Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **25** (1999), 5-10

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

ON A CONDITIONAL CAUCHY-TYPE FUNCTIONAL EQUATION INVOLVING POWERS

Emma Martín-Gutiérrez

Communicated by V. Kanev

Abstract. We solve the functional equation $f(x^m + y) = f(x)^m + f(y)$ in the realm of polynomials with integer coefficients.

Introduction. The aim of this paper is the study of the conditional functional equation

$$f(x^m + y) = f(x)^m + f(y)$$

for any fixed positive integer m, greater than 1, and where f is a map from the polynomial ring $Z[X]$ into itself. First we cover the case $m = 2$. This functional equation belongs to the general framework which appears in section 16.6 of [1].

1. The functional equation $f(x^2 + y) = f(x)^2 + f(y)$ with f defined over $Z[X]$. In this section we show that the functional equation

$$f(x^2 + y) = f(x)^2 + f(y) \tag{1}$$

for $f : Z[X] \to Z[X]$ entails the Cauchy functional equation, but is not equivalent. In fact it has less solutions.

1991 *Mathematics Subject Classification:* 39B52
Key words: functional equation

Lemma 1.1. *Any map $f : Z[X] \to Z[X]$ satisfying* (1) *sends zero into zero, and also satisfies*

$$f(x^2) = f(x)^2 \tag{2}$$

and

$$f\left(\sum_{i=1}^{r} \alpha_i x_i^2 + y\right) = \sum_{i=1}^{r} \alpha_i f(x_i)^2 + f(y) \tag{3}$$

where $\alpha_1, \dots, \alpha_r$ are integers and r is arbitrary.

Proof. Setting $x = 0$ in (1), we get $f(0) = 0$, and setting then $y = 0$ in (1), we get (2).

We shall now show that

$$f(\alpha x^2 + y) = \alpha f(x)^2 + f(y) \tag{4}$$

holds for any nonzero α in Z and any x, y in $Z[X]$. From (4), by induction, (3) follows at once.

In order to establish (4), assume first α positive. Using (1) inductively, we may write

$$f(\alpha x^2 + y) = f(\underbrace{x^2 + \cdots + x^2}_{\alpha} + y) = \underbrace{f(x)^2 + \cdots + f(x)^2}_{\alpha} + f(y) = \alpha f(x)^2 + f(y)$$

which proves (4) for positive α.

Now, if $z = -x^2 + y$, i.e. $y = x^2 + z$, we see (by (1)) that $f(y) = f(x)^2 + f(z)$, i.e.

$$f(-x^2 + y) = f(z) = -f(x)^2 + f(y).$$

Proceeding inductively as before for positive α we now also obtain (4) for negative α and the lemma is proved. □

Proposition 1.2. *If $f : Z[X] \to Z[X]$ satisfies functional equation* (1) *then f also satisfies the Cauchy functional equation $f(x+y) = f(x) + f(y)$. And in case that $f(1) = 0$, then f is necessarily the zero map.*

Proof. By (2), if f satisfies equation (1) then the image of 1 is necessarily 0 or 1. On the other hand, for any $x \in Z[X]$, $2x = (1+x)^2 - x^2 - 1$, and then, by (3),

$$f(2X) = f(1+x)^2 - f(x)^2 - f(1)^2 = \big(f(1)^2 + f(x)\big)^2 - f(x)^2 - f(1)^2. \tag{5}$$

When $f(1) = 0$, we obtain $f(2x) = 0$ for any $x \in Z[X]$, and so (by the preceding lemma) $0 = f(2x)^2 = f(2^2x^2) = 4f(x)^2$, therefore $f(x)^2 = 0$, for all x in $Z[X]$, so that f is the zero map.

When $f(1) = 1$, from (5), we obtain $f(2x) = 2f(x)$ for any $x \in Z[X]$ and now we may show that f satisfies the Cauchy functional equation:

$$\begin{aligned}2f(x+y) &= f(2x+2y) = f\left((1+x)^2 - x^2 - 1 + 2y\right)\\ &= \left(f(1)^2 + f(x)\right)^2 - f(x)^2 - f(1)^2 + f(2y)\\ &= 1 + 2f(x) + f(x)^2 - f(x)^2 - 1 + f(2y)\\ &= 2f(x) + f(2y) = 2f(x) + 2f(y) = 2(f(x) + f(y))\end{aligned}$$

Obviously, if $f : Z[X] \to Z[X]$ is not the zero map, then $f(1) = 1$ and f satisfies the Cauchy functional equation, so it is $Z-$linear and therefore it will be determined by the images of $X, X^2, X^3, \ldots$, so that

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_01 + a_1f(X) + a_2f(X^2) + \cdots + a_nf(X^n)$$

However, if f satisfies (1), the images of $X, X^2, X^3, \ldots$ are related as the next lemma shows.

Lemma 1.3. *If $f : Z[X] \to Z[X]$ is a solution of* (1) *with* $f(1) = 1$ *then for any n,*

$$f(X^n) = f(X)^n.$$

P r o o f. From (2), $f(x^2) = f(x)^2$ for all $x \in Z[X]$, in particular for $x = X$. We proceed by induction on n, but have to distinguish the case n even (trivial) from n odd where we write $n = 2k+1$ and use $Z-$linearity (proved in Proposition 1.2) together with the expansions of $f\left((X^{2k} + X)^2\right) = f\left(X^{2k} + X\right)^2$. □

Thus, if f is not the trivial solution, then

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_01 + a_1u + a_2u^2 + \cdots + a_nu^n \tag{6}$$

where $u \in Z[X]$ may be chosen arbitrarily and since all maps given by (6) satisfy equation (1), we can assert the following.

Theorem 1.4. *A map $f : Z[X] \to Z[X]$ is a solution of the functional equation* (1) *if only if f is either the zero map, or, for any polynomial $P(X)$ of $Z[X]$,*

$$f(P(X)) = P(u)$$

where $u \in Z[X]$ may be chosen arbitrarily.

So, functional equation (1) has less solutions than the Cauchy equation, because in the latter it is possible to select the images of $X, X^2, X^3, \ldots$, arbitrarily.

2. The functional equation $f(x^m + y) = f(x)^m + f(y)$ for f defined over $Z[X]$. Here we show that the functional equation

$$f(x^m + y) = f(x)^m + f(y) \tag{7}$$

for any fixed m ($m > 2$) is equivalent to functional equation (1) when m is even, but has more solutions when m is odd.

To begin with, remark that a similar argument to that of Lemma 1.1 would now prove the following:

Lemma 2.1. *If $f : Z[X] \to Z[X]$ satisfies functional equation (7), for any m fixed ($m \geq 2$), then f sends zero into zero, and moreover, f satisfies:*

$$f(x^m) = f(x)^m \tag{8}$$

for all x in $Z[X]$, and

$$f\left(\sum_{i=1}^{r} \alpha_i x_i^m + y\right) = \sum_{i=1}^{r} \alpha_i f(x_i)^m + f(y) \tag{9}$$

where $\alpha_1, \dots, \alpha_r$ are integers and r is arbitrary.

Observe that by (8), $f(1) = f(1)^m$, so that the image of 1 may be 0, 1 or even -1 when m is odd.

Proposition 2.2. *With the preceding hypotheses,in accordance with $f(1)$ being $0, 1$ or -1, we have that $f(\alpha + y)$ is either $f(y)$, $\alpha + f(y)$ or $-\alpha + f(y)$, respectively.*

P r o o f. Immediate, setting $r = 1$ and $x_1 = 1$ in (9). □

Remark that by setting $y = 0$ in the preceding proposition, we get that the image of α (for any integer α) is either 0, α, or $-\alpha$, depending on $f(1)$ being $0, 1$, or -1, respectively.

Now we shall show that the solutions of (7) satisfy the Cauchy functional equation, but before we need the following lemma.

Lemma 2.3. *For any integer $m \geq 1$, there are integers $\alpha, \delta, \alpha_0, \dots, \alpha_{m-1}$, with $\alpha \neq 0$ and $\sum_{r=0}^{m-1} \alpha_r = 0$ such that,*

$$\alpha T + \delta = \sum_{r=0}^{m-1} \alpha_r (T + r)^m,$$

where T is an indeterminate. As a consequence, for any x in $Z[X]$, the following relation holds:

$$\alpha x + \delta = \sum_{r=0}^{m-1} \alpha_r (x + r)^m.$$

P r o o f. If g is a polynomial in the indeterminate T and $\Delta g(T) :=$

$g(T+1) - g(T)$, then iterating the operator Δ, we get easily by induction

$$\Delta^s g(T) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} g(T+s-i)$$

Using this result and directly computing $\Delta^s g(T)$ with $s = m-1$ and $g(T) = T^m$, we get (cf [2, Theorem 402]):

$$m!T + \delta = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (T+m-1-i)^m$$

which proves the lemma, because $\sum_{r=0}^{m-1} \alpha_r = \sum_{r=0}^{m-1} (-1)^{m-1-r} \binom{m-1}{r} = 0.$ $\square$

Using the preceding propositions and proceeding as in Proposition 1.2, (but working with αx instead of $2x$), we get

Proposition 2.4. *If $f : Z[X] \to Z[X]$ satisfies* (7)*, with m fixed ($m \geq 2$), then f also satisfies the Cauchy functional equation. And in the particular case that $f(1) = 0$, then f is necessarily the zero map.*

Therefore, if $f : Z[X] \to Z[X]$ is a non-trivial solution of (7), then

$f(a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n) = a_0 f(1) + a_1 f(X) + a_2 f(X^2) + \cdots + a_n f(X^n)$.

We next prove that the image $f(X)$ of X already determines the images of $X^2, X^3, \ldots$, but first we need the following result.

Lemma 2.5. *For any integer $m \geq 2$, there are integers $\beta, \gamma, \beta_0, \ldots, \beta_{m-1}$, with $\beta \neq 0$ such that for any $x \in Z[X]$, we have*

$$\beta x^2 + \gamma = \sum_{r=0}^{m-1} \beta_r (x+r)^m.$$

P r o o f. Similar that of Lemma 2.3, by setting $s = m-2$ and $g(T) = T^m$. $\square$

Proposition 2.6. *Let $f : Z[X] \to Z[X]$ be a solution of* (7) *for a fixed m, ($m > 2$). If $f(1) = 1$ then f also satisfies*

$$f(x^2 + y) = f(x)^2 + f(y)$$

and if $f(1) = -1$ then f satisfies

$$f(x^2 + y) = -f(x)^2 + f(y) \tag{10}$$

P r o o f. Analogous to the proof of Proposition 1.2, but working now with the expression βx^2 instead of $2x$. $\square$

Therefore, when $f(1) = 1$ and $u := f(X)$, we get

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_0 + a_1u + a_2u^2 + \cdots + a_nu^n$$

and when $f(1) = -1$, we obtain

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = -a_0 + a_1u - a_2u^2 + \cdots + (-1)^{n+1}a_nu^n$$

since (proceeding as in Proposition 1.3) when f satisfies (10), we obtain

$$f(X^n) = (-1)^{n+1}f(X)^n.$$

And as all these maps obviously satisfy the functional equation (7), for any $u \in Z[X]$, we finally get

Theorem 2.7. *The only maps $f : Z[X] \to Z[X]$ satisfying the functional equation*

$$f(x^m + y) = f(x)^m + f(y)$$

for a fixed m ($m \geq 2$), are the zero map and the maps given by

$$f(P(X)) = P(u)$$

or

$$f(P(X)) = -P(v)$$

(the latter occurring only in case m is odd) where $u = f(X)$ and $v = -f(X)$ may be chosen arbitrarily in $Z[X]$.

Acknowledgment. The author is grateful to professor J-L. García-Roig (Barcelona) for his suggestions and encouragement during the preparation of this paper.

REFERENCES

[1] J. Aczél, J. Dhombres. Functional Equations in Several Variables. Cambridge Univ. Press, 1989.

[2] G. H. Hardy, E. M. Wright. An introduction to the Theory of Numbers, 5th ed. Clarendon Press, Oxford, 1979.

Emma Martín Gutiérrez
E.T.S. de Arquitectura de La Coruña
Campus da Zapateira
15071 La Coruña
Spain

Received November 13, 1996