \|$ is not Borel.*

Proof. As $|||\ |||$ we take the gauge functional of the convex hull

$$\operatorname{conv}\left(\{e \in E : \|e\| \leq 1\} \cup \{x \in X : |||x||| \leq 1\}\right).$$

It is shown in the same manner as in [13, p. 68] that $|||\ |||$ is really a norm on E weaker than $\|\ \|$ and that $|||\ |||$ is a extension of the norm of X onto E. Since X is a closed subspace of E, if the identity operator $E, |||\ ||| \to E, \|\ \|$ were Borel, then its restriction onto X would be Borel too. $\square$

Proof of Theorem $2'$. Take, by Lemma 4, separable subspaces Y_1, G_1 of E and E^* respectively which 1-norm one another. If the quotient $E/G_1^\top$ is non-separable, then Lemma 5 finishes the proof. If it is separable, then by Lemma 6 the annihilator $G_1^\top$ is non-quasireflexive and we can choose a separable non-quasireflexive subspace $X_1 \subset G_1^\top$ [13, p. 76].

Let for an ordinal $\alpha < \omega_1$ we have constructed separable in norm subspaces Y_β, X_β, G_β, $1 \leq \beta < \alpha$ with the properties:

1) $Y_\gamma \subset Y_\beta$, $X_\gamma \subset Y_\beta$, $G_\gamma \subset G_\beta$ for $\gamma < \beta$,
2) Y_β and G_β 1-norms one another,
3) $X_\beta \subset G_\beta^\top$ and is non-quasireflexive.

We choose, by Lemma 4, separable subspaces $Y_\alpha \supset [X_\beta, Y_\beta : \beta < \alpha]$ and $G_\alpha \supset [G_\beta : \beta < \alpha]$ which 1-norm one another. If the quotient $E/G_\alpha^\top$ is non-separable, Lemma 5 finishes the proof. If it is separable, then, by Lemma 6, $G_\alpha^\top$ is non-quasireflexive and we can choose a separable non-quasireflexive subspace $X_\alpha \subset G_\alpha^\top$.

If the process do not finish on any-countable ordinal α, then we finish it on the ordinal ω_1 putting $X = [X_\alpha : 1 \leq \alpha < \omega_1]$. Let P_α be the projection of X onto $[X_\beta : \beta < \alpha]$ along $[X_\beta : \beta \geq \alpha]$. Then $\|P_\alpha\| = 1$, $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min(\alpha,\beta)}$, $X_\alpha = (P_{\alpha+1} - P_\alpha)X$. For every α we take, by Theorem 4, a total norm closed subspace $F_\alpha \subset (P^*_{\alpha+1} - P^*_\alpha)X^*$ such that $(F_\alpha)_{(\alpha)} \neq (P^*_{\alpha+1} - P^*_\alpha)X^*$. Let us introduce, as in [13, p. 189-190], a weaker norm $\|x\|_\alpha$ such that $(X_\alpha, \| \|_\alpha)^* \subset F_\alpha$. And finally we define on X the norm $|||x||| = \sup_\alpha \|(P_{\alpha+1} - P_\alpha)x\|_\alpha$. This norm is equivalent on X_α to the norm $\| \|_\alpha$ and, it is weaker than $\| \|$ on X.

Let us show that the identity operator $X, ||| \, ||| \to X, \| \|$ is not Borel. Indeed, in the opposite case the $\| \|$-ball $B(X)$ belongs to some class $\alpha < \omega_1$ of Borel subsets of $X, ||| \, |||$. But since the subspace X is $||| \, |||$-closed, $B(X_\alpha) = B(X) \cap X_\alpha$ belongs to the class α of Borel subsets of $X, ||| \, |||$. But then the identity operator $X_\alpha, ||| \, ||| \to X_\alpha, \| \|$ is of the class α and, by Theorem 3, $(F_\alpha)_{(\alpha)} = X_\alpha^*$. This contradicts to the choice of α. To end, it is enough to apply Lemma 7. $\square$

Definition 3. *Let X be a Banach space. A system $(x_j, f_j, j \in J)$, $x_j \in X$, $f_j \in X^*$, J is some set of indices, is called to be countably norming (countably 1-norming) Markushevich basis (M-basis in short) if $f_i(x_j) = \delta_{ij}$ (δ is the Kronecker symbol), $[x_j : j \in J] = X$, the subset $\{f_j : j \in J\}$ is total and the subspace F of elements $f \in X^*$, for which* $\operatorname{card}\{j \in J : f(x_j) \neq 0\} \leq \aleph_0$, *is norming (1-norming).*

Every WCG-space has a countably norming M-basis [15].

Theorem 2″. *Let a Banach space $E, \| \|$ contains a subspace X having a countably norming M-basis, which cannot be decomposed into a direct sum of separable and reflexive subspaces. Then we can introduce on E a weaker norm $||| \, |||$ such that the identity operator $E, ||| \, ||| \to E, \| \|$ is not Borel.*

P r o o f. Lemma 7 at once reduces the proof to the space X. The subspace F is 1-norming in the norm $\|x\|_0 = \sup\{|f(x)| : \|f\| \leq 1\}$, which is equivalent to the original norm $\| \|$. Now, by the basic property of countably norming M-bases, for every countable subsets $J_1, J_2 \subset J$ there exists a countable subset $J_1 \cup J_2 \subset J_3 \subset J$ such that the subspaces $[x_j : j \in J_3]$ and $[f_j : j \in J_3]$ 1-norm one another and $[f_j : j \in J_3]^\top = [x_j : j \in J_3]$ [15]. Thus we can prove Theorem 2′ for this space X in such a way that this proof do not stop on any countable step. Therefore we do not need to apply the continuum hypothesis. $\square$

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