Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **34** (2008), 455–466

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

STEFFENSEN METHODS FOR SOLVING GENERALIZED EQUATIONS

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Communicated by A. L. Dontchev

Abstract. We provide a local convergence analysis for Steffensen's method in order to solve a generalized equation in a Banach space setting. Using well known fixed point theorems for set–valued maps [13] and Hölder type conditions introduced by us in [2] for nonlinear equations, we obtain the superlinear local convergence of Steffensen's method. Our results compare favorably with related ones obtained in [11].

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution x^* of the generalized equation

$$0 \in F(x) + G(x), \tag{1.1}$$

2000 *Mathematics Subject Classification:* 65G99, 65K10, 47H04.
Key words: Steffensen's method, Banach space, set–valued mapping, generalized equations, Aubin continuity, divided difference, Newton's method.

where F is a continuous mapping from an open subset D of a Banach space X into itself, and G is a set–valued map from X into the subsets of X with closed graph. We approximate x^* using Steffensen's method [2], [4], [11]:

$$0 \in F(x_k) + [x_k, g(x_k); F](x_{k+1} - x_k) + G(x_{k+1}), \ (x_0 \in D), \ (k \in \mathbb{N}), \tag{1.2}$$

where $g : D \longrightarrow X$ is a continuous mapping, and $[x, y; F] \in \mathcal{L}(X)$ is a divided difference of order one satisfying

$$[x, y; F]\,(x - y) = F(x) - F(y) \ \text{ for all } \ x, y \in X \ \text{ with } \ x \neq y. \tag{1.3}$$

Note that if F is Fréchet–differentiable at x then $[x, x; F] = F'(x)$.

The main advantage of our method (1.2) is that it does not need to evaluate any Fréchet derivative. Moreover our method extends several methods and allows to have a finer error bounds on the distances $\|x_k - x^*\|$ $(k \geq 0)$. This last observation is very important in computational mathematics [4].

If $G \equiv 0$ in (1.2), then we obtain Steffensen's method, studied by us [2]–[4] and others [1], [11]. Moreover if $g(x) = x$ or $g(x_k) = x_{k-1}$, then we obtain the classical Newton's method and Secant method respectively [3], [6], [9]. Furthermore, if F is Fréchet–différentiable and $g(x) = x$, method (1.2) reduces to Newton's method, studied in [2], [4], [7], [8] under various conditions.

In particular, Hilout in [11] using condition introduced in [2] (see hypothesis $(\mathcal{H}1)$) provided a local convergence analysis for Steffensen's method (1.2). Here, we are motivated by optimization considerations. Using weaker conditions and under less computational cost, we also show the superlinear local convergence of Steffensen's method. Moreover our approach has the additional advantages:

(a) smaller radius of convergence;

and

(b) a larger choice of initial guesses x_0.

Finally, note that optimization problems, systems of linear and nonlinear complementarity problems, equilibrium problems, variational problems can be formulated like equation (1.1) [6], [13]–[17].

2. Preliminaries. In order to make the paper as self contained as possible we recall some terminology introduced in [3], [5], [7], [11]. The distance from a point x to a set A in the metric space (Z, ρ) is defined by $\operatorname{dist}(x, A) = \inf\{\rho(x, y),\ y \in A\}$. The excess e from the set A to the set C is given by

$e(A,C) = \sup\{\text{dist}\,(x,C),\ x \in A\}$. Let $\Lambda : X \rightrightarrows X$ be a set–valued map, we denote by $gph\,\Lambda = \{(x,y) \in X \times X,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$ is the inverse of Λ. We call $B_r(x)$ the closed ball centered at x with radius r.

Definition 2.1. *A set–valued Λ is said to be pseudo–Lipschitz around $(x_0, y_0) \in gph\,\Lambda$ with modulus M if there exist constants a and b such that*

$$\sup_{z\in\Lambda(y')\cap B_a(y_0)} \text{dist}\,(z, \Lambda(y'')) \le M\|y' - y''\|, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{2.1}$$

We have an equivalent definition in terms of excess by replacing the inequality (2.1) by

$$e(\Lambda(y') \cap B_a(y_0), \Lambda(y'')) \le M\|y' - y''\|, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{2.2}$$

The pseudo–Lipschitzness property has been introduced by Aubin [5] and he was the first to define this concept as a continuity property. This property is also called “Aubin continuity”. For more characterizations and applications of this concept, the reader could be referred to ([17] and the references given there).

We will need the following lemma (due to Dontchev and Hager [7]).

Lemma 2.2. *Let (Z, ρ) be a complete metric space, let ϕ a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \le \lambda < 1$ and*
(a) $\text{dist}\,(\eta_0, \phi(\eta_0)) \le r(1 - \lambda)$,
(b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \le \lambda\, \rho(x_1, x_2),\ \ \forall x_1, x_2 \in B_r(\eta_0)$,
then ϕ has a fixed–point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

Lemma 2.2 is a generalization of a fixed–point theorem given in [13] where in assertion (b) of the Lemma 2.2 the excess e is replaced by the Pompeiu–Hausdorff distance. In the sequel, the distance ρ in Lemma 2.2 is replaced by the norm.

We suppose that, for every distinct points x and y in a neighborhood V of x^*, there exists a first order divided difference of F at these points. We will need the following assumptions on a neighborhood V of x^*:

($\mathcal{H}0$) $\|g(x) - g(x^*)\| \leq \alpha_0 \|x - x^*\|$, $\alpha_0 \in [0,1]$ for all $x \in V$ and $g(x^*) = x^*$,

($\mathcal{H}1$) There exist ν_0, $\nu_1 > 0$ such that for all x, y in V

$$\|[x, x^*; F] - [x, g(x); F]\| \leq \nu_0 \|x^* - g(x)\|^p,$$

$$\|[x, y; F] - [x, g(x); F]\| \leq \nu_1 \|y - g(x)\|^p, \; p \in [0,1],$$

The assumption ($\mathcal{H}1$) is called a (ν_0, ν_1, p)–Hölder continuity property of divided difference. Note that if $p = 1$ then F has a Lipschitz continuous divided difference.

($\mathcal{H}2$) The set–valued map $(F(x^*) + G)^{-1}$ is M–pseudo–Lipschitz around $(0, x^*)$,

($\mathcal{H}3$) For all $x, y \in V$, we have $\|[x, y; F]\| \leq d$, $\|F(x) - F(x^*)\| \leq d_0 \, \|x - x^*\|$, and $Md < 1$.

Remark 2.3. The assumption ($\mathcal{H}3$) implies that F is d–Lipschitz on V.

Remark 2.4. Hernández and Rubio [9, 10] show a semilocal result of convergence of the Secant method to solve a nonlinear equation using ω–conditioned divided difference, i.e., one replaces in ($\mathcal{H}1$) the right term of the inequality by $\omega(\|x - u\|, \|y - v\|)$ where ω from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}_+$ is a continuous nondecreasing function in both variables.

3. Local convergence analysis. We need to introduce some standard notations. First, let us define the set–valued map $Q : X \rightrightarrows X$ by

$$Q(x) = F(x^*) + G(x). \tag{3.1}$$

For $k \in \mathbb{N}$ and x_k defined in (1.2), we consider the quantity

$$Z_k(x) := F(x^*) - F(x_k) - [x_k, g(x_k); F](x - x_k). \tag{3.2}$$

Finally, define the set–valued map $\psi_k : X \rightrightarrows X$ by

$$\psi_k(x) := Q^{-1}(Z_k(x)). \tag{3.3}$$

We provide the main local convergence result:

Theorem 3.1. *We suppose that assumptions* $(\mathcal{H}0)$–$(\mathcal{H}3)$ *are satisfied. For every constant* $C > \dfrac{M\,\nu_0\,\alpha_0^p}{1 - M\,d} = C_0$, *one can find* $\delta > 0$ *such that for every starting point* x_0 *in* $B_\delta(x^*)$ *(*x_0 *and* x^* *distinct), there exists a sequence* (x_k) *defined by* (1.2) *which satisfies*

$$\|x_{k+1} - x^*\| \leq C\|x_k - x^*\|^{p+1}. \tag{3.4}$$

The proof of Theorem 3.1 is by induction on k, we first state a result which is the starting point of our algorithm.

Proposition 3.2. *Under the assumptions of Theorem* 3.1, *there exists* $\delta > 0$ *such that for every starting point* x_0 *in* $B_\delta(x^*)$ *(*x_0 *and* x^* *distinct), the set–valued map* ψ_0 *has a fixed point* x_1 *in* $B_\delta(x^*)$ *satisfying*

$$\|x_1 - x^*\| \leq C\|x_0 - x^*\|^{p+1}. \tag{3.5}$$

Remark 3.3. The point x_1 is a fixed point of ψ_0 if and only if the following holds

$$0 \in F(x_0) + [x_0, g(x_0); F](x_1 - x_0) + G(x_1). \tag{3.6}$$

An easy computation of x_k shows that the set–valued mapping ψ_k has a fixed point x_{k+1} in X. This process is useful to prove the existence of (x_k) satisfying (1.2).

P r o o f o f P r o p o s i t i o n 3.2. By hypothesis $(\mathcal{H}2)$ there exist positive numbers M, a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M\|y' - y''\|, \ \forall y', y'' \in B_b(0). \tag{3.7}$$

Fix $\delta > 0$ such that $B_\delta(x^*) \subseteq V \subseteq D$ and

$$\delta < \delta_0 = \min \left\{ a \ ; \ \sqrt[p+1]{\frac{b}{4 \ \nu_1 \ (1+\alpha_0)^p}} \ ; \ \frac{1}{\sqrt[p]{C}} \ ; \ \frac{b}{2\, d_0} \right\}. \tag{3.8}$$

The main idea of the proof of Proposition 3.2 is to show that both assertions (a) and (b) of Lemma 2.2 hold; where $\eta_0 := x^*$, ϕ is the function ψ_0 defined by (3.3) and where r and λ are numbers to be set. According to the definition of the excess e, we have

$$\text{dist}\,(x^*, \psi_0(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \psi_0(x^*) \Big). \tag{3.9}$$

Note that for $x \in B_\delta(x^*)$ using $(\mathcal{H}0)$ we can have

$$\| g(x) - x^* \| \leq \| g(x) - g(x^*) \| \leq \alpha_0 \| x - x^* \| \leq \| x - x^* \| \leq \delta,$$

which implies $g(x) \in B_\delta(x^*)$. Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distinct) we have

$$\| Z_0(x^*) \| = \| F(x^*) - F(x_0) - [x_0, g(x_0); F](x^* - x_0) \|.$$

By assumptions $(\mathcal{H}0)$–$(\mathcal{H}1)$ we deduce

$$\begin{aligned} \| Z_0(x^*) \| &= \| \Big([x_0, x^*; F] - [x_0, g(x_0); F] \Big) (x^* - x_0) \| \\ &\leq \| [x_0, x^*; F] - [x_0, g(x_0); F] \| \ \| x^* - x_0 \| \\ &\leq \nu_0 \ \| x^* - g(x_0) \|^p \ \| x^* - x_0 \| \\ &\leq \nu_0 \ \alpha^p \ \| x^* - x_0 \|^{p+1} \end{aligned} \tag{3.10}$$

Then (3.8) yields, $Z_0(x^*) \in B_b(0)$.
Using (3.7) we have

$$\begin{aligned} e\Big(Q^{-1}(0) \cap B_\delta(x^*), \psi_0(x^*) \Big) &= e\Big(Q^{-1}(0) \cap B_\delta(x^*), Q^{-1}[Z_0(x^*)] \Big) \\ &\leq M \ \nu_0 \ \alpha^p \ \| x^* - x_0 \|^{p+1} \end{aligned} \tag{3.11}$$

By the inequality (3.9), we get

$$\text{dist}\,(x^*, \psi_0(x^*)) \quad \leq \quad M\ \nu_0\ \alpha^p\ \|x^* - x_0\|^{p+1}. \tag{3.12}$$

Since $C(1 - M\,d) > M\ \nu_0\ \alpha_0^p$, there exists $\lambda \in [M\,d, 1[$ such that $C(1-\lambda) \geq M\ \nu_0\ \alpha_0^p$ and

$$\text{dist}\,(x^*, \psi_0(x^*)) \leq C\ (1-\lambda)\ \|x_0 - x^*\|^{p+1}. \tag{3.13}$$

By setting $r := r_0 = C\ \|x_0 - x^*\|^{p+1}$ we can deduce from the inequality (3.13) that the assertion (a) in Lemma 2.2 is satisfied.
Now, we show that condition (b) of Lemma 2.2 is satisfied.
By (3.8) we have $r_0 \leq \delta \leq a$ and moreover for $x \in B_\delta(x^*)$ we have

(3.14)
$$\begin{array}{rcl} \|Z_0(x)\| & = & \|F(x^*) - F(x_0) - [x_0, g(x_0); F](x - x_0)\| \\ & \leq & \|F(x^*) - F(x)\| + \|F(x) - F(x_0) - [x_0, g(x_0); F](x - x_0)\| \\ & \leq & \|F(x^*) - F(x)\| + \|[x_0, x; F] - [x_0, g(x_0); F]\|\ \|x - x_0\| \end{array}$$

Using the assumptions ($\mathcal{H}0$)–($\mathcal{H}1$) and ($\mathcal{H}3$) we obtain

$$\begin{array}{rcl} \|Z_0(x)\| & \leq & d\ \|x^* - x\| + \nu_1\ \|x - g(x_0)\|^p\ \|x - x_0\| \\ & \leq & d\ \|x^* - x\| + \nu_1\ (\|x - x^*\| + \|x^* - g(x_0)\|)^p\ \|x - x_0\| \\ & \leq & d\ \delta + \nu_1\ (1 + \alpha_0)^p\ \delta^p\ (2\delta) = d\ \delta + 2\ \nu_1\ (1+\alpha_0)^p\ \delta^{p+1}. \end{array} \tag{3.15}$$

Then by (3.8) we deduce that for all $x \in B_\delta(x^*)$ we have $Z_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$ we have

$$e(\psi_0(x') \cap B_{r_0}(x^*), \psi_0(x'')) \leq e(\psi_0(x') \cap B_\delta(x^*), \psi_0(x'')),$$

which yields by (3.7)

$$\begin{array}{rcl} e(\psi_0(x') \cap B_{r_0}(x^*), \psi_0(x'')) & \leq & M\ \|Z_0(x') - Z_0(x'')\| \\ & \leq & M\ \|[x_0, g(x_0); f]\|\ \|x'' - x'\| \end{array} \tag{3.16}$$

Using ($\mathcal{H}3$) and the fact that $\lambda \geq M\,d$, we obtain

$$e(\psi_0(x') \cap B_{r_0}(x^*), \psi_0(x'')) \leq M\ d\,\|x'' - x'\| \leq \lambda\ \|x'' - x'\| \tag{3.17}$$

and thus condition (b) of Lemma 2.2 is satisfied. Since both conditions of Lemma 2.2 are fulfilled, we can deduce the existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ψ_0. This finishes the proof of Proposition 3.2. □

Proof of Theorem 3.1. Keeping $\eta_0 = x^*$ and setting $r := r_k = C\|x^* - x_k\|^{p+1}$, the application of Proposition 3.2 to the map ψ_k gives the existence of a fixed point x_{k+1} for ψ_k which is an element of $B_{r_k}(x^*)$. This last fact implies the inequality (3.4), which is the desired conclusion. □

Example 3.4. *Simple example illustrating the algorithm presented in this paper is given by a variational inequalities problems, i.e., if K is a convex set in $\mathbb{R}^n$ and h is a function from K to $\mathbb{R}^n$, the variational inequality problem consists of seeking k_0 in K such that*

$$\text{(3.18)} \qquad \textit{For each } k \in K, \quad (h(k_0), k - k_0) \ \geq \ 0$$

where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{R}^n$.

Let $\mathcal{I}_K$ be a convex indicator function of K and ∂ denotes the subdifferential operator. Then the problem (3.18) *is equivalent to problem*

$$\text{(3.19)} \qquad 0 \in h(k_0) + H(k_0)$$

with $H = \partial \mathcal{I}_K$. H is also called the normal cone of K. The variational inequality problem (3.18) *is equivalent to* (3.19) *which is a generalized equation in the form* (1.1)*. Consequently, the problem* (3.18) *can be studied using our method* (1.2)*.*

Remark 3.5. In order for us to compare our results with the corresponding ones in [11], let us introduce stronger conditions used in [11] to prove a result similar to Theorem 3.1

$(\mathcal{H}0)'$ g is α–Lipschitz on V, $\alpha \in [0,1]$ and $g(x^*) = x^*$,

$(\mathcal{H}1)'$ There exists $\nu > 0$ such that for all x, y, u and w in V

$$\|[x,y;F] - [u,w;F]\| \leq \nu(\|x-u\|^p + \|y-w\|^p),\ p \in [0,1],$$

$(\mathcal{H}3)'$ For all $x, y \in V$, we have $|||[x, y; f]||| \leq d$ and $M\, d < 1$.

Define also parameters δ_0' and C_0' by

$$\delta_0' = \min\left\{a\ ;\ \sqrt[p+1]{\frac{b}{4\ \nu\ (1+\alpha)^p}}\ ;\ \frac{1}{\sqrt[p]{C'}}\ ;\ \frac{b}{2\, d}\right\}, \tag{3.20}$$

and

$$C_0' = \frac{M\, \nu\, \alpha^p}{1 - M\, d}. \tag{3.21}$$

Clearly,

$$\nu_0 \leq \nu_1 \leq \nu, \tag{3.22}$$

$$\alpha_0 \leq \alpha, \tag{3.23}$$

and

$$d_0 \leq d, \tag{3.24}$$

hold in general and $\frac{\nu}{\nu_0}$, $\frac{\nu}{\nu_1}$, $\frac{\alpha}{\alpha_0}$ and $\frac{d}{d_0}$ can be arbitrarily large [3], [4]. It then follows from the definition of C, C' ($C' > C_0'$), (3.8) and (3.20)–(3.24) that

$$C \leq C', \tag{3.25}$$

and

$$\delta_0' \leq \delta_0. \tag{3.26}$$

Moreover in case any of (3.22)–(3.24) holds as a strict inequality, then so do (3.25) and (3.26). Hence, the claims us made in the introduction have been justified.

4. Variant of method and conclusion. In this section we consider a variant of Steffensen–type algorithm (1.2) by replacing in the first argument of divided difference x_k by $y_k = \beta\ x_k + (1-\beta)\ x_{k-1}$, more precisely, we associate to (1.1) the following algorithm ($k = 1, 2, \dots$)

$$\begin{cases} x_0 \text{ and } x_1 \text{ are given as starting points} \\ y_k = \beta\ x_k + (1-\beta)\ x_{k-1};\ \beta \text{ is fixed in } [0,1[\\ 0 \in F(x_k) + [y_k, g(x_k); F]\,(x_{k+1} - x_k) + G(x_{k+1}) \end{cases} \tag{4.1}$$

Note that this method is considered in [12] in the particular case $g(x) = x$. The local convergence result of algorithm (4.1) is as follows

Theorem 4.1. *Suppose that* $(\mathcal{H}0)$–$(\mathcal{H}3)$ *are checked. For every* $C' > \dfrac{M\nu_1[2(1-\beta)^p + \alpha_0^p]}{1 - M\,d}$*, there exist* $\gamma > 0$ *such that, for every distinct starting points* x_0 *and* x_1 *in* $B_\gamma(x^*)$ *and a sequence* (x_k) *defined by (4.1) which satisfies:*

$$\|x_{k+1} - x^*\| \leq C' \|x_k - x^*\| \max\,\{\|x_k - x^*\|^p, \|x_{k-1} - x^*\|^p\}. \tag{4.2}$$

The proof of Theorem 4.1 is almost identical to Theorem 3.1. It is enough to make some modifications by replacing the mappings (3.2) and (3.3) by $Z'_k(x) := F(x^*) - F(x_k) - [y_k, g(x_k); F](x - x_k)$ and $\psi'_k(x) := Q^{-1}(Z'_k(x))$ respectively and choosing the constant γ such that

$$\gamma < \min\left\{a\ ;\ \sqrt[p+1]{\frac{b}{4\,\nu_1\,\left(2^p\,(1-\beta)^p + \alpha_0^p + 1\right)}}\ ;\ \frac{1}{\sqrt[p]{C'}}\ ;\ \frac{b}{2\,d_0}\right\}. \tag{4.3}$$

□

Remark 4.2. A remark identical to Remark 3.5 can now follow for Theorem 4.1.

Conclusion. We provided a local convergence, for Steffensen–type methods for solving generalized equations. Method (1.2) generalizes the Steffensen's method restricted to nonlinear equations [2].

For $\beta = 1$, our method (4.1) is no longer valid, but if F is Fréchet differentiable (4.1) is equivalent to Newton–type method (see [6]) to solve (1.1), we have then the quadratically convergence result for $p = 1$.

Our results have improved the corresponding ones in [11].

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Received April 25, 2007
Revised July 20, 2007