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# TEST FOR INDEPENDENCE OF THE VARIABLES WITH MISSING ELEMENTS IN ONE AND THE SAME COLUMN OF THE EMPIRICAL CORRELATION MATRIX

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Abstract. We consider variables with joint multivariate normal distribution and suppose that the sample correlation matrix has missing elements, located in one and the same column. Under these assumptions we derive the maximum likelihood ratio test for independence of the variables. We obtain also the maximum likelihood estimations for the missing values.

**1. Introduction.** We consider variables with joint multivariate normal distribution and suppose that the sample correlation matrix has missing elements, located in one and the same column. This might be due to a loss during the keeping or the transportation to the researcher. Another reason is when a researcher "A" has the full matrix of the observations on $n$ variables $X_1, \ldots, X_n$, but he decides to include in considerations an additional variable $X_{n+1}$. Assume that

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his colleague "B" works in a competitive firm and has the data matrix for the variables $X_{k+1}, \dots, X_n, X_{n+1}$. Suppose the researcher "A" succeeds to get from "B" only the empirical correlation coefficients $r_{i\,n+1}, i = k+1, \dots, n$, instead of the data vector of the observations on $X_{n+1}$, which is $m \times 1$ and $m$ is enough greater than $n-k$ ($m \gg n-k$). In this situation "A" will have the complete empirical correlation matrix for the variables $X_1, \dots, X_n, X_{n+1}$, with the exception of the coefficients $r_{i\,n+1}, i = 1, \dots, k$.

Recently, the question of evaluating the return to a given program or treatment has received considerable attention in the economics, statistics and medical literatures [2, 5, 7, 8]. The derived densities of the outcome gain distributions necessarily depend on the unidentified cross-regime correlation coefficient.

Roy's model of self-selection has been applied extensively in economics to explain individual choice between two alternatives or sectors. Since an agent's earnings are observed only in the sector of choice under partial observability, the correlation coefficient between the disturbance terms in the earnings equations is not identified. Similar investigations with one missing element in the correlation matrix have been made by Heckman and Honore [1], Vijverberg [11], Koop and Poirier [4], Sareen [6].

In the present paper we assume that the researcher is interested in $n+1$ random variables, which are multivariate normal distributed but he has only the empirical correlation matrix $\mathbf{R}$, in which the elements $r_{i\,n+1}, i = 1, \dots, k$ are missing. In this case we obtain the maximum likelihood estimations for the correlation coefficients $\rho_{i\,n+1}, i = 1, \dots, k$, which are elements of the theoretical correlation matrix $\mathbf{P}$. We derive also the maximum likelihood ratio test for the hypothesis for the independence of the $n+1$ random variables, i.e.

$$H_0 : \mathbf{P} = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix.

**2. The likelihood ratio test derivation.** We begin with one auxiliary Theorem.

**Theorem 1.** *Let $\omega_{ij}$, $1 \le i \le j \le n$ be random variables with joint distribution - the Wishart distribution $W_n(m, \mathbf{\Sigma})$. Consider the set of random variables $V = \{\tau_i,\ i = 1, \dots, n,\ \nu_{ij},\ 1 \le i < j \le n\}$, such that*

$$\tau_i = \omega_{ii},\ i = 1, \dots, n,$$

$$\nu_{ij} = \frac{\omega_{ij}}{\sqrt{\omega_{ii}\omega_{jj}}},\ 1 \le i < j \le n.$$

*The joint density function of the random variables from* $V$ *has the form*

$$f_V(t_1,\dots,t_n,y_{ij},1\le i<j\le n)$$
$$= \frac{(t_1\dots t_n)^{\frac{m}{2}-1}}{2^{\frac{nm}{2}}\pi^{\frac{n(n-1)}{4}}(\det(\mathbf{\Sigma}))^{\frac{m}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{m-i+1}{2}\right)}(\det(\mathbf{Y}_n))^{\frac{m-n-1}{2}}e^{-\frac{1}{2}tr(\mathbf{T}\mathbf{Y}_n\mathbf{T}\mathbf{\Sigma}^{-1})}I_E,$$

*where*

– $\Gamma(\cdot)$ *is the well known Gamma function;*

– $\mathbf{T}$ *and* $\mathbf{Y}_n$ *are the matrices*

$$\mathbf{T} = \mathrm{diag}(\sqrt{t_1},\dots,\sqrt{t_n}) = \begin{pmatrix} \sqrt{t_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{t_n} \end{pmatrix},$$

$$\mathbf{Y}_n = \begin{pmatrix} 1 & y_{12} & \cdots & y_{1n} \\ y_{12} & 1 & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1n} & y_{2n} & \cdots & 1 \end{pmatrix};$$

– $E$ *is the set of all points* $(t_1,\dots,t_n,\ y_{ij},\ 1\le i<j\le n)$ *in the real space* $R^{n(n+1)/2}$, *such that* $t_i > 0,\ i=1,\dots,n$ *and the matrix* $\mathbf{Y}_n$ *is positive definite, and* $I_E$ *here and after denotes the indicator of a given set* $E$.

P r o o f. The inverse transformation formulas are

$$\omega_{ii} = \tau_i,\ i=1,\dots,n,$$

$$\omega_{ij} = \nu_{ij}\sqrt{\tau_i\,\tau_j},\ 1\le i<j\le n.$$

The Jacobian of the transformation is

$$J = \frac{\partial(\omega_{11},\dots,\omega_{nn},\omega_{12},\dots,\omega_{1n},\omega_{23},\dots,\omega_{n-1n})}{\partial(\tau_1,\dots,\tau_n,\nu_{12},\dots,\nu_{1n},\nu_{23},\dots,\nu_{n-1n})}$$

$$= \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sqrt{\tau_1\tau_2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \sqrt{\tau_{n-1}\tau_n} \end{pmatrix}.$$

Consequently

$$\det(J) = (\tau_1 \dots \tau_n)^{\frac{n-1}{2}}.$$

The joint density function of $\omega_{ij}$, $1 \le i \le j \le n$ has the form

$$f_{\omega_{ij}, 1\le i\le j\le n}(w_{ij}, 1 \le i \le j \le n) = \frac{(\det(\mathbf{W}))^{\frac{m-n-1}{2}} e^{-\frac{1}{2}tr(\mathbf{W}\mathbf{\Sigma}^{-1})}}{2^{\frac{nm}{2}} \pi^{\frac{n(n-1)}{4}} (\det(\mathbf{\Sigma}))^{\frac{m}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{m-i+1}{2}\right)} \, I_{\{\mathbf{W}>0\}},$$

where $\mathbf{W}$ is the symmetric matrix with elements $w_{ij}, 1 \le i \le j \le n$ and $\{\mathbf{W} > 0\}$ here and after denotes the set of all values for the unknown elements of a given real matrix $\mathbf{W}$, such that the matrix $\mathbf{W}$ is positive definite. Hence, the joint density function of the random variables from the set $V = \{\tau_i,\ i = 1, \dots, n,\ \nu_{ij},\ 1 \le i < j \le n\}$ will have the form

$$\begin{aligned} f_V(t_1, \dots, t_n, y_{ij}, 1 \le i < j \le n) \\ &= \frac{(\det(\mathbf{T}\mathbf{Y}_n\mathbf{T}))^{\frac{m-n-1}{2}} e^{-\frac{1}{2}tr(\mathbf{T}\mathbf{Y}_n\mathbf{T}\mathbf{\Sigma}^{-1})}}{2^{\frac{nm}{2}} \pi^{\frac{n(n-1)}{4}} (\det(\mathbf{\Sigma}))^{\frac{m}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{m-i+1}{2}\right)} (t_1 \dots t_n)^{\frac{n-1}{2}} I_E \end{aligned}$$

$$= \frac{(t_1 \dots t_n)^{\frac{m}{2}-1}}{2^{\frac{nm}{2}} \pi^{\frac{n(n-1)}{4}} (\det(\mathbf{\Sigma}))^{\frac{m}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{m-i+1}{2}\right)} (\det(\mathbf{Y}_n))^{\frac{m-n-1}{2}} e^{-\frac{1}{2}tr(\mathbf{T}\mathbf{Y}_n\mathbf{T}\mathbf{\Sigma}^{-1})} I_E. \quad \square$$

**Corollary 1.** *Under the conditions of Theorem 1, let* $\mathbf{\Sigma} = \mathbf{D}$, *where* $\mathbf{D}$ *is a diagonal matrix* $\mathbf{D} = \mathrm{diag}(\sigma_1^2, \dots, \sigma_n^2)$. *Then the set of random variables* $\{\tau_1, \dots, \tau_n\}$ *is independent of the set* $\{\nu_{ij},\ 1 \le i < j \le n\}$. *The variables* $\tau_1, \dots, \tau_n$ *are mutually independent and* $\tau_i \sim \sigma_i^2\chi^2(m)$, $i = 1, \dots, n$. *The random* *variables $\nu_{ij}$, $1 \leq i < j \leq n$ has joint distribution $\Psi(m,n)$ with density function of the form*

$$f_{\nu_{ij},1\leq i<j\leq n}(y_{ij}, 1 \leq i < j \leq n) = \frac{\left[\Gamma\left(\frac{m}{2}\right)\right]^{n-1}}{\pi^{\frac{n(n-1)}{4}} \prod_{i=1}^{n-1} \Gamma\left(\frac{m-i}{2}\right)} (\det(\mathbf{Y}_n))^{\frac{m-n-1}{2}} I_{\{\mathbf{Y}_n>0\}}.$$

Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ be a sample from $N_{n+1}(\mu, \mathbf{\Sigma})$, $\mu$ is unknown. Assume that we wish to test the hypothesis
$H_0 : \mathbf{\Sigma} = \mathrm{diag}(\sigma_1^2, \dots, \sigma_{n+1}^2)$, $\sigma_i^2$ are unknown, $i = 1, \dots, n+1$
which is equivalent to $H_0 : \mathbf{P} = \mathbf{I}$) against the alternative
$H_1$ : no constraints on $\mathbf{\Sigma}$.

Suppose at first, that we don't hold the observations themselves, but we have $m$ and the empirical covariance matrix $\mathbf{S} = \{s_{ij}\}_{i,j=1}^{n+1}$, in which the elements $s_{in+1}, i = 1, \dots, k$ are unidentified. It is well known, that the joint distribution of the elements of the matrix $(m-1)\mathbf{S}$ is Wishart - $W_{n+1}(m-1, \mathbf{\Sigma})$. Consider the variables

$$\tau_i = (m-1)s_{ii}, \ i = 1, \dots, n+1$$

and

$$\nu_{ij} = \frac{(m-1)s_{ij}}{\sqrt{(m-1)s_{ii}(m-1)s_{jj}}} = r_{ij}, \ 1 \leq i < j \leq n+1,$$

where $r_{ij}$ is the corresponding element of the empirical correlation matrix $\mathbf{R}$.

• Let the hypothesis $H_0$ be true. According to Corollary 1, the distribution of the elements of the empirical correlation matrix $\mathbf{R}$ is $\Psi(m-1, n+1)$. The random variables $\tau_i = (m-1)s_{ii}$, $i = 1, \dots, n+1$ are mutually independent, they are independent of the correlation coefficients $r_{ij}$, $1 \leq i < j \leq n+1$ and $\tau_i = (m-1)s_{ii} \sim \sigma_i^2\chi^2(m-1)$, $i = 1, \dots, n+1$.

The next proposition can be found in [9].

**Proposition 1.** *Let $\nu_{ij}$, $1 \leq i < j \leq n$ be random variables with distribution $\Psi(m,n)$. The joint density function of the random variables from the set $V = \{\nu_{ij}, 1 \leq i < j \leq n\} \backslash \{\nu_{1n}, \dots, \nu_{kn}\}$, where $k$ is an integer, $1 \leq k \leq n-2$,* *has the form*

$$f_V(y_{ij},\ 1\le i<j\le n{-}1,\ y_{k+1n},\dots,y_{n-1n})=\frac{\left[\Gamma\left(\frac{m}{2}\right)\right]^{n-1}}{\pi^{\frac{n(n-1)-2k}{4}}\Gamma\left(\frac{m-n+k+1}{2}\right)\prod\limits_{i=1}^{n-2}\Gamma\left(\frac{m-i}{2}\right)}$$

$$\times\frac{(\det(\mathbf{Y}_n(\{n\})))^{\frac{m-n}{2}}(\det(\mathbf{Y}_n(\{1,\dots,k\})))^{\frac{m-n+k-1}{2}}}{(\det(\mathbf{Y}_n(\{1,\dots,k,n\})))^{\frac{m-n+k}{2}}}I_{\{\mathbf{Y}_n(\{n\})>0\}}\,I_{\{\mathbf{Y}_n(\{1,\dots,k\})>0\}},$$

*where* $\mathbf{A}(\{i_1,\dots,i_l\})$ *here and after denotes the matrix, which is obtained from the matrix* $\mathbf{A}$*, after deleting the rows and columns with numbers* $i_1,\dots,i_l$*.*

From this Propositions it follows that under $H_0$, the joint density function of the empirical correlation coefficients from the set $U=\{r_{ij},\ 1\le i<j\le n,\ r_{k+1n+1},\ \dots,\ r_{nn+1}\}$ will have the form

$$\begin{aligned}&f_U(y_{ij},\ 1\le i<j\le n,\ y_{k+1n+1},\dots,y_{nn+1})\\&\quad=\frac{\left[\Gamma\left(\frac{m-1}{2}\right)\right]^{n}}{\pi^{\frac{n(n+1)-2k}{4}}\Gamma\left(\frac{m-n+k-1}{2}\right)\prod\limits_{i=1}^{n-1}\Gamma\left(\frac{m-i-1}{2}\right)}I_{\{\mathbf{Y}_{n+1}(\{n+1\})>0\}}\,I_{\{\mathbf{Y}_{n+1}(\{1,\dots,k\})>0\}}\\&\quad\times\frac{(\det(\mathbf{Y}_{n+1}(\{n+1\})))^{\frac{m-n-2}{2}}(\det(\mathbf{Y}_{n+1}(\{1,\dots,k\})))^{\frac{m-n+k-3}{2}}}{(\det(\mathbf{Y}_{n+1}(\{1,\dots,k,n+1\})))^{\frac{m-n+k-2}{2}}}.\end{aligned}$$

Consequently, under $H_0$ the joint density function of our "observations": $(m-1)s_{11},\ \dots,\ (m-1)s_{n+1n+1},\ r_{12},\ \dots,\ r_{n-1n},\ r_{k+1n+1},\ \dots,\ r_{nn+1}$, i.e. the likelihood function $L_0$, will have the form

$$\begin{aligned}L_0&=\frac{(m-1)^{\frac{(n+1)(m-3)}{2}}}{2^{\frac{(n+1)(m-1)}{2}}\pi^{\frac{n(n+1)-2k}{4}}\Gamma\left(\frac{m-n+k-1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{m-i}{2}\right)}\\&\qquad\times\frac{(s_{11}\dots s_{n+1n+1})^{\frac{m-3}{2}}}{(\sigma_1\dots\sigma_{n+1})^{m-1}}e^{-\frac{(m-1)}{2}\sum\limits_{i=1}^{n+1}\frac{s_{ii}}{\sigma_i^2}}\\&\qquad\times\frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}}(\det(\mathbf{R}(\{1,\dots,k\})))^{\frac{m-n+k-3}{2}}}{(\det(\mathbf{R}(\{1,\dots,k,n+1\})))^{\frac{m-n+k-2}{2}}}.\end{aligned}$$

The maximum likelihood estimations for the unknown parameters $\sigma_1^2, \ldots, \sigma_{n+1}^2$ are $s_{11}, \ldots, s_{n+1n+1}$ respectively. The maxima $L_0^*$ of the likelihood function is

$$L_0^* = \frac{(m-1)^{\frac{(n+1)(m-3)}{2}}}{2^{\frac{(n+1)(m-1)}{2}} \pi^{\frac{n(n+1)-2k}{4}} \Gamma\left(\frac{m-n+k-1}{2}\right) \prod_{i=1}^{n} \Gamma\left(\frac{m-i}{2}\right)} \times \frac{e^{-\frac{(n+1)(m-1)}{2}}}{(s_{11} \ldots s_{n+1n+1})} \frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}} (\det(\mathbf{R}(\{1,\ldots,k\})))^{\frac{m-n+k-3}{2}}}{(\det(\mathbf{R}(\{1,\ldots,k,n+1\})))^{\frac{m-n+k-2}{2}}}.$$

• Let the hypothesis $H_1$ be true. Applying Theorem 1 for $m = m-1$ and $n = n+1$, we get the joint density function $f_1$ of the "observations": $(m-1)s_{11}, \ldots, (m-1)s_{n+1n+1}, r_{12}, \ldots, r_{n-1n}, r_{k+1n+1}, \ldots, r_{nn+1}$ and the unknown realizations $r_{1n+1}, \ldots, r_{kn+1}$ of the correlation coefficients $\rho_{1n+1}, \ldots, \rho_{kn+1}$:

$$f_1 = \frac{(m-1)^{\frac{(n+1)(m-3)}{2}} (s_{11} \ldots s_{n+1n+1})^{\frac{(m-3)}{2}}}{2^{\frac{(n+1)(m-1)}{2}} \pi^{\frac{n(n+1)}{4}} (\det(\mathbf{\Sigma}))^{\frac{(m-1)}{2}} \prod_{i=1}^{n+1} \Gamma\left(\frac{m-i}{2}\right)} \times (\det(\mathbf{R}))^{\frac{m-n-3}{2}} e^{-\frac{(m-1)}{2} tr(\mathbf{TRT\Sigma}^{-1})} I_{\{\mathbf{R}>0\}},$$

where $\mathbf{T}$ is the diagonal matrix $\mathbf{T} = \mathrm{diag}(\sqrt{s_{11}}, \ldots, \sqrt{s_{n+1n+1}})$. The likelihood function $L_1$ under $H_1$ is an integral of $f_1$ with respect to $r_{1n+1}, \ldots, r_{kn+1}$:

$$L_1 = \int_{-1}^{1} \ldots \int_{-1}^{1} f_1 \, dr_{1n+1} \ldots dr_{kn+1}.$$

Let us denote by $\sigma^{ij}$, $1 \le i \le j \le n+1$ the elements of the matrix $\mathbf{\Sigma}^{-1}$. It is easy to see that the trace

$$tr(\mathbf{TRT\Sigma}^{-1}) = \Xi + 2\sum_{i=1}^{k} \sigma^{in+1} \sqrt{s_{ii} s_{n+1n+1}} \, r_{in+1},$$

where

$$\text{(1)} \quad \Xi = \sum_{i=1}^{n+1} s_{ii} \, \sigma^{ii} + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sigma^{ij} \sqrt{s_{ii} s_{jj}} \, r_{ij} + 2 \sum_{i=k+1}^{n} \sigma^{in+1} \sqrt{s_{ii} s_{n+1n+1}} \, r_{in+1}.$$

Hence

$$L_1 = K(\det(\mathbf{\Sigma}^{-1}))^{\frac{(m-1)}{2}} e^{-\frac{(m-1)}{2}\Xi}\, J, \tag{2}$$

where

$$K = \frac{(m-1)^{\frac{(n+1)(m-3)}{2}} (s_{11}\dots s_{n+1n+1})^{\frac{(m-3)}{2}}}{2^{\frac{(n+1)(m-1)}{2}} \pi^{\frac{n(n+1)}{4}} \prod\limits_{i=1}^{n+1} \Gamma\left(\frac{m-i}{2}\right)},$$

$$J = \int\limits_{-1}^{1} \dots \int\limits_{-1}^{1} (\det(\mathbf{R}))^{\frac{m-n-3}{2}} e^{-(m-1)\sum\limits_{j=1}^{k}(\sigma^{jn+1}\sqrt{s_{jj}s_{n+1n+1}}r_{jn+1})} \times I_{\{\mathbf{R}>0\}}\, dr_{1n+1}\dots dr_{kn+1}.$$

The maximum likelihood estimations $\hat{\sigma}^{ij}$ for $\sigma^{ij}$, $1 \le i \le j \le n+1$ satisfy the system of equations

$$\begin{aligned} &\frac{\partial L_1}{\partial \sigma^{ii}} = 0, \quad i = 1, \dots, n+1 \\ &\frac{\partial L_1}{\partial \sigma^{ij}} = 0, \quad 1 \le i < j \le n+1 \end{aligned} \tag{3}$$

It is easy to see that

$$\frac{\partial \det(\mathbf{\Sigma}^{-1})}{\partial \sigma^{11}} = \det(\mathbf{\Sigma}^{-1}(\{1\})).$$

Hence the first equation of the system (3) is

$$\frac{\partial L_1}{\partial \sigma^{11}} = K\, J \frac{(m-1)}{2} (\det(\mathbf{\Sigma}^{-1}))^{\frac{(m-3)}{2}} e^{-\frac{(m-1)}{2}\Xi} \left[\det(\mathbf{\Sigma}^{-1}(\{1\})) - \det(\mathbf{\Sigma}^{-1}) s_{11}\right] = 0.$$

Consequently,

$$\frac{\det(\mathbf{\Sigma}^{-1}(\{1\}))}{\det(\mathbf{\Sigma}^{-1})} = s_{11}.$$

The left hand side of the above equation equals to the (1,1) element of the inverse matrix of the matrix $\mathbf{\Sigma}^{-1}$, i.e. the (1,1) element of the matrix $\mathbf{\Sigma}$. Consequently we have

$$\hat{\sigma}_{11} = s_{11}.$$

We conclude similarly that

$$\hat{\sigma}_{ii} = s_{ii}, \;\; i = 2, \ldots, n+1.$$

In the further considerations we need the following Theorem.

**Theorem 2.** *Let* $\mathbf{A} = \{a_{ij}\}_{i,j=1}^{n}$, $a_{ij} = a_{ji}$ *be a* $n \times n$ *symmetric matrix. For each element* $a_{pq}, p \neq q$ *of the matrix* $\mathbf{A}$*, the determinant* $\det(\mathbf{A})$ *can be written as*

$$\text{(4)} \quad \det(\mathbf{A}) = -\det(\mathbf{A}(\{p,q\}))a_{pq}^2 + 2(-1)^{q-p}\det(A(\{p\},\{q\})^0)a_{pq} + \det(A_{p,q}^0),$$

*where*

– $\mathbf{A}(\{p\},\{q\})^0$ *is the matrix, which is obtained from the matrix* $\mathbf{A}$ *after deleting its p'th row, q'th column and replacing the element* $a_{qp}$ *with zero;*

– $\mathbf{A}_{p,q}^0$ *is the matrix, which is obtained from the matrix* $\mathbf{A}$*, after replacing the elements* $a_{pq}$ *and* $a_{qp}$ *with zero.*

Proof. It is easy to see that

$$\text{(5)} \qquad \det(\mathbf{A}) = a_{pq}(-1)^{q-p}\det(\mathbf{A}(\{p\},\{q\})) + \det(\mathbf{A}^0),$$

where $\mathbf{A}^0$ is the matrix, which is obtained from the matrix $\mathbf{A}$ after replacing the element $a_{pq}$ with zero. Analogically

$$\text{(6)} \qquad \det(\mathbf{A}^0) = a_{qp}(-1)^{p-q}\det(\mathbf{A}(\{q\},\{p\})^0) + \det(\mathbf{A}_{p,q}^0)$$

and

$$\text{(7)} \quad \det(\mathbf{A}(\{p\},\{q\})) = a_{qp}(-1)^{p-q-1}\det(\mathbf{A}(\{p,q\})) + \det(\mathbf{A}(\{p\},\{q\})^0).$$

Since the matrix $\mathbf{A}$ is symmetric, from (5)–(7) we obtain (4). □

Put in the above Theorem $\mathbf{A} = \mathbf{\Sigma}^{-1}$, $n = n+1$, $p = 1$, $q = 2$. Then

$$\det(\mathbf{\Sigma}^{-1}) = -\det(\mathbf{\Sigma}^{-1}(\{1,2\}))(\sigma^{12})^2 - 2\det(\mathbf{\Sigma}^{-1}(\{1\},\{2\})^0)\sigma^{12} + \det((\mathbf{\Sigma}^{-1})_{1,2}^0).$$

Consequently,

$$\frac{\partial \det(\mathbf{\Sigma}^{-1})}{\partial \sigma^{12}} = -2\det(\mathbf{\Sigma}^{-1}(\{1,2\}))\sigma^{12} - 2\det(\mathbf{\Sigma}^{-1}(\{1\},\{2\})^{0})$$

$$= -2\det(\mathbf{\Sigma}^{-1}(\{1\},\{2\})).$$

Hence

$$\frac{\partial L_1}{\partial \sigma^{12}} = K\,J(m-1)(\det(\mathbf{\Sigma}^{-1}))^{\frac{(m-3)}{2}} e^{-\frac{(m-1)}{2}\Xi} \times \left[-\det(\mathbf{\Sigma}^{-1}(\{1\},\{2\})) - \det(\mathbf{\Sigma}^{-1})\sqrt{s_{11}s_{22}}\,r_{12}\right] = 0.$$

Therefore we get

$$\frac{-\det(\mathbf{\Sigma}^{-1}(\{1\},\{2\}))}{\det(\mathbf{\Sigma}^{-1})} = \sqrt{s_{11}s_{22}}\,r_{12} = s_{12}.$$

The left hand side of the above equation equals to the (1,2) element of the inverse matrix of the matrix $\mathbf{\Sigma^{-1}}$, i.e. the (1,2) element of the matrix $\mathbf{\Sigma}$. Consequently we get

$$\hat{\sigma}_{12} = s_{12}.$$

We conclude by analogy that

$$\hat{\sigma}_{ij} = s_{ij},\ \ 1 \le i < j \le n,$$

$$\hat{\sigma}_{in+1} = s_{in+1},\ \ i = k+1, \dots, n.$$

Let us apply Theorem 2 for every one of the elements $\sigma^{in+1}$, $i = 1, \dots, k$ of the matrix $\mathbf{\Sigma}^{-1}$. We get

$$\det(\mathbf{\Sigma}^{-1}) = -\det(\mathbf{\Sigma}^{-1}(\{i, n+1\}))(\sigma^{in+1})^2 + 2(-1)^{n-i+1}\det(\mathbf{\Sigma}^{-1}(\{i\},\{n+1\})^{0})\sigma^{in+1} + \det((\mathbf{\Sigma}^{-1})^{0}_{i,n+1}), \quad i = 1, \dots, k.$$

Hence

$$\frac{\partial \det(\mathbf{\Sigma}^{-1})}{\partial \sigma^{in+1}} = -2\det(\mathbf{\Sigma}^{-1}(\{i,n+1\}))\sigma^{in+1} + 2(-1)^{n-i+1}\det(\mathbf{\Sigma}^{-1}(\{i\},\{n+1\})^{0})$$

$$= 2(-1)^{n-i+1}[\det(\mathbf{\Sigma}^{-1}(\{i\},\{n+1\})^0) + (-1)^{n-i}\det(\mathbf{\Sigma}^{-1}(\{i,n+1\}))\sigma^{in+1}]$$
$$= 2(-1)^{n-i+1}\det(\mathbf{\Sigma}^{-1}(\{i\},\{n+1\})).$$

The integral $J$ depends of $\sigma^{in+1}$, $i=1,\dots,k$ and

$$\frac{\partial J}{\partial \sigma^{in+1}} = -(m-1)\sqrt{s_{n+1n+1}}\, J_i,$$

where

$$J_i = \sqrt{s_{ii}}\int_{-1}^{1}\dots\int_{-1}^{1}(\det(\mathbf{R}))^{\frac{m-n-3}{2}} r_{in+1} e^{-(m-1)\sum_{j=1}^{k}(\sigma^{jn+1}\sqrt{s_{jj}s_{n+1n+1}}r_{jn+1})}$$
$$\times I_{\{\mathbf{R}>0\}} dr_{1n+1}\dots dr_{kn+1}, \quad i=1,\dots,k.$$

Therefore

$$\frac{\partial L_1}{\partial \sigma^{in+1}} = K\,(m-1)(\det(\mathbf{\Sigma}^{-1}))^{\frac{(m-3)}{2}} e^{-\frac{(m-1)}{2}\Xi}$$
$$\times [(-1)^{n-i+1}\det(\mathbf{\Sigma}^{-1}(\{i\},\{n+1\}))\,J - \det(\mathbf{\Sigma}^{-1})\sqrt{s_{n+1n+1}}\,J_i] = 0,$$
$$i=1,\dots,k.$$

Consequently, for $i=1,\dots,k$ we have

$$\frac{(-1)^{n-i+1}\det(\mathbf{\Sigma}^{-1}(\{i\},\{n+1\}))}{\det(\mathbf{\Sigma}^{-1})} = \frac{\sqrt{s_{n+1n+1}}\,J_i}{J}.$$

The left hand side of the above equation equals to the $(i,\, n+1)$ element of the matrix $\mathbf{\Sigma}$, i.e. the element $\sigma_{in+1}$. Consequently for $\hat{\sigma}_{in+1}$, $i=1,\dots,k$ we get the equations

$$\hat{\sigma}_{in+1} = \sqrt{s_{n+1n+1}}\frac{\hat{J}_i}{\hat{J}}, \quad i=1,\dots,k, \tag{8}$$

where $\hat{J}_i$ and $\hat{J}$ are the integrals $J_i$ and $J$ respectively, in which we are substituted the maximum likelihood estimations $\hat{\sigma}^{in+1}$, $i=1,\dots,k$ for the unknown parameters $\sigma^{in+1}$, $i=1,\dots,k$.

Let us denote by $\hat{\mathbf{\Sigma}}^{-1}$ the maximum likelihood estimation of the unknown matrix $\mathbf{\Sigma}^{-1}$. For the matrix $\hat{\mathbf{\Sigma}}$ we obtained that

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} s_{11} & \cdots & s_{1k} & \cdots & s_{1n} & \hat{\sigma}_{1n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ s_{1k} & \cdots & s_{kk} & \cdots & s_{kn} & \hat{\sigma}_{kn+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & \cdots & s_{kn} & \cdots & s_{nn} & s_{nn+1} \\ \hat{\sigma}_{1n+1} & \cdots & \hat{\sigma}_{kn+1} & \cdots & s_{nn+1} & s_{n+1n+1} \end{pmatrix}.$$

Let us denote by $\mathbf{A}(\{i_1,\dots,i_l\},\{j_1,\dots,j_l\})$ the matrix, which is obtained from the matrix $\mathbf{A}$ after deleting the rows with numbers $i_1,\dots,i_l$ and the columns with numbers $j_1,\dots,j_l$. The matrix $\mathbf{A}(\{i_1,\dots,i_l\},\{i_1,\dots,i_{l-1},j\})^0$ is the matrix, which is obtained from the matrix $\mathbf{A}(\{i_1,\dots,i_l\},\{i_1,\dots,i_{l-1},j\})$ after replacing the element $a_{ji_l}$ with zero.

We shall prove that for $\hat{\sigma}_{in+1}$, defined by

$$\text{(9)}\quad \hat{\sigma}_{in+1} = \frac{(-1)^{n-i+1}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,i\},\{1,\dots,i-1,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,i,n+1\}))}, \quad i=1,\dots,k,$$

the equations (8) holds. The formulae (9) are recurrent. They determine at first $\hat{\sigma}_{kn+1}$, then $\hat{\sigma}_{k-1n+1}$ and so on, and finally - $\hat{\sigma}_{1n+1}$. The next Theorem gives an equivalent presentation of $\hat{\sigma}_{in+1}$, $i=1,\dots,k$.

**Theorem 3.** *Let $\hat{\sigma}_{in+1}$, $i=1,\dots,k$ are defined by the equalities* (9). *Then*

$$\text{(10)}\quad \hat{\sigma}_{in+1} = \frac{(-1)^{n-k+1}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k\},\{1,\dots,i-1,i+1,\dots,k,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k,n+1\}))},$$

$$i=1,\dots,k.$$

P r o o f. The proof is by induction. The presentation (10) holds for $i=k$. Suppose that it holds for $i=k,k-1,\dots,q+1$, where $q$ is an integer, $1\le q\le k-1$. We shall prove that it holds for $i=q$ too. From (9) we have that

$$\text{(11)}\qquad \hat{\sigma}_{qn+1} = \frac{(-1)^{n-q+1}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q\},\{1,\dots,q-1,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q,n+1\}))}.$$

It can be easily seen that

$$\text{(12)}\quad \det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q\},\{1,\ldots,q-1,n+1\})^0) = \det(\mathbf{A}) \\ + \hat{\sigma}_{q+1n+1}(-1)^{n-q+1}\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q,n+1\},\{1,\ldots,q-1,q+1,n+1\})),$$

where $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{pmatrix} s_{qq+1} & s_{q+1q+1} & \cdots & s_{q+1n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{qn} & s_{q+1n} & \cdots & s_{nn} \\ 0 & 0 & \cdots & s_{nn+1} \end{pmatrix}.$$

If we move in the matrix $\mathbf{A}$ the last row after its first row, the determinant will eventually change its sign, more precisely

$$\text{(13)}\qquad \det(\mathbf{A}) = (-1)^{n-q+1}\det\begin{pmatrix} s_{qq+1} & s_{q+1q+1} & \cdots & s_{q+1n} \\ 0 & 0 & \cdots & s_{nn+1} \\ s_{qq+2} & s_{q+1q+2} & \cdots & s_{q+2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{qn} & s_{q+1n} & \cdots & s_{nn} \end{pmatrix}.$$

Let us denote the last matrix by $\mathbf{B}$. Applying the Sylvester's determinant identity to the matrix $\mathbf{B}$ (Karlin, 1968) we have

$$\begin{aligned} \text{(14)}\quad & \det(\mathbf{B})\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q+1,n+1\})) \\ &= (-1)^{n-q+1}\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q+1\},\{1,\ldots,q,n+1\})^0) \\ &\times \det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q,n+1\},\{1,\ldots,q-1,q+1,n+1\})) \\ &- (-1)^{n-q+1}\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q+1\},\{1,\ldots,q-1,q+1,n+1\})^0) \\ &\qquad \times \det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q,n+1\})). \end{aligned}$$

Replacing (13) and (14) in (12) and using the representation (9) for $\hat{\sigma}_{q+1n+1}$, we obtain that

$$\begin{aligned} \text{(15)}\quad \det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q\},\{1,\ldots,q-1,n+1\})^0) &= -\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q,n+1\})) \\ &\times \frac{\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q+1\},\{1,\ldots,q-1,q+1,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\ldots,q+1,n+1\}))}. \end{aligned}$$

Consequently from the equality (11) it follows that

$$(16) \qquad \hat{\sigma}_{qn+1} = \frac{(-1)^{n-q}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q+1\},\{1,\dots,q-1,q+1,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q+1,n+1\}))}.$$

According to the representation (9) for $\hat{\sigma}_{q+1n+1}$,

$$(17) \qquad \hat{\sigma}_{q+1n+1} = \frac{(-1)^{n-q}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q+1\},\{1,\dots,q,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q+1,n+1\}))}.$$

The right hand sides of formulas (16) and (17) are almost the same. The difference is only in the first column of the matrix in the numerator. The other elements of the matrices are identical and do not depend of the elements in the first columns of the two matrices – $s_{qq+2},\dots,s_{qn}$ and $s_{q+1q+2},\dots,s_{q+1n}$ respectively. According to the induction assumption

$$\hat{\sigma}_{q+1n+1} = \frac{(-1)^{n-k+1}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k\},\{1,\dots,q,q+2,\dots,k,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k,n+1\}))}.$$

Therefore we conclude that

$$\hat{\sigma}_{qn+1} = \frac{(-1)^{n-k+1}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k\},\{1,\dots,q-1,q+1,\dots,k,n+1\})^0)}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k,n+1\}))}.$$

Consequently the presentation (10) holds for $i = q$, hence it is true by induction. $\square$

Let us determine for $\hat{\sigma}_{in+1}$, $i = 1,\dots,k$, given by (9) the corresponding elements $\hat{\sigma}^{in+1}$, $i = 1,\dots,k$ of the matrix $\hat{\mathbf{\Sigma}}^{-1}$. By definition

$$\hat{\sigma}^{in+1} = \frac{(-1)^{n-i+1}\det(\hat{\mathbf{\Sigma}}(\{i\},\{n+1\}))}{\det(\hat{\mathbf{\Sigma}})}, \quad i = 1,\dots,k.$$

It is easy to see that

$$\begin{aligned}
&(-1)^n\det(\hat{\mathbf{\Sigma}}(\{1\},\{n+1\})) \\
&\quad = (-1)^n[\hat{\sigma}_{1n+1}(-1)^{n-1}\det(\hat{\mathbf{\Sigma}}(\{1,n+1\})) + \det(\hat{\mathbf{\Sigma}}(\{1\},\{n+1\})^0)] \\
&\quad = (-1)^n\left[(-1)\frac{\det(\hat{\mathbf{\Sigma}}(\{1\},\{n+1\})^0}{\det(\hat{\mathbf{\Sigma}}(\{1,n+1\}))}\det(\hat{\mathbf{\Sigma}}(\{1,n+1\})) + \det(\hat{\mathbf{\Sigma}}(\{1\},\{n+1\})^0)\right] \\
&\quad = 0.
\end{aligned}$$

Consequently

$$\hat{\sigma}^{1n+1} = 0.$$

We shall prove by induction that

$$\hat{\sigma}^{in+1} = 0, \quad i = 1, \ldots, k. \tag{18}$$

Assume that $\hat{\sigma}^{in+1} = 0$ for $i = 1, \ldots, q-1$, where $q$ is an integer, $q \leq k$ and consider the matrices

$$\begin{array}{ll} \mathbf{A} = \hat{\mathbf{\Sigma}}(\{q, \ldots, n+1\}), & \mathbf{D} = \hat{\mathbf{\Sigma}}(\{1, \ldots, q-1\}), \\ \mathbf{B} = \hat{\mathbf{\Sigma}}(\{q, \ldots, n+1\}, \{1, \ldots, q-1\}), & \mathbf{C} = \hat{\mathbf{\Sigma}}(\{1, \ldots, q-1\}, \{q, \ldots, n+1\}). \end{array}$$

The matrix $\hat{\mathbf{\Sigma}}$ can be written as a block matrix

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}.$$

It can be easily verified that

$$\hat{\mathbf{\Sigma}}^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{U} \end{pmatrix}.$$

The element in the lower left corner of the matrix

$$\mathbf{U} = \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} + \mathbf{D}^{-1}$$

is exactly the element $\hat{\sigma}^{qn+1}$ and we have to prove that it equals to zero. At first we shall show that the element $d^{1n-q}$ in the lower left corner of the matrix $\mathbf{D}^{-1}$ equals to zero. Indeed,

$$d^{1n-q} = \frac{(-1)^{n-q-1} \det(\mathbf{D}(\{1\}, \{n-q\}))}{\det(\mathbf{D})}$$

and it is easy to see that

$$\begin{aligned}\det(\mathbf{D}(\{1\},\{n-q\})) &= \det(\hat{\mathbf{\Sigma}}(\{1,\dots,q\},\{1,\dots,q-1,n+1\}))\\ &= \hat{\sigma}_{qn+1}(-1)^{n-q}\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q,n+1\}))\\ &+ \det(\hat{\mathbf{\Sigma}}(\{1,\dots,q\},\{1,\dots,q-1,n+1\})^0)\\ &= -\det(\hat{\mathbf{\Sigma}}(\{1,\dots,q\},\{1,\dots,q-1,n+1\})^0\\ &+ \det(\hat{\mathbf{\Sigma}}(\{1,\dots,q\},\{1,\dots,q-1,n+1\})^0 = 0.\end{aligned}$$

According to the induction assumption, all elements in the last row of the matrix $\mathbf{Z} = -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}$ are equal to zero. Hence the elements in the last row of the matrix $\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} = \mathbf{Z}\mathbf{B}\mathbf{D}^{-1}$ are equal to zero too. Adding the matrix $\mathbf{D}^{-1}$ to the last one we obtain the matrix $\mathbf{U}$ and conclude that the element $\hat{\sigma}^{qn+1}$ in its lower left corner is equal to zero. Consequently the equalities (18) are true by induction.
The integrals $\hat{J}$ and $\hat{J}_i$ then become

$$\hat{J} = \int_{-1}^{1}\dots\int_{-1}^{1}(\det(\mathbf{R}))^{\frac{m-n-3}{2}}\, I_{\{\mathbf{R}>0\}}\, dr_{1n+1}\dots dr_{kn+1}$$

and

$$J_i = \sqrt{s_{ii}}\int_{-1}^{1}\dots\int_{-1}^{1}(\det(\mathbf{R}))^{\frac{m-n-3}{2}} r_{in+1}\, I_{\{\mathbf{R}>0\}}\, dr_{1n+1}\dots dr_{kn+1}.$$

Applying Proposition 1 for $m = m-1$, $n = n+1$ we get that
(19)

$$\hat{J} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^k \Gamma\left(\frac{m-n-1}{2}\right)}{\Gamma\left(\frac{m-n+k-1}{2}\right)} \frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}}(\det(\mathbf{R}(\{1,\dots,k\})))^{\frac{m-n+k-3}{2}}}{(\det(\mathbf{R}(\{1,\dots,k,n+1\})))^{\frac{m-n+k-2}{2}}}$$

Let $i$ is an integer, $1 \le i \le k$. If $i \ge 2$, from Proposition 1 we obtain that

(20)

$$\begin{aligned}\int_{-1}^{1}\dots\int_{-1}^{1}(\det(\mathbf{R}))^{\frac{m-n-3}{2}}\, I_{\{\mathbf{R}>0\}}\, dr_{1n+1}\dots dr_{i-1n+1} &= \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^{i-1}\Gamma\left(\frac{m-n-1}{2}\right)}{\Gamma\left(\frac{m-n+i-2}{2}\right)}\\ \times \frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}}(\det(\mathbf{R}(\{1,\dots,i-1\})))^{\frac{m-n+i-4}{2}}}{(\det(\mathbf{R}(\{1,\dots,i-1,n+1\})))^{\frac{m-n+i-3}{2}}} I_{\{\mathbf{R}(\{1,\dots,i-1\})>0\}}.\end{aligned}$$

For $1 \le i \le k-1$ let us move in the matrix $\mathbf{R}$ the $i$'th row after the $k$'th row and do the same with the $i$'th column. Denote the new matrix by $\mathbf{R}'$. It is easy to see that

$$\det(\mathbf{R}'(\{1,\ldots,i-1\})) = \det(\mathbf{R}(\{1,\ldots,i-1\})).$$

It can be easily shown that the matrix $\mathbf{R}'(\{1,\ldots,i-1\})$ is positive definite iff the matrix $\mathbf{R}(\{1,\ldots,i-1\})$ is so. The proof uses the fact that the two matrices have one and the same eigenvalues. From (20) it follows that

$$\begin{aligned}
&\int_{-1}^{1}\cdots\int_{-1}^{1} (\det(\mathbf{R}'(\{1,\ldots,i-1\})))^{\frac{m-n+i-4}{2}} I_{\{\mathbf{R}'(\{1,\ldots,i-1\})>0\}}\, dr_{i+1n+1}\ldots dr_{kn+1} \\
&\qquad = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^{k-i}\Gamma\left(\frac{m-n+i-2}{2}\right)}{\Gamma\left(\frac{m-n+k-2}{2}\right)} I_{\{\mathbf{R}(\{1,\ldots,i-1,i+1,\ldots,k\})>0\}} \\
&\times \frac{(\det(\mathbf{R}(\{1,\ldots,i-1,n+1\})))^{\frac{m-n+i-3}{2}}(\det(\mathbf{R}(\{1,\ldots,i-1,i+1,\ldots,k\})))^{\frac{m-n+k-4}{2}}}{(\det(\mathbf{R}(\{1,\ldots,i-1,i+1,\ldots,k,n+1\})))^{\frac{m-n+k-3}{2}}}.
\end{aligned}$$

Consequently for $1 \le i \le k$,

$$\begin{aligned}
&\int_{-1}^{1}\cdots\int_{-1}^{1} (\det(\mathbf{R}))^{\frac{m-n-3}{2}} I_{\{\mathbf{R}>0\}}\, dr_{1n+1}\ldots dr_{i-1n+1}dr_{i+11n+1}\ldots dr_{kn+1} \\
&\qquad = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^{k-1}\Gamma\left(\frac{m-n-1}{2}\right)}{\Gamma\left(\frac{m-n+k-2}{2}\right)} I_{\{\mathbf{R}(\{1,\ldots,i-1,i+1,\ldots,k\})>0\}} \\
&\qquad \times \frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}}(\det(\mathbf{R}(\{1,\ldots,i-1,i+1,\ldots,k\})))^{\frac{m-n+k-4}{2}}}{(\det(\mathbf{R}(\{1,\ldots,i-1,i+1,\ldots,k,n+1\})))^{\frac{m-n+k-3}{2}}}.
\end{aligned}$$

Hence

$$\hat{J}_i = \sqrt{s_{ii}}\,\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^{k-1}\Gamma\left(\frac{m-n-1}{2}\right)}{\Gamma\left(\frac{m-n+k-2}{2}\right)}$$
$$\times\frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}}}{(\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k,n+1\})))^{\frac{m-n+k-3}{2}}}$$
$$\times\int\limits_{-1}^{1} r_{in+1}(\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k\})))^{\frac{m-n+k-4}{2}} I_{\{\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k\})>0\}} dr_{in+1}.$$

Let us denote the above integral by $H$. The next Proposition can be found in [10].

**Proposition 2.** *Let $\mathbf{A}$ be a real symmetric matrix of size $n$, and let $i,j$ be fixed integers such that $1 \le i < j \le n$. The matrix $\mathbf{A}$ is positive definite iff the matrices $\mathbf{A}(\{i\})$ and $\mathbf{A}(\{j\})$ are positive definite and the element $a_{ij}$ satisfies the inequalities*

$$\frac{(-1)^{j-i}\det(A(\{i\},\{j\})^0) - \sqrt{\det(A(\{i\}))\det(A(\{j\}))}}{\det(A(\{i,j\}))} < a_{ij}$$
$$< \frac{(-1)^{j-i}\det(A(\{i\},\{j\})^0) + \sqrt{\det(A(\{i\}))\det(A(\{j\}))}}{\det(A(\{i,j\}))}.$$

Let in the integral $H$ we do the substitution

$$r_{in+1} = \frac{(-1)^{n-k+1}\det(\mathbf{R}(\{1,\dots i-1,i+1,\dots,k,n+1\},\{1,\dots,k\})^0)}{\det(\mathbf{R}(\{1,\dots,k,n+1\}))}$$
$$+\,t\,\frac{\sqrt{\det(\mathbf{R}(\{1,\dots i-1,i+1,\dots,k,n+1\}))\det(\mathbf{R}(\{1,\dots,k\}))}}{\det(\mathbf{R}(\{1,\dots,k,n+1\}))}.$$

¿From Proposition 2 it follows that the new variable $t$ will runs from –1 to 1. Applying Theorem 2 we obtain the representation

$$\det(\mathbf{R}(\{1,\dots i-1,i+1,\dots,k\})) = -\det(\mathbf{R}(\{1,\dots,k,n+1\}))r_{in+1}^2$$
$$+\,2(-1)^{n-k+1}\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k,n+1\},\{1,\dots,k\})^0)r_{in+1}$$
$$+\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k\})^0_{i,n+1}). \tag{21}$$

From the Sylvester's determinant identity we have

$$\begin{aligned}(22)\quad &\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k\})^0_{i,n+1})\det(\mathbf{R}(\{1,\dots,k,n+1\}))\\ &=\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k,n+1\}))\det(\mathbf{R}(\{1,\dots,k\}))\\ &\qquad-(\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k,n+1\},\{1,\dots,k\})^0))^2.\end{aligned}$$

Using (21) and (22) it can be shown that

$$\begin{aligned}&\det(\mathbf{R}(\{1,\dots i-1,i+1,\dots,k\}))\\ &=\frac{\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k,n+1\}))\det(\mathbf{R}(\{1,\dots,k\}))}{\det(\mathbf{R}(\{1,\dots,k,n+1\}))}(1-t^2).\end{aligned}$$

The integral $H$ now can be found and we obtain that

$$\begin{aligned}\hat{J}_i =&\sqrt{s_{ii}}\,(-1)^{n-k+1}\ \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^k\Gamma\left(\frac{m-n-1}{2}\right)}{\Gamma\left(\frac{m-n+k-1}{2}\right)}\\ &\times\det(\mathbf{R}(\{1,\dots,i-1,i+1,\dots,k,n+1\},\{1,\dots,k\})^0)\\ &\times\frac{(\det(\mathbf{R}(\{n+1\})))^{\frac{m-n-2}{2}}(\det(\mathbf{R}(\{1,\dots,k\})))^{\frac{m-n+k-3}{2}}}{(\det(\mathbf{R}(\{1,\dots,k,n+1\})))^{\frac{m-n+k}{2}}}.\end{aligned}$$

At last it can be easily checked that $\hat{\sigma}_{in+1}$, $i=1,\dots,k$, defined by (9), or equivalently by (10), give a solution of the equations (8). To prove that the likelihood function $L_1$ reaches exactly maxima, one can calculate the Hess matrix of the second derivatives of $L_1$ with respect to $\sigma^{in+1}$, $i=1,\dots,k$ and ascertain that it is a negative definite matrix for $\hat{\sigma}_{in+1}$, $i=1,\dots,k$, defined by (9).

We now have to substitute in $L_1$ the obtained maximum likelihood estimations for the unknown parameters. The determinant of the matrix $\hat{\mathbf{\Sigma}}$ can be found using the next statement.

**Theorem 4.** *Let* $\mathbf{A}$ *be a symmetric matrix of size* $n$. *Let the element* $a_{1n}$ *satisfies the equality*

$$a_{1n}=\frac{(-1)^{n-1}\det(\mathbf{A}(\{1\},\{n\})^0)}{\det(\mathbf{A}(\{1,n\}))}. \tag{23}$$

*Then*

$$\det(\mathbf{A})=\frac{\det(\mathbf{A}(\{1\}))\det(\mathbf{A}(\{n\}))}{\det(\mathbf{A}(\{1,n\}))}. \tag{24}$$

P r o o f. From Theorem 2 it follows that

$$\det(\mathbf{A}) = -\det(\mathbf{A}(\{1,n\}))a_{1n}^2 + 2(-1)^{n-1}\det(\mathbf{A}(\{1\},\{n\})^0)a_{1n} + \det(\mathbf{A}_{1,n}^0). \tag{25}$$

From the Sylvester's determinant identity we have

$$\det(\mathbf{A}_{1,n}^0)\det(\mathbf{A}(\{1,n\})) = \det(\mathbf{A}(\{1\}))\det(\mathbf{A}(\{n\})) - (\det(\mathbf{A}(\{1\},\{n\})^0))^2. \tag{26}$$

Applying (26) and the representation (23) for $a_{1n}$ in (25), we get (24). □

In the matrix $\hat{\mathbf{\Sigma}}$ the element $\hat{\sigma}_{1n+1}$, according to the presentation (9) satisfies the condition (23). Therefore from Theorem 4 it follows that

$$\det(\hat{\mathbf{\Sigma}}) = \frac{\det(\hat{\mathbf{\Sigma}}(\{1\}))\det(\hat{\mathbf{\Sigma}}(\{n+1\}))}{\det(\hat{\mathbf{\Sigma}}(\{1,n+1\}))}.$$

Consider the matrix $\hat{\mathbf{\Sigma}}(\{1\})$. The element $\hat{\sigma}_{2n+1}$, according to the presentation (9) satisfies the equality (23). Hence

$$\det(\hat{\mathbf{\Sigma}}(\{1\})) = \frac{\det(\hat{\mathbf{\Sigma}}(\{1,2\}))\det(\hat{\mathbf{\Sigma}}(\{1,n+1\}))}{\det(\hat{\mathbf{\Sigma}}(\{1,2,n+1\}))}.$$

Consequently

$$\det(\hat{\mathbf{\Sigma}}) = \frac{\det(\hat{\mathbf{\Sigma}}(\{1,2\}))\det(\hat{\mathbf{\Sigma}}(\{n+1\}))}{\det(\hat{\mathbf{\Sigma}}(\{1,2,n+1\}))}.$$

Proceeding this way, we obtain finally

$$\begin{aligned}\det(\hat{\mathbf{\Sigma}}) &= \frac{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k\}))\det(\hat{\mathbf{\Sigma}}(\{n+1\}))}{\det(\hat{\mathbf{\Sigma}}(\{1,\dots,k,n+1\}))} \\ &= s_{11}\dots s_{n+1n+1}\frac{\det(\mathbf{R}(\{1,\dots,k\}))\det(\mathbf{R}(\{n+1\}))}{\det(\mathbf{R}(\{1,\dots,k,n+1\}))}.\end{aligned}$$

Using the equalities (18), the expression $\Xi$, defined by (1) can be written in the form

$$\begin{aligned}\Xi &= \sum_{i=1}^{n+1} s_{ii}\,\hat{\sigma}^{ii} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s_{ij}\hat{\sigma}^{ij} + 2\sum_{i=k+1}^{n} s_{in+1}\hat{\sigma}^{in+1} + 2\sum_{i=1}^{k}\hat{\sigma}_{in+1}\hat{\sigma}^{in+1} \\ &= tr\{\hat{\mathbf{\Sigma}}\,\hat{\mathbf{\Sigma}}^{-1}\} = n+1.\end{aligned}$$

So for the maxima $L_1^*$ of the likelihood function $L_1$ we obtain

$$L_1^* = \frac{(m-1)^{\frac{(n+1)(m-3)}{2}}}{2^{\frac{(n+1)(m-1)}{2}} \pi^{\frac{n(n+1)-2k}{4}} \Gamma\left(\frac{m-n+k-1}{2}\right) \prod_{i=1}^{n} \Gamma\left(\frac{m-i}{2}\right)}$$

$$\times \frac{e^{-\frac{(n+1)(m-1)}{2}}}{(s_{11} \dots s_{n+1n+1})} \frac{(\det(\mathbf{R}(\{1,\dots,k,n+1\})))^{\frac{n-k+1}{2}}}{(\det(\mathbf{R}(\{1,\dots,k\})))^{\frac{n-k+2}{2}} (\det(\mathbf{R}(\{n+1\})))^{\frac{n+1}{2}}}.$$

Finally we calculate the likelihood ratio

$$\frac{L_0^*}{L_1^*} = \left( \frac{\det(\mathbf{R}(\{1,\dots,k\})) \det(\mathbf{R}(\{n+1\}))}{\det(\mathbf{R}(\{1,\dots,k,n+1\}))} \right)^{\frac{m-1}{2}}.$$

Obviously, for the testing of $H_0$ against $H_1$ the elements $s_{11}, \dots, s_{n+1n+1}$ of the covariance matrix are unnecessary. So instead of the covariance matrix $\mathbf{S}$ with missing elements we need only the correlation matrix $\mathbf{R}$ with the same elements missing.

Instead of the likelihood ratio we can use the log-likelihood

$$-2 \log \left( \frac{L_0^*}{L_1^*} \right) = (m-1) \log \left( \frac{\det(\mathbf{R}(\{1,\dots,k,n+1\}))}{\det(\mathbf{R}(\{1,\dots,k\})) \det(\mathbf{R}(\{n+1\}))} \right).$$

Then an asymptotic rejection region can be given by computing the $1-\alpha$ quantile $\chi^2_{1-\alpha;\frac{n(n+1)}{2}}$ of the $\chi^2\left(\frac{n(n+1)}{2}\right)$ distribution:

$$(m-1) \log \left( \frac{\det(\mathbf{R}(\{1,\dots,k,n+1\}))}{\det(\mathbf{R}(\{1,\dots,k\})) \det(\mathbf{R}(\{n+1\}))} \right) > \chi^2_{1-\alpha;\frac{n(n+1)}{2}}.$$

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