# Serdica

## Mathematical Journal

# Сердика

## Математическо списание

# INTEGER POINTS CLOSE TO A SMOOTH CURVE

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*Communicated by S. L. Troyanski*

ABSTRACT. We review the existing estimates for the number of integer points close to a smooth curve and improve on some of these.

**1. Introduction.** In the last few years a number of papers containing estimates for the number of integer points close to a smooth curve appeared. In this paper we review these estimates and obtain some minor improvements of the existing results.

Let $\delta$, $T$ and $M$ be positive real numbers and $f : [M, 2M] \mapsto R$ be any real valued function. Consider the set

$$S = \{u \in (M, 2M] : ||f(u)|| \le \delta\},$$

where $u$ is an integer and $||x||$ is the distance from $x$ to the nearest integer. We want to get estimates for the size of $S$. We will consider functions that satisfy

$$(1) \qquad \frac{T}{c_r M^r} \le |f^{(r)}(x)| \le \frac{c_r T}{M^r}$$

for all $x \in [M, 2M]$ where $c_r \geq 1$ is a constant depending only on $r$, for values of $r$ that we will specify later on.

W.l.o.g. we can assume that $\delta \leq \frac{1}{2}$ and $M \geq 1$, otherwise the problem is trivial. Also all functions we consider have finitely many zeros of $f'$ and the case $T < 1$ is simple too. So, throughout the paper we consider the case $T \geq 1$ only.

By using Vaaler's results [18] one can get estimates for $|S|$ by using the method of exponential sums. A typical estimate that can obtained in such a way is:

$$(2) \qquad |S| = \delta M + O(T^{\frac{k}{k+1}} M^{\frac{l}{k+1}})$$

where $(k, l)$ is an exponential pair. Following a seminal work by Bombieri and Iwaniec [3] a number of improvements in the method of exponential sums were obtained. An active research in this area is still going on and the estimates are far from final. We call the first term in right hand side of (2) a $\delta$-term and the second term – a smoothness term. While the $\delta$-term in (2) is optimal, one can get better smoothness term in a number of cases. This was done in a number of papers that used no estimates of exponential sums. These papers were based on combinatorial ideas and on investigation of the geometry of the set $S$. The first result of such nature belongs to M. N .Huxley [10]. His paper contains several estimates for the size of the set $S$. One of that estimates, Theorem 1 [10] was improved further on in a joint paper of M. N. Huxley and P. Sargos [13].

**Theorem 1** (Huxley and Sargos). *Let $f : [M, 2M] \mapsto R$ be a function with continuous n-th derivative and let condition (1) hold for $n$, where $n$ is an integer greater than 1. Then*

$$(3) \qquad |S| \ll M^{\frac{n-1}{n+1}} T^{\frac{2}{n(n+1)}} + M\delta^{\frac{2}{n(n-1)}} + M\left(\frac{\delta}{T}\right)^{\frac{1}{n}}.$$

An estimate with a better main $\delta$-term was obtained in Filaseta and Trifonov [7].

**Theorem 2** (Filaseta and Trifonov). *$f : [M, 2M] \mapsto R$ be a function with continuous n-th derivative, $n \geq 3$ and suppose that condition (1) holds for $r = n - 2, n - 1, n$. Let $\delta < k \min(TM^{-n+2}, T^{\frac{n-4}{n-2}} M^{-n+3} + TM^{-n+1})$ for some sufficiently small constant $k$ depending on $n$ and on the constants $c_r$, $r = n - 2, n - 1, n$. Then*

$$(4) \qquad |S| \ll M^{\frac{n-1}{n+1}} T^{\frac{2}{n(n+1)}} + M\delta^{\frac{2}{(n-1)(n-2)}} + M(\delta TM^{1-n})^{\frac{1}{n^2-3n+4}}.$$

The restriction on $\delta$ in Theorem 2 can be relaxed. We describe shortly one way to do it.

Let $A_0(x_0, y_0), \ldots, A_n(x_n, y_n)$ be $n + 1$ points in the plane such that $x_i \neq x_j$ when $i \neq j$. We define the divided difference of the points $A_0, \ldots, A_n$, $f[A_0, \ldots, A_n]$ as the coefficient of $x^n$ of the polynomial of degree $n$ that interpolates the points $A_0, \ldots, A_n$. It is well-known that there exists a unique polynomial with these properties. Let $f : R \mapsto R$ and let $x_0, \ldots, x_n$ be $n + 1$ distinct real numbers. We define $f[x_0, \ldots, x_n] = f[A_0, \ldots, A_n]$, where $A_i = (x_i, f(x_i))$ for $i = 0, \ldots, n$.

Let $m$ and $n$ be integers such that $m \geq n \geq 2$, and let $T = \{A_i(x_i, y_i), i = 0, \ldots, m\}$ be a set of $m + 1$ points in the plane such that $x_i \neq x_j$ when $i \neq j$, and let the $n$-th divided difference of any $n + 1$ distinct points of $T$ be positive (negative). Then we say that the set $T$ is strictly $n$-convex (concave). We prove the following modification of Theorem 2.

**Theorem 3.** *Let $f : [M, 2M] \mapsto R$ be a function with continuous $n$-th derivative and let condition (1) hold for $r = n-1, n$. Let $S_1$ be a strictly $n$-convex subset of $S$. Then*

$$|S_1| \ll M^{\frac{n-1}{n+1}} T^{\frac{2}{n(n+1)}} + M\delta^{\frac{2}{(n-1)(n-2)}} + M(\delta T M^{1-n})^{\frac{1}{n^2 - 3n + 4}}.$$

So, we have replaced the restriction on $\delta$ by a $n$-convexity condition. We show that in the cases $n = 3$ and $n = 4$ this gives us a wider range for $\delta$. Namely, we prove the following:

**Corollary 1.** *Let $n = 3$. Then Theorem 2 holds true provided that*

$$(5) \qquad\qquad \delta < kT^{\frac{1}{2}}M^{-1}.$$

**Corollary 2.** *Let $n = 4$. Then Theorem 2 holds true provided that*

$$(6) \qquad\qquad \delta < kT^{\frac{1}{3}}M^{-\frac{4}{3}}.$$

We also prove certain facts about the 2- and 3-convexity of $S$.

Note that the smoothness term in Theorems 1, 2 and 3 is one and same. The progress has been in improving the $\delta$-terms and the conditions on $\delta$. As a result, in the applications we consider the worst term is the smoothness term. There are several approaches that can be followed to improve that term too. Swinnerton-Dyer [17] proved that $|S| \ll M^{\frac{3}{5}}$ when $\delta = 0$, $T = M$, and condition (1) holds for $r = 3$. Later on Huxley [10] extended his work in the case $\delta \geq 0$ and $T > 1$. A further improvement was obtained by Huxley and Trifonov [14].

**Theorem 4** (Huxley and Trifonov).    *Let $f : [M, 2M] \mapsto R$ be a function with continuous third derivative and assume (1) for $r = 2$ and 3. Let $T \geq M$ and $\Delta = \frac{T}{M^2} \min_{x \in (1,2]} |f''(x)|$. Suppose that $\Delta < 1$ and that $0 \leq \delta \leq \sqrt{\Delta}$. Then*

$$
\begin{aligned}
|S| \quad \ll \quad & T^{\frac{3}{10}} M^{\frac{3}{10}} (\log M)^{\frac{1}{2}} + T^{\frac{4}{11}} M^{\frac{2}{11}} (\log M)^{\frac{5}{11}} + T^{\frac{3}{8}} M^{\frac{1}{4}} \delta^{\frac{1}{8}} (\log M)^{\frac{5}{8}} + \\
& T^{\frac{1}{7}} M^{\frac{4}{7}} \delta^{\frac{1}{7}} (\log M)^{\frac{5}{7}} + T^{\frac{1}{5}} M^{\frac{3}{5}} \delta^{\frac{2}{5}} \log M + \delta M.
\end{aligned}
$$

(7)

This Theorem is a improvement on Theorems 1,2 and 3 in the case $n = 3$. Unfortunately no such result is known for $n = 4$. A result of such type would lead to new results about the distribution of square-full numbers in short intervals.

Theorem 4 was recently improved further on by Huxley [12]. We provide some details on the new results of M.Huxley in Section 2.

Another case when one can get a better smoothness term is the case when the function $f(u) = \frac{x}{u^s}$ where $s$ is a rational number.

**Theorem 5.** (Filaseta and Trifonov).    *Let $k \geq 2$ be an integer, let $x \geq 1$, and let $s$ be a rational number, $s \neq i$ for $i = -(k-1), -(k-2), \ldots, k-2, k-1$. Let $f(u) = x/u^s$ and let $\delta < cM^{-(k-1)}$ where $c$ is a sufficiently small constant depending on $s$ and $k$. Denote $T = \frac{x}{M^s}$. Then*

$$
|S| \ll T^{\frac{1}{2k+1}} M^{\frac{k}{2k+1}} + \delta T^{\frac{1}{6k+3}} M^{\frac{6k^2+2k-1}{6k+3}}.
$$

(8)

The smoothness term in (8) is much better than the previous ones. Unfortunately this estimate works only when $\delta$ is "very small".

Third interesting case is $\delta = 0$. In a paper published in 1989 Bombieri and Pila [4] proved a number of estimates for the number of integer points on a smooth curve. However, so far there was not any successful extension of their methods in the case $\delta > 0$. We believe that such results will appear soon.

A new approach has been recently introduced by Konyagin [15]. It is based on properties of lattices.

**Theorem 6** (Konyagin). *Let $r \geq 2$ be an integer, $W \geq 1$, $f : [1, 2] \to R$ a function with continuous $r + 1$-st derivative on $[1, 2]$, and $\delta$ a positive number. Denote $F(u) = Tf(u/M)$ where $|T| \geq 1$ and $M \geq 1$. Suppose that $1/2 \leq F^{(r)}(u) \leq 1$ for all $u \in [M, M + L]$ where $1 \leq L \leq M$. Let $\tilde{S} = \{u \in (M, M + L] \cap Z :$ there exist integers $v$ and $w$ such that $1 \leq w \leq W$ and $|F(u) - v/w| < \delta\}$. Then*

$$|\tilde{S}| \ll L\left(\left(\frac{TW^2}{M^r}\right)^{\frac{1}{2r-1}} + \left(\frac{W^2}{M}\right)^{\frac{1}{2r}} + \left(\delta W^4\right)^{\frac{1}{3r-2}} + \left(\frac{\delta W^{2r+2}}{T}\right)^{\frac{1}{2r^2+r-1}}\right) +$$

$$(9) \qquad \left(\frac{\delta W^2 M^r L^{r-1} r^r}{T}\right)^{\frac{1}{2r-1}} + r$$

*where the constant in $\ll$ does not depend on $T$, $M$, $L$, $W$ and $r$.*

So far Konyagin's approach works well only when $r$ is big, but there are number of refinements of the method that can be done.

We outline briefly the content of the paper. In Section 2 we compare the various estimates we listed. In Section 3 we prove Theorem 3 and Corollaries 1 and 2. In Section 4 we review some applications.

**Notation**

We mention briefly some notation that will be used throughout the paper:

$\|x\|$ is the distance from the real number $x$ to the nearest integer;

$[x[$ denotes the nearest integer to $x$;

$[x]$ denotes as usual the integer part of $x$ and $\{x\} = x - [x]$ is the fractional part of $x$;

$f(u) \ll g(u)$, $g(u) \gg f(u)$, or $f(u) = O(g(u))$ will mean that there exists a constant $c$ such that $|f(u)| \leq c|g(u)|$ whenever $u$ is sufficiently large;

$f(u) = o(g(u))$ will mean that $\lim_{u \to \infty}(f(u)/g(u)) = 0$.

**2. Comparison of the estimates for $|S|$.** We consider the case that is important for most applications, namely $M \leq T \leq M^2$. In case $T \leq M$ the usual technique is to apply the existing estimates to the inverse function of $f$. For simplicity we drop all log factors when comparing the estimates for $|S|$. Denote $T = M^\alpha, 1 \leq \alpha \leq 2$ and $\delta = M^{-\beta}, \beta \geq 0$. Let compare Theorems 1, 2, 3, 4 and 6 (with $W = 1$). Figure 1 corresponds to the case $1 \leq \alpha \leq \frac{3}{2}$. The coordinates of

the point $A$ are $(\frac{23}{17}, \frac{7}{17})$ and the coordinates of $B$ are $(\frac{16}{13}, \frac{5}{13})$. In regions I, II, III and IV Theorem 4 provides better estimates for $|S|$ than Theorems 1, 2, 3 and 6 do. The best estimates one can get by using the above theorems are as follows: in region I $|S| \ll T^{\frac{3}{10}} M^{\frac{3}{10}} (\log M)^{\frac{1}{2}}$, in region II $|S| \ll T^{\frac{1}{7}} M^{\frac{4}{7}} \delta^{\frac{1}{7}} (\log M)^{\frac{5}{7}}$, in region III $|S| \ll T^{\frac{3}{8}} M^{\frac{1}{4}} \delta^{\frac{1}{8}} (\log M)^{\frac{5}{8}}$ and in region IV $|S| \ll T^{\frac{1}{5}} M^{\frac{3}{5}} \delta^{\frac{2}{5}} (\log M)$ (all this estimates come from Theorem 4). The new estimates of Huxley [12] give the



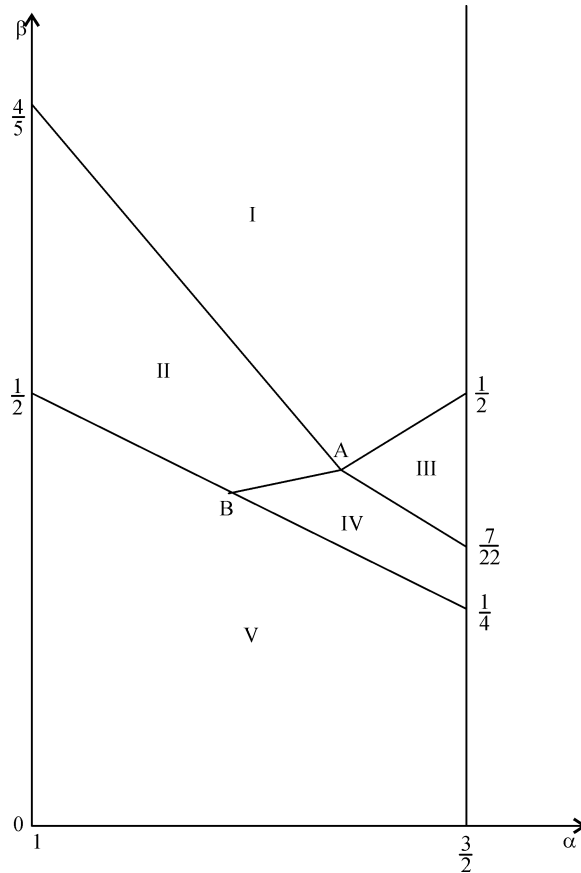Fig. 1. Comparisons in the case $M \leq T \leq M^{\frac{3}{2}}$.

same estimates as the listed above, but with better exponents of the log factors. In case Theorem 5 holds true it provides better estimates than the above when $7\alpha + 30\beta \geq 45$.

In region V Theorems 2, 3 and 4 do not apply since the conditions on $\delta$ are not satisfied. For the greater part of region V and for tiny part of region II (in

the lower left corner) the method of exponential sums provides the best estimates for $|S|$ (see Huxley [11]). We are not going into more detail since none of the applications we envision needs estimates in regions II and V. Another feature of the new estimates of Huxley [12] is that they hold in region V as well.

Now let compare Theorems 1, 2, 3, 4 and 6 when $\frac{3}{2} \leq \alpha \leq 2$. As before we disregard log factors when comparing the estimates for $|S|$. See Figure 2.
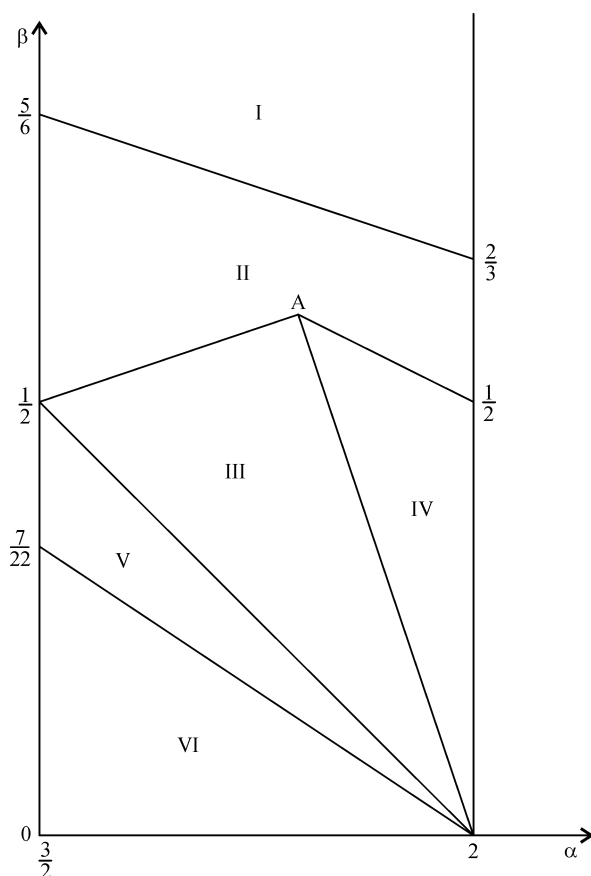


Fig. 2. Comparisons in the case $M^{\frac{3}{2}} \leq T \leq M^2$.

The coordinates of the point $A$ are $(\frac{9}{5}, \frac{3}{5})$. In regions I and II Corollaries 1 and 2 provide better estimates than Theorems 1, 2, 4 and 6. In region I we have $|S| \ll T^{\frac{1}{10}} M^{\frac{3}{5}}$ and in region II $|S| \ll T^{\frac{1}{6}} M^{\frac{1}{2}}$. In region VI and in most parts of regions III, IV, V and in the right bottom part of region II the method of exponential sums provides better estimates than the above theorems (see Huxley

[11]). The upper boundary of the region where exponential sums provide best estimates for $|S|$ is inside the triangle with vertices the points $(\frac{3}{2}, \frac{3}{8}), (\frac{3}{2}, \frac{1}{2})$ and $(2, \frac{2}{3})$. In the applications we consider we need only estimates in regions I and II. For the sake of completeness let us mention the best estimates one can get in regions III, IV and V by using Theorems 1, 2, 3, 4 and 6. In region III $|S| \ll T^{\frac{1}{4}} M^{\frac{1}{2}} \delta^{\frac{1}{4}}$ (Corollary 1), in region IV $|S| \ll M \delta^{\frac{1}{3}}$ (Theorem 1), and in region V $|S| \ll T^{\frac{3}{8}} M^{\frac{1}{4}} \delta^{\frac{1}{8}} (\log M)^{\frac{5}{8}}$ (Theorem 4). The new estimates of Huxley [12] are better than the listed above in parts of regions II, III, IV and V. In case Theorem 5 applies it provides the best estimates for $|S|$ when $\alpha + 30\beta \geq 36$.

### 3. Proofs of Theorem 3 and Corollaries 1 and 2.

Theorem 3 is only a modification of Theorem 6 from the paper of Filaseta and Trifonov [7]. The the difference between the theorems is that instead of restriction on $\delta$ we have a $n$-convexity (concavity) condition in Theorem 3. We prove Theorem 3 by modifying the proof of Theorem 6 [7]. The only places in the proof of Theorem 6 [7] where the restriction on $\delta$ is used are Lemmas 6 and 7. Lemma 6 is used only to prove Lemma 7. We will show that the $n$-convexity (concavity) of the set $|S_1|$ implies Lemma 7. Let introduce some notation. Let $0 = \beta_0 < \beta_1 < \ldots < \beta_n$ be integers and let $u_i = u + \beta_i \in S_1 \subset S$ for $i = 0, \ldots, n$. Set

$$G^{(n)} = G^{(n)}(\beta_0, \beta_1, \ldots, \beta_n) = \prod_{0 \leq i < j \leq n} (\beta_j - \beta_i).$$

For $j \in \{0, 1, \ldots, n\}$ we define $G_j^{(n)} = G^{(n-1)}(\beta_0, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_n)$, $M_j^{(n)} = (-1)^{n-j} G_j^{(n)}$ and

$$D_j^{(n)} = \prod_{\substack{0 \leq i \leq n \\ i \neq j}} (\beta_j - \beta_i) = \frac{G^{(n)}}{M_j^{(n)}}.$$

We use the following well-known property of divided differences

$$(10) \qquad f[u, u + \beta_1, \ldots, u + \beta_n] = \sum_{j=0}^{n} \frac{f(u + \beta_j)}{D_j^{(n)}} = \sum_{j=0}^{n} \frac{f(u + \beta_j) M_j^{(n)}}{G^{(n)}}.$$

We define a function $h : S \mapsto Z$ in the following way - if $x \in S$ then $h(x) = [f(x)[$.

Let $c > 0$ and $u + c + \beta_i \in S_1$ for $i = 0, \ldots, n$. Define

$$m_1 = \sum_{j=0}^{n} [f(u + \beta_j)][M_j^{(n)}(\beta_0, \ldots, \beta_n) = \sum_{j=0}^{n} h(u + \beta_j) M_j^{(n)}(\beta_0, \ldots, \beta_n),$$

$$m_2 = \sum_{j=0}^{n} h(u + c + \beta_j) M_j^{(n)}(\beta_0, \ldots, \beta_n),$$

$$l_1 = \sum_{j=0}^{n-1} (h(u + c + \beta_j) - h(u + \beta_j)) M_j^{(n-1)}(\beta_0, \ldots, \beta_{n-1}) \text{ and}$$

$$l_2 = \sum_{j=0}^{n-1} (h(u + c + \beta_{j+1}) - h(u + \beta_{j+1})) M_j^{(n-1)}(\beta_0, \ldots, \beta_{n-1}).$$

Note that (10) implies

$$
\begin{align}
(11) \qquad m_1 &= G^{(n)}(\beta_0, \ldots, \beta_n) h[u, \ldots, u + \beta_n], \\
(12) \qquad m_2 &= G^{(n)}(\beta_0, \ldots, \beta_n) h[u + c, \ldots, u + c + \beta_n], \\
(13) \qquad l_1 &= G^{(n-1)}(\beta_0, \ldots, \beta_{n-1})(h[u + c, \ldots, u + c + \beta_{n-1}] \\
& \qquad - h[u, \ldots, u + \beta_{n-1}]) \text{ and} \\
(14) \qquad l_2 &= G^{(n-1)}(\beta_0, \ldots, \beta_{n-1})(h[u + c + \beta_1, \ldots, u + c + \beta_n] \\
& \qquad - h[u + \beta_1, \ldots, u + \beta_n]).
\end{align}
$$

The part of Lemma 7 [7] using Lemma 6 [7] and the restriction on $\delta$ is the statement that $m_1$, $m_2$, $l_1$ and $l_2$ are all different from zero. From the definition of $G$ follows that all $G^{(n)}$ and $G^{(n-1)}$ in equations (11), (12), (13) and (14) are nonzero. Also $h[u, \ldots, u + \beta_n] \neq 0$ because of the strict $n$-convexity (concavity) of $S_1$ and $h[u + c, \ldots, u + c + \beta_n] \neq 0$ for the same reason. Also, $n$-convexity (concavity) implies strict monotonicity of the $n-1$st divided differences of $h$. It is well-known that if $S_1$ is strictly $n$-convex and $u_i, v_i \in S_1, u_i < v_i$ for $i = 0, \ldots, n - 1$ then

$$(15) \qquad h[u_0, \ldots, u_{n-1}] < h[v_0, \ldots, v_{n-1}].$$

Hence the difference of the divided differences in (13) and (14) is nonzero, which proves the part of Lemma 7 [7] we need. This completes the proof of Theorem 3. □

Proof of Corollary 1. In [10] Huxley proves that if

(16)                              $$\delta^2 < \min_{1 \le x \le 2} |f''(x)|$$

and $f$ satisfies condition (1) with $r = 2$, then the set $S$ is either convex or concave, depending on the sign of $f''$. He also proves that $S = S_1 \cup S_2$, where $S_1$ is a strictly convex (concave) set and $|S_2| \ll \delta M + M \left(\frac{\delta}{T}\right)^{\frac{1}{2}} \ll \delta M + M^{\frac{1}{2}}$. Hence $|S| \ll |S_1| + \delta M + M^{\frac{1}{2}}$. Theorem 3 with $n = 3$ gives us $|S_1| \ll T^{\frac{1}{6}} M^{\frac{1}{2}} + \delta M + M^{\frac{1}{2}} T^{\frac{1}{4}} \delta^{\frac{1}{4}}$.   $\square$

Proof of Corollary 2. First, note that when $T \le M^{\frac{3}{2}}$ Theorem 3 with $n = 4$ gives a worse estimate for $|S|$ than Theorem 1 with $n = 3$. Also, when $T \ge M^4$ Theorem 3 gives us an estimate that is worse that the trivial one $|S| \ll M$. So, till the end of this proof we assume $M^{\frac{3}{2}} \le T \le M^4$.

The proof consists of several steps.

**Lemma 1.**    $|S| \ll |S_3| + \delta M + M^{\frac{1}{2}}$ where $S_3 \subset S$ is strictly convex (concave) and the distance between any two elements of $S_3$ is $\gg \left(\frac{M^2}{T}\right)^{\frac{1}{3}}$.

Proof of Lemma 1. Indeed, since the conditions of Corollary 1 hold, we have as before $S = S_1 \cup S_2$, where $S_1$ is a strictly convex (concave) set and $|S_2| \ll \delta M + M^{\frac{1}{2}}$. Let $x_1 < x_2 < x_3$ be three elements of $S_1$. The convexity (concavity) of $S_1$ implies that the divided difference $h[x_1, x_2, x_3] \ne 0$. From (10) we get $(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)h[x_1, x_2, x_3] = h(x_1)(x_3 - x_2) - h(x_2)(x_3 - x_1) + h(x_3)(x_2 - x_1) = n$, where $n$ is a nonzero integer. Since $|h(x_i) - f(x_i)| < \delta$ for $i = 1, 2, 3$ we obtain $(x_2 - x_1)(x_3 - x_2)(x_3 - x_2)|h[x_1, x_2, x_3] - f[x_1, x_2, x_3]| \le 2\delta(x_3 - x_1)$. So,

(17)        $$(x_2 - x_1)(x_3 - x_2)(x_3 - x_2)f[x_1, x_2, x_3] = n + 2\delta\theta(x_3 - x_1),$$

where $|\theta| < 1$.

We consider two cases. First, let $x_3 - x_1 \le \frac{1}{4\delta}$. Then the right-hand side of (17) has absolute value greater than $\frac{1}{2}$. Since $|f[x_1, x_2, x_3]| = |f''(\xi)| \asymp \frac{T}{M^2}$ we get $x_3 - x_1 \gg \left(\frac{M^2}{T}\right)^{\frac{1}{3}}$.

Second, let $x_3 - x_1 > \frac{1}{4\delta}$. Note that (5) implies $\frac{1}{4\delta} \gg \left(\frac{M^2}{T}\right)^{\frac{1}{3}}$. So, we proved that if $x_1, x_2, x_3$ are three consecutive elements of $S_1$ then $x_3 - x_1 \gg \left(\frac{M^2}{T}\right)^{\frac{1}{3}}$. Define $S_3$ to be the set one gets by taking every second element of $S_1$.   $\square$

**Lemma 2.** *Let $a < b < c$ and $d$ be real numbers such that the distance between any two of them is at least $m$. Then*

$$(18) \qquad (d-a)^2(d-b)^2(d-c)^2 \geq \frac{1}{2}m^3(c-a)(b-a)(c-b)$$

Proof of Lemma 2. Since $|d-a| + |d-c| \geq c-a$ without loss of generality we can assume that $|d-a| \geq \frac{c-a}{2}$. Also, $|d-b| + |d-c| \geq c-b$ so $|d-b||d-c| \geq \max(\frac{c-b}{2}m, m^2)$. Therefore $(d-a)^2(d-b)^2(d-c)^2 \geq \frac{(c-a)^2}{4} \cdot \frac{c-b}{2}m \cdot m^2 \geq (c-a)(b-a) \cdot \frac{1}{2}(c-b)m^3$. $\square$

Note: One can check that there can be no equality in (18), but the constant $\frac{1}{2}$ can not be replaced by any constant greater that $\frac{2}{3}$ (consider the case $a = 0$, $b = 2m$, $c = 3m$, and $d = m$).

Without loss of generality we can assume that $f'''(x) \geq 0$ for every $x \in [1, 2)$. If not, we consider $-f$ instead of $f$. We claim that $S_3$ is 3-convex. In other words we show that for any $x_0 < x_1 < x_2 < x_3$ in $S_3$ the divided difference $h[x_0, x_1, x_2, x_3] \geq 0$.

Since $h(x_j) = f(x_j) + \theta_j \delta$, $|\theta_j| < 1$ for $j = 0, 1, 2, 3$ we get

$$G^{(3)}(x_0, x_1, x_2, x_3)h[x_0, x_1, x_2, x_3] = \sum_{j=0}^{3} h(x_j)M_j^{(3)} = \sum_{j=0}^{3}(f(x_j) + \theta_j\delta)M_j^{(3)} \geq$$

$$G^{(3)}(x_0, x_1, x_2, x_3)f[x_0, x_1, x_2, x_3] - \delta\sum_{j=0}^{3}|M_j^{(3)}|.$$

By using Lemma 2 with $m \gg \left(\frac{M^2}{T}\right)^{\frac{1}{3}}$ we get $G^{(3)}(x_0, x_1, x_2, x_3) \gg \left(\frac{M^2}{T}\right)^{\frac{1}{2}}|M_j^{(3)}|^{\frac{3}{2}}$ for $j = 0, 1, 2, 3$. Hence $G^{(3)}(x_0, x_1, x_2, x_3) \gg \left(\frac{M^2}{T}\right)^{\frac{1}{2}}\left(\sum_{j=0}^{3}|M_j^{(3)}|\right)^{\frac{3}{2}}$. Denote $y = \delta\sum_{j=0}^{3}|M_j^{(3)}|$. Since $f[x_0, x_1, x_2, x_3] = \frac{f'''(\xi)}{3!} \gg \frac{T}{M^3}$ we get $G^{(3)}h[x_0, x_1, x_2, x_3] \geq c_1\left(\frac{M^2}{T}\right)^{\frac{1}{2}}\frac{T}{M^3}\frac{1}{\delta^{\frac{3}{2}}}y^{\frac{3}{2}} - y.$

Note that $\min_{y \in [0,\infty)} sy^{\frac{3}{2}} - y = -\frac{4}{27s^2} > -1$ for any constant $s > \frac{2}{3\sqrt{3}}$. So, if the constant $k$ in (6) is sufficiently small ($k < 3\left(\frac{c_1}{2}\right)^{\frac{2}{3}}$) then $G^{(3)}h[x_0, x_1, x_2, x_3] > -1$. Since $G^{(3)}h[x_0, x_1, x_2, x_3]$ is an integer then $G^{(3)}h[x_0, x_1, x_2, x_3] \geq 0$. Noting that $G^{(3)}(x_0, x_1, x_2, x_3) > 0$ we get $h[x_0, x_1, x_2, x_3] \geq 0$. Therefore $S_3$ is 3-convex.

Our next aim is to show that $|S_3| \ll |S_4|+$ remainder terms, where $S_4$ is a strictly 3-convex subset of $S_3$ and the remainder terms are "small".

Let $S_3$ consist of the lattice points $A_i$ with coordinates $(x_i, h(x_i))$ $i = 1, \ldots, t$ where $x_1 < x_2 < \cdots < x_t$. Define a finite sequence $n_l$ as follows.

(i) $n_0 = 1$

(ii) Suppose that $n_{l-1}$ has been defined. Then $n_l$ is the unique integer such that the points $A_i$ for $n_{l-1} \leq i \leq n_l$ lie on some real algebraic curve with equation $y = P_l(x) = a_l x^2 + b_l x + c_l$, but the points $n_{l-1} \leq i \leq n_l + 1$ do not, if such an integer $n_l$ exists. Otherwise the sequence terminates with $n_{l-1}$. It is clear that $n_l - n_{l-1} \geq 2$. We call the parabola $P_l$ a minor arc if $n_l - n_{l-1} = 2$ and we call it a major arc otherwise. Let $q_l$ be the least positive integer such that $q_l P_l(x) \in Z[x]$. Then $a_l = \frac{r_l}{q_l}$ where $r_l$ is an integer. Now we use the following lemma of Huxley and Sargos [13].

**Lemma 3** (Huxley and Sargos).    *Let $f : [a, a + N] \mapsto R$ be a function with continuous third derivative and let $|f'''(x)| \geq \lambda > 0$ for every $x$ in the domain of $f$. Let $\delta$ be a positive real number less than $\frac{1}{2}$ and let $P(x) = ax^2 + bx + c$ where $a$, $b$ and $c$ are any real numbers. Define the set $T$ in the following way*

$$T = \{x \in [a, a + N] : |f(x) - P(x)| < \delta\}.$$

*Then $T$ is a union of no more than 3 disjoint open intervals, and the length of each of this intervals does not exceed $6(\delta/\lambda)^{1/3}$.*

Now, let consider a major arc $P_l$ that contains at least 10 lattice points $(n_l - n_{l-1} \geq 9)$. According to Lemma 3 the abscissae of at least 4 of these points are in an open interval with length $\leq 6(\delta/\lambda)^{1/3}$. Note that the equation of the parabola $P_l$ is uniquely determined by any 3 distinct points on it. Lagrange's interpolation formula implies that if $P(x)$ is a parabola that passes through the lattice points $(x, y)$, $(x + a, y_1)$ and $(x + b, y_2)$ where $0 < a < b$ then $ab(b - a)P(x) \in Z[x]$. So, $q_l$ is a divisor of a positive integer that is $\ll \frac{\delta M^3}{T}$. Therefore if $n_l - n_{l-1} \geq 9$ then $0 < q_l \ll \frac{\delta M^3}{T}$. Let $B$ be a positive constant. Since $T \geq M^{\frac{3}{2}} \geq M$ if the constant $k$ in (6) is sufficiently small we get $q_l \leq \frac{1}{B\delta}$. Let $\nu(\mathcal{P}_0)$ denote the number of lattice points in $S_3$ lying on parabolas $P_l$ such that $n_l - n_{l-1} \geq 9$ and $q_l \leq \frac{1}{B\delta}$. Huxley and Sargos [13] proved that

$$\nu(\mathcal{P}_0) \ll M\delta^{\frac{1}{3}} + \max_l(n_l - n_{l-1} + 1).$$

Note that $\mathcal{M} = \max_l(n_l - n_{l-1} + 1)$ is the maximal number of lattice points on a major arc. From Lemma 3 we get $\mathcal{M} \ll 3(6M\left(\frac{\delta}{T}\right)^{\frac{1}{3}} + 1) \ll M\delta^{\frac{1}{3}}$. Also, we proved that $n_l - n_{l-1} \geq 9$ implies $q_l \leq \frac{1}{B\delta}$. Now, for each major arc $P_l$ we perform the following operation - we discard all elements of $S_3$ from the major arc with the exception of $A_{n_{l-1}}$, $A_{n_{l-1}+1}$ and $A_{n_l}$ (i.e. we keep only the endpoints and one middle point). Let the set we get in this way be $S_4$. It is clear that $S_4$ contains only minor arcs. Let estimate the size of $S_4$. We proved that there are $\ll M\delta^{\frac{1}{3}}$ elements of $S_3$ on major arcs containing at least 10 lattice points. So, by discarding points from such major arcs we decrease the size of $S_3$ by $\ll M\delta^{\frac{1}{3}}$. Also, from every major arc containing less than 10 lattice points we have kept an intermediate point and the endpoints. Hence $|S_3| \leq 5|S_4| + |S_5|$ where $|S_5| \ll M\delta^{\frac{1}{3}}$. Combining this inequality with Lemma 1 we get

$$(19) \qquad |S| \ll |S_4| + \delta M^{\frac{1}{3}} + M^{\frac{1}{2}}.$$

$S_4$ is a subset of $S_3$ therefore it is 3-convex. Also, $S_4$ contains no major arcs which means that there is no parabola that contains 4 consecutive elements of $S_4$. One can easily prove the following lemma.

**Lemma 4.** *Let $V$ be a finite set of points $(x_i, y_i)$, $i = 0, \ldots, k$ in the plane with $x_0 < x_1 < \cdots < x_k$. If $V$ is a 3-convex set and if no parabola $y = ax^2 + bx + c$ contains four consecutive elements of $V$ then $V$ is strictly 3-convex.*

Lemma 4 implies that $S_4$ is strictly 3-convex. We apply Theorem 3 with $n = 4$ to $S_4$ and we get

$$|S| \ll M^{\frac{3}{5}}T^{\frac{1}{10}} + M\delta^{\frac{1}{3}} + M^{\frac{5}{8}}T^{\frac{1}{8}}\delta^{\frac{1}{8}} + \left(M\delta^{\frac{1}{3}} + M^{\frac{1}{2}}\right).$$

Note that we can drop the last two terms in the right-hand side of the above inequality - the first one for obvious reasons, and also $M^{\frac{1}{2}} \ll M^{\frac{3}{5}}T^{\frac{1}{8}}$ since both $T$ and $M$ are $\geq 1$. $\square$

Note 1: Inequality (19) allows us to work with a set of lattice points that is strictly 3-convex. This is a step in extending Swinnerton-Dyer's work in the case $n = 4$. Also, one can use an argument similar to the one of Lemma 1 to get a subset of $S_4$ that has spacing $\gg \left(\frac{M^3}{T}\right)^{\frac{1}{6}}$ between any two consecutive elements, and size that has the same order as the size of $S_4$.

Note 2: It is clear that one can relax the condition on $\delta$ in Theorem 2 for $n > 4$ as well, by using an argument similar to the argument of the proof of Corollary 2.

## 4. Applications.

### 4.1. Distribution of squarefree and k-free numbers in short intervals.
Let $k \geq 2$ be an integer. We say that a positive integer $n$ is $k$-free if no $k$-th power of a prime divides $n$. The density of $k$-free numbers is $\frac{1}{\zeta(k)}$, that is the interval $[1, x]$ contains $\frac{x}{\zeta(k)}(1 + o(1))$ $k$-free integers for $x \geq 1$. The following result states that a short interval contains the right proportion of $k$-free numbers, provided the interval is not "too short".

**Theorem** (Filaseta and Trifonov).     *There exists a constant $c = c(k)$ such that if $h = cx^{\frac{1}{2k+1}} \log x$ and $x$ is sufficiently large, then the interval $(x, x+h]$ contains a $k$-free number. Furthermore, if $h = x^{\frac{1}{2k+1}}(\log x)g(x)$ where $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, then the number of $k$-free numbers in $(x, x+h]$ is $\frac{h}{\zeta(k)} + o(h)$.*

We sketch briefly the proof. Denote $I = (x, x+h]$. Let count the number of integers in $I$ that are not $k$-free. For any integer $u \geq 1$ the number of integers in $I$ divisible by $u^k$ is $\left[\frac{x+h}{u^k}\right] - \left[\frac{x}{u^k}\right]$. By simple sieve-of-Erathostenes one gets that the number of integers in $I$ that are divisible by a k-th power of a prime $p < \log\log x$ is $h(1 - \frac{1}{\zeta(k)} + o(1))$.

Also, $\left[\frac{x+h}{u^k}\right] - \left[\frac{x}{u^k}\right] \leq \frac{h}{u^k} + 1$, so the number of integers in $I$, divisible by a $k$-th power of a prime $p$ with $\log\log x \leq p < h\sqrt{\log x}$ is $o(h) + \pi(h\sqrt{\log x}) = o(h)$.

The essential part of the proof is estimating the number of integers in $I$ that are divisible by a k-th power of a prime $p > h\sqrt{\log x}$. We do that by using estimates for the number of integer points close to a certain curve. Note that for $u > h$, $\left[\frac{x+h}{u^k}\right] - \left[\frac{x}{u^k}\right]$ is 0 or 1, and it is 1 if, and only if, $\{\frac{x}{u^k}\} \in (1 - \frac{h}{u^k}, 1)$. Let $S(N, 2N)$ be the set $\{u \in (N, 2N] \cap Z : \{\frac{x}{u^k}\} \in (1 - \frac{h}{N^k}, 1)\}$. It is clear that for $u > (2x)^{\frac{1}{k}}$ both $\left[\frac{x+h}{u^k}\right]$ and $\left[\frac{x}{u^k}\right]$ are 0. Let $N \in [h\sqrt{\log x}, (2x)^{\frac{1}{k}}]$. An upper bound for $|S(N, 2N)|$ is the number of integer points within $\delta = \frac{h}{N^k}$ of the curve $f(u) = \frac{x}{u^k}$, $u \in (N, 2N]$. Apply Theorem 5 with $T = \frac{x}{N^k}$, $s = -k$, $\delta = \frac{h}{N^k}$ and $M = N$. One can check that the conditions of the Theorem hold true when $x$ is sufficiently large and we get

$$|S(N, 2N)| \leq |S| \ll x^{\frac{1}{2k+1}} + h\left(\frac{x}{N^{2k+1}}\right)^{\frac{1}{6k+1}}.$$

Let $a$ be the largest integer such that $2^a \leq h\sqrt{\log x}$, and let $b$ be the largest integer such that $2^b < (2x)^{\frac{1}{k}}$. Then $\sum_{i=a}^{b} |S(2^i, 2^{i+1})| \ll x^{\frac{1}{2k+1}} + o(h) = o(h)$ for $h$ as in Theorem 7. Hence the number of integers divisible by a k-th power of a prime $p > h\sqrt{\log x}$ is $o(h)$ and this completes the proof of Theorem 7.

Note that we get the term $x^{\frac{1}{2k+1}}$ in the estimate for $|S(N, 2N)|$ for all $N \in (h\sqrt{\log x}, (2x)^{\frac{1}{k}}]$, so we have a wide range of critical values of $N$. In the case of squarefree numbers, $k = 2$, one can narrow somewhat the critical range. Let $N = x^\phi$. If $\frac{1}{5} \leq \phi < \frac{1}{4}$ then Corollary 2 holds and we get $|S(N, 2N)| \leq |S| \ll x^{\frac{1+4\phi}{10}} + N^{\frac{1}{3}} h^{\frac{1}{3}} + x^{\frac{1}{8}} N^{\frac{1}{8}} h^{\frac{1}{8}} \ll x^{\frac{1+4\phi}{10}} + o\left(\frac{h}{\log x}\right)$ for $h = x^\theta$, $\theta > \frac{5}{28}$.

Now, let $\frac{1}{3} < \phi < \frac{2}{5}$. In this case we consider the inverse function of $f$ and we estimate the number of integer points close to a curve when $f(m) = x^{\frac{1}{2}} m^{-\frac{1}{2}}$, $T = N$, $M = O(\frac{x}{N^2})$ and $\delta = O(\frac{hN}{x})$. Theorem 5 with $k = 2$ implies $|S(N, 2N)| \leq |S| \ll x^{\frac{2-3\phi}{5}} + h x^{\frac{12-38\phi}{15}}$. Also, if $\frac{1}{3} < \phi \leq \frac{1}{2}$ Theorem 4 implies $|S(N, 2N)| \leq |S| \ll x^{\frac{3-3\phi}{10}} + o\left(\frac{h}{\log x}\right)$ for $h \gg x^\theta$, $\theta > \frac{1}{6}$.

So, the the critical range for $\phi$ is $[\frac{1}{4}, \frac{1}{3}]$.

There are statistical reasons for the estimate

(20)
$$|S| \ll \delta M + M^{\frac{1}{2}}$$

to hold, provided that $T \geq M$ and $f$ is sufficiently smooth function. We call this estimate "ideal estimate". So far, this estimate is out of reach in the case $T \geq M$ if one uses only the existence of finitely many derivatives of $f$ and their sizes. If we assume the "ideal estimate" then one can prove Theorem 7 with the exponent $\frac{1}{2k+1}$ replaced by $\frac{1}{2k+2}$, the worst case being $N = x^{\frac{1}{k+1}}$.

**4.2. Distribution of squarefull numbers in short intervals.** A positive integer is squarefull if each prime factor occurs to the second power or higher. Let $Q(x)$ be the number of squarefull numbers $\leq x$. Bateman and Grosswald [2] proved that

$$Q(x) = \frac{\zeta(\frac{3}{2})}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)} x^{\frac{1}{3}} + o\left(x^{\frac{1}{6}}\right).$$

The estimate of Bateman and Grosswald implies that the number of squarefull numbers in the interval $(x, x + h]$ is $Q(x + h) - Q(x) \sim \frac{\zeta(\frac{3}{2})}{2\zeta(3)} x^\theta$ for $h = x^{\frac{1}{2}+\theta}$

with $\frac{1}{2} > \theta \geq \frac{1}{6}$. This result was improved several times by different authors, the latest one due to Huxley and Trifonov [14].

**Theorem 8** (Huxley and Trifonov).   *Let $\epsilon > 0$ be given. For $x$ and $h$ positive integers with $x$ sufficiently large in terms of $\epsilon$, and $h \geq \frac{1}{\epsilon} x^{\frac{5}{8}} (\log x)^{\frac{5}{16}}$,*

$$(21) \qquad Q(x+h) - Q(x) = \frac{\zeta(\frac{3}{2})}{2\zeta(3)} \frac{h}{\sqrt{x}} + O\left(\frac{\epsilon h}{\sqrt{x}}\right).$$

The asymptotic formula (21) as additive in $h$, so we may assume that $h \leq x^{\frac{2}{3}}$.

The reduction of this problem to the problem of estimating the number of integer points close to a curve is more difficult than in the case of squarefree numbers. The most efficient way to do this so far, was discovered independently by Heath-Brown [9] and Liu [16]. By using their approach Huxley and Trifonov [14] proved:

**Lemma 5.**   *Let $x$ be a positive integer, and put $V = \left[ x^{\frac{1}{5}} \right]$, $W = \left[ \sqrt{\frac{x}{V^3}} \right]$. Let $\epsilon$ be a small positive number, and $h$ a positive integer with $x^{\frac{1}{2}} \leq \epsilon^3 h \leq \epsilon^5 x$. Then*

$$Q(x+h) - Q(x) = \frac{\zeta(\frac{3}{2})}{2\zeta(3)} \frac{h}{\sqrt{x}} + O\left(\frac{\epsilon h}{\sqrt{x}}\right) + O(R_1 + R_2)$$

*where*
*$R_1$ is the number of pairs of positive integers $(m, k)$ with: $\frac{\epsilon h}{\sqrt{x}} < m \leq W$, $x < m^2 k^3 \leq x + h$ and $R_2$ is the number of pairs of positive integers $(m, k)$ with: $\frac{\epsilon h}{\sqrt{x}} < k \leq V$, $x < m^2 k^3 \leq x + h$.*

Let estimate $R_1$. Let $m \in (N, 2N]$ with $\frac{\epsilon h}{\sqrt{x}} < N \leq W$. Then $m^2 k^3 \in (x, x+h]$ for some integer $k$ if and only if $\left[ \sqrt[3]{\frac{x+h}{m^2}} \right] - \left[ \sqrt[3]{\frac{x}{m^2}} \right] = 1$ (note that $\sqrt[3]{\frac{x+h}{m^2}} - \sqrt[3]{\frac{x}{m^2}} \leq \frac{1}{3} h x^{-\frac{2}{3}} \leq \frac{1}{3}$). If the last identity holds then $\left\{ \sqrt[3]{\frac{x}{m^2}} \right\} \in (1 - \frac{h}{x^{\frac{2}{3}} N^{\frac{2}{3}}}, 1)$. Define $S_1(N, 2N) = \{ m \in (N, 2N] \cap Z : \left\{ \sqrt[3]{\frac{x}{m^2}} \right\} \in (1 - \frac{h}{x^{\frac{2}{3}} N^{\frac{2}{3}}}, 1) \}$. The number of integers $m \in (N, 2N]$ such that there exists an integer $k$ with $m^2 k^3 \in (x, x+h]$ is bounded by $|S_1(N, 2N)|$. To estimate $|S_1(N, 2N)|$ we use Theorem 4 with $f(m) = \sqrt[3]{\frac{x}{m^2}}$, $T = \frac{x^{\frac{1}{3}}}{N^{\frac{2}{3}}}$, $M = N$ and $\delta = \frac{h}{x^{\frac{2}{3}} N^{\frac{2}{3}}}$. Theorem 4 holds and we get

$|S_1(N, 2N)| \ll x^{\frac{13}{105}} (\log x)^{\frac{5}{7}}$. Therefore $R_1 \ll x^{\frac{13}{105}} (\log x)^{\frac{12}{7}} = o\left(\frac{\epsilon h}{\sqrt{x}}\right)$ for $h \geq x^{\frac{5}{8}}$ and $x$ sufficiently large.

We estimate $R_2$ in a similar way. Let $k \in (N, 2N]$ with $\frac{\epsilon h}{\sqrt{x}} < N \leq V$. The number of integers $k \in (N, 2N]$ such that $m^2 k^3 \in (x, x+h]$ for some integer $m$ is bounded by $|S_2(N, 2N)|$, where $S_2(N, 2N)$ is the set $\{k \in (N, 2N] \cap Z : \{\sqrt{\frac{x}{k^3}}\} \in (1 - \frac{h}{x^{\frac{1}{2}} N^{\frac{3}{2}}}, 1)\}$. We use different theorems to estimate the size of $S_2(N, 2N)$, depending on the size of $N$. First, let $N \geq x^{\frac{1}{6}} (\log x)^{\frac{5}{4}}$. In this case we use Theorem 4 with $f(k) = \sqrt{\frac{x}{k^3}}$, $T = \frac{x^{\frac{1}{2}}}{N^{\frac{3}{2}}}$, $M = N$ and $\delta = \frac{h}{x^{\frac{1}{2}} N^{\frac{3}{2}}}$. Theorem 4 holds and we get

$$(22) \qquad |S_2(N, 2N)| \ll \frac{x^{\frac{3}{20}}}{N^{\frac{3}{20}}} (\log x)^{\frac{1}{2}} + \frac{x^{\frac{1}{8}} h^{\frac{1}{8}}}{N^{\frac{1}{2}}} (\log x)^{\frac{5}{8}} + x^{\frac{13}{105}} (\log x)^{\frac{5}{7}}.$$

Now, let $\frac{\epsilon h}{\sqrt{x}} < N \leq x^{\frac{1}{6}} (\log x)^{\frac{5}{4}}$. In this case we use Corollary 1. Condition (5) on $\delta$ holds and we get

$$(23) \qquad |S_2(N, 2N)| \ll x^{\frac{1}{12}} N^{\frac{1}{4}} + x^{-\frac{1}{2}} N^{-\frac{1}{2}} h + h^{\frac{1}{4}} N^{-\frac{1}{4}}.$$

Combining (22) and (23) we obtain $R_2 \ll \frac{\epsilon h}{\sqrt{x}}$, for $h \geq \frac{1}{\epsilon} x^{\frac{5}{8}} (\log x)^{\frac{5}{16}}$. This completes the proof of Theorem 8. $\square$

To improve Theorem 8 it is sufficient to obtain better estimates for the size of $S_2(N, 2N)$ when $N \asymp x^{\frac{1}{8}}$ and when $N \asymp x^{\frac{1}{6}}$. When $N \asymp x^{\frac{1}{8}}$ any nontrivial estimate for $|S_2(N, 2N)|$ will do, for instance one can use exponential sums. The real obstacle lies in the case $N \asymp x^{\frac{1}{6}}$. The most natural thing to do here is to extend the work of Swinnerton-Dyer [17].

The "ideal estimate" (20) implies Theorem 8 with exponent $\frac{1}{10}$ instead of $\frac{1}{8}$, the worst case being $m \asymp k \asymp x^{\frac{1}{5}}$.

### 4.3. Estimates of the least prime factor of a binomial coefficient.
The material in this subsection is based on a work of Konyagin [15].

Let $g(k)$ be the least integer $n > k + 1$ such that all prime factors of $\binom{n}{k}$ are $> k$. Ecklund, Erdős, and Selfridge [5] proved that $g(k) > k^{1+c_1}$ with $c_1 > 0$. This result was improved by Erdős, Lacampagne and Selfridge [6] to $g(k) > c_2 k^2 / \log k$ and by Granville and Ramare [8] who proved $\log g(k) \gg (\log^3 k / \log\log k)^{\frac{1}{2}}$ for $k \geq 3$. Recently Konyagin obtained the following:

**Theorem 9** (Konyagin).    *For any positive integer $k$*

$$(24) \qquad\qquad \log g(k) \gg \log^2 k.$$

We sketch the proof of the Theorem. If all prime factors of $\begin{pmatrix} n \\ k \end{pmatrix}$ are $> k$, then for any prime $p \le k$ we have $\left\{ \frac{n}{p} \right\} \ge \left\{ \frac{k}{p} \right\}$. Let $0 < \beta < 1$. One can easily see that for every integer $w \ge 2$ and every prime $p \in \left( \frac{k}{w}, \frac{k+k^{1-\beta}}{w} \right)$, $\left\{ \frac{k}{p} \right\} > 1 - wk^{-\beta}$ and hence $\left\{ \frac{n}{p} \right\} > 1 - wk^{-\beta}$. Therefore there exists an integer $v$ such that $0 < \frac{v}{w} - \frac{n}{wp} < k^{-\beta}$.

We choose such $\beta$ and $\gamma \in (0, \beta)$ that there are $\gg k^{1-\beta}$ integers $u \in (k, k + k^{1-\beta})$ which can be represented as $wp$ with $p$ prime and $w \le W = k^\gamma$. Also, we will use Theorem 6 to show that if $n > ck^2 / \log k$ and there are $\gg k^{1-\beta}$ integers $u$ such that $n/u$ is within $k^{-\beta}$ of rational number with denominator $\le W$ then $\log n \gg \log^2 k$.

First, we choose $0 < \alpha < 1/2$ such that for sufficiently large $x$ and $x^{1-\alpha} \le y \le x$ there are $\gg y/\log x$ primes on the interval $(x, x + y)$. Baker and Harman [1] proved that $\alpha = 0.465$ is admissible. Let us take $\beta$ with $0 < \beta < 0.9\alpha$ and $(3 + \beta)/6 < 1 - \alpha$. Set $\gamma = \beta/10$, $W = k^\gamma$. A simple counting argument shows that when $k$ is sufficiently large then the interval $(k, k + k^{1-\beta})$ contains $\gg k^{1-\beta}$ integers which are product of prime and integer $w$ with $2 \le w \le W$. In other words

$$(25) \qquad\qquad |\tilde{S}| \gg k^{1-\beta}$$

where $\tilde{S} = \{u \in (k, k + k^{1-\beta}] \cap Z : \text{there exists a prime } p \text{ and an integer } w \text{ such that } 2 \le w = W \text{ and } u = wp\}$.

Let $k + 1 < n < \exp(c_3 \log^2 k)$ where $c_3$ is a positive constant. We will prove that at least one prime factor of $\begin{pmatrix} n \\ k \end{pmatrix}$ is $\le k$. From [6] follows that we can assume $n > c_2 k^2 / \log k$. Denote $f(u) = 1/u$, $T = n/k$, $M = k$, $L = k^{1-\beta}$ and $\delta = k^{-\beta}$. Let $r$ be the least positive integer such that $\frac{nr!}{k^{r+1}} \le k^{-\beta}$. One can show that $2 \le r \le 2c_3 \log k$. Theorem 6 holds with this choice of $r$ and we get $|\tilde{S}| \ll c_3 k^{1-\beta} + o\left(k^{1-\beta}\right)$ where the constant in $\ll$ does not depend on $k$ and $c_3$. We get a contradiction with (25) when $c_3$ is sufficiently small and this completes the proof of Theorem 9.

More applications based on estimates of the number of lattice points close to a curve can be found in [7].

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