Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **34** (2008), 543–556

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

TAYLOR SPECTRUM AND CHARACTERISTIC FUNCTIONS OF COMMUTING 2-CONTRACTIONS

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Communicated by S. L. Troyanski

Abstract. In this paper, we give a description of Taylor spectrum of commuting 2-contractions in terms of characteritic functions of such contractions. The case of a single contraction obtained by B. Sz. Nagy and C. Foias is generalied in this work.

1. Introduction. Let $\mathcal{H}$ be a Hilbert space. A 2-tuple $A = (A_1, A_2)$ of bounded operators on $\mathcal{H}$ is called contractive (or 2-contraction) if $\|A_1h_1+A_2h_2\|^2 \leq \|h_1\|^2 + \|h_2\|^2$ for all h_1, h_2 in $\mathcal{H}$. It is equivalent [1] to the condition: $A_1A_1^\star + A_2A_2^\star \leq 1_{\mathcal{H}}$. If additionaly, operators A_1 and A_2 commute, then A is called a commuting 2-contraction. To every commuting 2-contraction $A = (A_1, A_2)$ corresponds an analytic operator-valued function $\theta_A : ID^2 \to B(\mathcal{D}_A, \mathcal{D}_{A^*})$ called characteristic function of A and defined by :

$$\theta_A(z_1, z_2) = -A + D_{A^*}(1_{\mathcal{H}} - z_1A_1^* - z_2A_2^*)^{-1}(z_1.1_{\mathcal{H}}, z_2.1_{\mathcal{H}})D_A \tag{1.1}$$

where

2000 *Mathematics Subject Classification:* 47A10, 47A13.
Key words: Characteritic function, 2-contraction, Taylor spectrum.

(a) $\mathbb{D}^2 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_1|^2 < 1\right\}$,

(b) $B(\mathcal{D}_A, \mathcal{D}_{A^*})$ is the set of all bounded operators from $\mathcal{D}_A$ into $\mathcal{D}_{A^*}$,

(c) $D_{A^*} = (I - A_1 A_1^\star - A_2 A_2^\star)^{\frac{1}{2}} : \mathcal{H} \to \mathcal{H}$ and $\mathcal{D}_{A^*}$ is the closure of the range of D_{A^*},

(d) $D_A = \begin{bmatrix} 1_H - A_1^* A_1 & -A_1^* A_2 \\ -A_2^* A_1 & 1_H - A_2^* A_2 \end{bmatrix}^{\frac{1}{2}} : \mathcal{H}^2 \to \mathcal{H}^2$ and is the closure of the range of D_A,

(e) $(z_1 1_{\mathcal{H}}, z_2 1_{\mathcal{H}}) : \mathcal{H}^2 \to \mathcal{H}; \qquad (z_1 1_{\mathcal{H}}, z_2 1_{\mathcal{H}}) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = z_1 h_1 + z_2 h_2.$

Characteristic function of commuting n-contraction has been introduced in [3] as a generalization of characteristic function of a single contraction [9]. A lot of its remarquable properties have been established in ([2], [4],[9]). In particular, it is shown (like in the single case) that the characteristic function is a unitary invariant. It means that characteristic functions of two pure or completely noncoisometric n-contractions ([2], [4]) $A = (A_1, \ldots, A_n)$ and $A' = (A'_1, \ldots, A'_n)$ coincide if and only if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $A'_i = U^{-1} A_i U$ for every $i = 1, \ldots, n$.

In the case of a completly nonunitary single contraction, the spectrum can be described in terms of the characteristic function (see theoerm 4.1 [9]). The aim of this paper is to give a description of Taylor spectrum in the case of commuting pure 2-contractions by means of characteristic function (1.1).

In section 2 we briefly remind the definition of Taylor spectrum. Section 3 contains characterizations of different components of Taylor spectrum. In section 4, we investigate the behavior of Taylor spectrum under the action of involutive automorphims of unit ball.

2. Taylor spectrum. Let $A = (A_1, A_2, \ldots, A_n)$ be a pure n-contraction. According to ([5], [6], [7], [10]), the Taylor spectrum of A can be defined as follows. Let $\Lambda(\mathcal{H})$ be the exterior algebra on n generators $e_1, \ldots, e_n$ with identity $e_0 = 1$ and coefficients in H. In other words,

$$\Lambda(\mathcal{H}) = \left\{x \otimes e_{i_1} \wedge \cdots \wedge e_{i_p} : x \in \mathcal{H};\ 1 \le i_1 \cdots < i_p \le n\ ;\ 1 \le p \le n\right\}$$

with the collapsing property : $e_i \wedge e_j + e_j \wedge e_i = 0$. One has

$$\begin{aligned} \Lambda(\mathcal{H}) &= \oplus_{k=1}^{n} \Lambda^k(\mathcal{H}); \\ \Lambda^k(\mathcal{H}) &= \{x \otimes e_{i_1} \wedge \cdots \wedge e_{i_k} : x \in H,\ 1 \leq i_1 \cdots \lessdot i_k \leq n\}; \\ \Lambda^0(\mathcal{H}) &= \mathcal{H}. \end{aligned}$$

Consider in $\Lambda(\mathcal{H})$ operator:

$$B_A : \Lambda(\mathcal{H}) \to \Lambda(\mathcal{H}) : \qquad B_A\left(x \otimes e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{k=1}^{n} A_k(x) \otimes e_k \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

It is not difficult to see that $B_A^2 = 0$ and $Ran\, B_A \subseteq Ker\, B_A$. Decomposition $\Lambda(\mathcal{H}) = \oplus_{k=1}^{n} \Lambda^k(\mathcal{H})$ gives a rise to a cochain $K(A, \mathcal{H})$, the so-called Koszul complex $K(A, \mathcal{H})$ associated to A on $\mathcal{H}$ as follows:

$$K(A_1, A_2, \mathcal{H}) : \{0\} \to \mathcal{H} = \Lambda^0(\mathcal{H}) \overset{B_A^0}{\to} \quad \cdots \quad \overset{B_A^{n-1}}{\longrightarrow} \Lambda^n(\mathcal{H}) \to \{0\}$$

where B_A^k is the restriction of B_A to the subspace $\Lambda^k(\mathcal{H})$. Complex $K(A, \mathcal{H})$ is said to be exact (or regular) if:

$$\begin{aligned} &\{0\} = \ker\, B_A^0, && Ran\, B_A^0 = \ker\, B_A^1, \ldots, \\ &Ran\, B_A^{n-2} = \ker\, B_A^{n-1}, && Ran\, B_A^{n-1} = \Lambda^n(H). \end{aligned}$$

Definition 1. *The Taylor spectrum of n-contraction is the set:*

$$\sigma_T(A) = \{z = (z_1, z_{2,\ldots,} z_n) \in \mathbb{C}^n : K(A_1 - z_1, \ldots, A_n - z_n; H) \ \textit{is not exact}\}.$$

Let us now suppose that $n = 2$. Then,

$$\begin{aligned} \Lambda(\mathcal{H}) &= \Lambda^0(\mathcal{H}) \oplus \Lambda^1(\mathcal{H}) \oplus \Lambda^2(\mathcal{H}) \\ &= (\mathcal{H} \otimes e_0) \oplus ((\mathcal{H} \otimes e_1) \oplus (\mathcal{H} \otimes e_2)) \oplus (\mathcal{H} \otimes e_1 \wedge e_2). \end{aligned}$$

According to this direct sum, operator B_A admits the matrix representation

$$B_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & -A_2 & A_1 & 0 \end{bmatrix}$$

and then operators $B^0_{(A_1,A_2)}$ and $B^1_{(A_1,A_2)}$ have the forms:

$$\left\{ \begin{array}{ll} B^0_{(A_1,A_2)}(x) = A_1(x) \oplus A_2(x), & (x, y \in \mathcal{H}), \\ B^1_{(A_1,A_2)}(x \oplus y) = -A_2(x) + A_1(y), & (x, y \in \mathcal{H}). \end{array} \right. \tag{2.1}$$

According to Definition 1 and formula (2.1), one has

$$\sigma_T(A_1, A_2) = \sigma_T^{(1)}(A_1, A_2) \cup \sigma_T^{(2)}(A_1, A_2) \cup \sigma_T^{(3)}(A_1, A_2),$$

where

$$(z_1, z_2) \in \sigma_T^{(1)}(A_1, A_2) \Leftrightarrow \exists\, x \in \mathcal{H} : x \neq 0, \quad (A_1 - z_1)x = (A_2 - z_2)x = 0, \tag{2.2}$$

$$(z_1, z_2) \in \sigma_T^{(2)}(A_1, A_2) \Leftrightarrow \exists\ (x_1, x_2) \in \mathcal{H}^2 : \left\{ \begin{array}{c} (A_1 - z_1)x_1 - (A_2 - z_2)x_2 = 0 \\ (x_1, x_2) \neq ((A_1 - z_1)h\ ,\ (A_2 - z_2)h), \forall h \in \mathcal{H} \end{array} \right., \tag{2.3}$$

$$(z_1, z_2) \in \sigma_T^{(3)}(A_1, A_2) \Leftrightarrow \exists\, y \in \mathcal{H} : y \neq (A_1 - z_1)x_1 - (A_2 - z_2)x_2, \quad \forall (x_1, x_2) \in \mathcal{H}^2. \tag{2.4}$$

Remark 1. $\sigma_T^{(1)}(A_1, A_2)$ is called the ponctual joint Taylor spectrum. Taylor joint spectrum generalizes the one variable notion of spectrum. It is a nonempty compact subset of $\mathbb{C}^n$. The reader can find an excellent account of the Taylor spectrum and its relations with other multiparameter spectral theories in [6]. Note also that in [2] a description of Harte spectrum by means of characteristic function is given.

Throughout this paper, we will suppose that if $A = (A_1, A_2)$ is a commuting 2-contraction, then operator $D_{A^*} = (I - A_1A_1^\star - A_2A_2^\star)^{\frac{1}{2}}$ is one to one. Note that pure (and more generally completely non coisometric) commutig 2-contractions ([2], [3], [4]) satisfy this condition. Indeed, if $A = (A_1, A_2)$ is a pure 2-contraction then, the decreasing sequence of positive bounded operators $((A_1A_1^\star + A_2A_2^\star)^n)_{n\in IN}$ admits a strong limit $A_\infty = 0$. Because of that,

$$\begin{array}{rcl} D_{A^*}(x) & = & 0 \Rightarrow D^2_{A^*}(x) = (I - A_1A_1^\star - A_2A_2^\star)x = 0 \\ & \Rightarrow & x = (A_1A_1^\star + A_2A_2^\star)x \Rightarrow x = (A_1A_1^\star + A_2A_2^\star)^n x\ , \forall n = 0, 1, 2, \ldots \\ & \Rightarrow & x = A_\infty(x) = 0. \end{array}$$

Using relations $AD_A = D_{A^*}A$ and $A^*D_{A^*} = D_AA^*$ ([3]), it can be proven that:

$D_{A^*} = (I - A_1A_1^\star - A_2A_2^\star)^{\frac{1}{2}}$ is one to one $\Leftrightarrow D_A = \begin{bmatrix} 1_H - A_1^*A_1 & -A_1^*A_2 \\ -A_2^*A_1 & 1_H - A_2^*A_2 \end{bmatrix}^{\frac{1}{2}}$ is one to one.

3. Characterization of Taylor spectrum.

Lemma 1. *Let $A = (A_1, A_2)$ be a commuting 2-contraction such that D_{A^*} is one to one. Then,*

$$A(x,y) = z_1x + z_2y, \left((z_1,z_2) \in \mathbb{C}^2,\ (x,y) \in \mathcal{H}^2\right) \Leftrightarrow \theta_A(z_1,z_2)\left(D_A\begin{pmatrix} x \\ y \end{pmatrix}\right) = 0.$$

P r o o f. It follows directly from relation

$$(3.1)\quad \theta_A(z_1,z_2)\,D_A\begin{pmatrix} x \\ y \end{pmatrix} = D_{A^*}\left(1_{\mathcal{H}} - z_1A_1^* - z_2A_2^*\right)^{-1}\left[(z_1x + z_2y) - (A_1x + A_2y)\right].$$

□

Lemma 2. *Let $A = (A_1, A_2)$ be a commuting pure 2-contraction such that D_{A^*} is one to one, $(z_1,z_2) \in \mathbb{C}^2$ and $x \in \mathcal{H}$. Then,*

$$A^*(x) = \begin{pmatrix} \overline{z_1}.x \\ \overline{z_2}.x \end{pmatrix} \Leftrightarrow (\theta_A(z_1,z_2))^*\,D_{A^*}(x) = 0.$$

P r o o f. Since in this case $D_{A^*}^2(x) = (1_{\mathcal{H}} - \overline{z_1}A_1 - \overline{z_2}A_2)\,x$, then the necessary condition is a direct conequence of relation,

$$(3.2)\quad (\theta_A(z_1,z_2))^*\,D_{A^*}(x) = D_A\left(-\begin{pmatrix} A_1^*x \\ A_2^*x \end{pmatrix} + \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}\left(\left(1_H - \overline{z_1}A_1 - \overline{z_2}A_2\right)^{-1}\right)D_{A^*}^2(x)\right).$$

P r o o f o f t h e s u f f i s a n t c o n d i t i o n. Hence $(\theta_A(0,0))^*\,D_{A^*}(x) = -\begin{pmatrix} A_1^*x \\ A_2^*x \end{pmatrix}$, one can whithout loosing the generality suppose that $(z_1,z_2) \neq (0,0)$.

$(\theta_A(z_1,z_2))^* D_{A^*}(x) = 0$

$$\Rightarrow \quad D_A\left(-\begin{pmatrix} A_1^* x \\ A_2^* x \end{pmatrix} + \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right) D_{A^*}^2(x)\right) = 0$$

$$\Rightarrow \quad \begin{cases} -A_1^* x + \overline{z_1}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \\ -A_2^* x + \overline{z_2}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} -A_1^* x + \overline{z_1}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \\ -A_2^* x + \overline{z_2}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} -\overline{z_2}A_1^* x + \overline{z_2 z_1}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \\ -\overline{z_1}A_2^* x + \overline{z_1 z_2}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \overline{z_1}A_2^* x = \overline{z_2}A_1^* x \\ -\overline{z_2}A_1^* x + \overline{z_2 z_1}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \overline{z_1}A_2^* x = \overline{z_2}A_1^* x \\ -\overline{z_2}A_1^* x + \overline{z_2}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(\overline{z_1}I - \overline{z_1}A_1A_1^* - \overline{z_2}A_2A_1^*)\, x = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \overline{z_1}A_2^* x = \overline{z_2}A_1^* x \\ \overline{z_2}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)((\overline{z_1}I - A_1^*))\, x = 0 \end{cases}$$

$$\Rightarrow \quad ((\overline{z_1}I - A_1^*))\, x = 0 \Rightarrow A_1^* x = \overline{z_1} x.$$

On the other hand,

1. $\overline{z_1}A_2^* x = \overline{z_2}A_1^* x,\ \ A_1^* x = \overline{z_1} x$ and $z_1 \neq 0 \Rightarrow A_2^* x = \overline{z_2} x.$
2. Putting $A_1^* x = \overline{z_1} x$ and $z_1 = 0$ in the relation

$$-A_2^* x + \overline{z_2}\left((1_H - \overline{z_1}A_1 - \overline{z_2}A_2)^{-1}\right)(I - A_1A_1^* - A_2A_2^*)\, x = 0,$$

and multiplying by $(1_H - \overline{z_2}A_2)^{-1}$, one obtains

$$(1_H - \overline{z_2}A_2)\, A_2^* x = \overline{z_2}\,(I - A_2A_2^*)\, x.$$

This last relation is equivalent to $A_2^* x = \overline{z_2} x$.

□

Proposition 1. *Let $A = (A_1, A_2)$ be a commuting pure 2-contraction such that D_{A^*} is one to one. Then, $(z_1, z_2) \in \sigma_T^{(1)}(A_1, A_2)$ if and only if equation $\theta_A(z_1, z_2) X = 0$ admits at least two nontrivial solutions $D_A(X_1)$ and $D_A(X_2)$ such that, $X_1 = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}$, $x \in \mathcal{H}$.*

P r o o f. One has

$$
\begin{aligned}
&(z_1, z_2) \in \sigma_T^{(1)}(A_1, A_2)\\
&\quad\Leftrightarrow \quad \exists x \in \mathcal{H} : x \neq 0, (A_1 - z_1)\, x = (A_2 - z_2)\, x = 0\\
&\quad\Leftrightarrow \quad \exists x \in \mathcal{H} : x \neq 0,\ A_1(x) = z_1.x \ \text{ and } A_2(x) = z_2.x\\
&\quad\Leftrightarrow \quad \exists x \in \mathcal{H} : x \neq 0,\ A\begin{pmatrix} x \\ 0 \end{pmatrix} = z_1.x + z_2.0 \ \text{ and } \ A\begin{pmatrix} 0 \\ x \end{pmatrix} = z_1.0 + z_2.x\\
&\quad\Leftrightarrow \quad \theta_A(z_1, z_2)\left(D_A\begin{pmatrix} x \\ 0 \end{pmatrix}\right) = 0 \quad \text{and} \quad \theta_A(z_1, z_2)\left(D_A\begin{pmatrix} 0 \\ x \end{pmatrix}\right) = 0.
\end{aligned}
$$

To end the proof, it is sufficient to remark that

$$
x \neq 0 \Leftrightarrow D_A\begin{pmatrix} x \\ 0 \end{pmatrix} \neq 0 \Leftrightarrow D_A\begin{pmatrix} 0 \\ x \end{pmatrix} \neq 0.
$$

□

Proposition 2. *Let $A = (A_1, A_2)$ be a commuting pure 2-contraction such that D_{A^*} is one to one. Then, $(z_1, z_2) \in \sigma_T^{(2)}(A_1, A_2)$ if and only if the equation $\theta_A(z_1, z_2) Y = 0$ admits at least one non trivial solution $Y = D_A\,(X)$ such that $X \neq \begin{pmatrix} (A_1 - z_1)\, h \\ (z_2 - A_2)\, h \end{pmatrix}$, $\forall h \in \mathcal{H}$.*

P r o o f. One has,

$$
(z_1, z_2) \in \sigma_T^{(2)}(A_1, A_2) \Leftrightarrow \left\{
\begin{array}{c}
\exists\, (x_1, x_2) \in \mathcal{H}^2 : \quad (A_1 - z_1)\, x_1 - (A_2 - z_2)\, x_2 = 0,\\
(x_1, x_2) \neq ((A_1 - z_1)\, h\ ,\ (A_2 - z_2)\, h) \ \ \forall h \in \mathcal{H},\\
(x_1, x_2) \neq (0, 0)\,.
\end{array}\right.
$$

It means that

$$
\begin{aligned}
&(z_1, z_2) \in \sigma_T^{(2)}(A_1, A_2) \Leftrightarrow\\
&\qquad\qquad \left\{
\begin{array}{c}
\exists\, (x_1, x_2) \in \mathcal{H}^2 : \quad A_1(x_1) + A_2(-x_2) = z_1.x_1 + z_2.(-x_2)\,,\\
(x_1, x_2) \neq ((A_1 - z_1)\, h\ ,\ (A_2 - z_2)\, h)\,;\ \forall h \in \mathcal{H},\\
(x_1, x_2) \neq (0, 0)\,.
\end{array}\right.
\end{aligned}
$$

According to Lemma 1, one has finally

$$(z_1, z_2) \in \sigma_T^{(2)}(A_1, A_2) \Leftrightarrow \begin{cases} \exists (x_1, x_2) \in \mathcal{H}^2 : \theta_A(z_1, z_2)\left(D_A\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}\right) = 0, \\ (x_1, x_2) \neq ((A_1 - z_1)h\ ,\ (A_2 - z_2)h);\ \forall h \in \mathcal{H}, \\ (x_1, x_2) \neq (0, 0). \end{cases}$$

□

Proposition 3. *Let $A = (A_1, A_2)$ be a commuting 2-contraction such that D_{A^*} is one to one and $(z_1, z_2) \in \mathbb{C}^2$. Assume that equation $(\theta_A(z_1, z_2))^* D_{A^*}(y) = 0$ admits at least one non trivial solution. Then,*

$$(z_1, z_2) \in \sigma_T^{(3)}(A_1, A_2).$$

Proof. Suppose that $(\theta_A(z_1, z_2))^* D_{A^*}(y) = 0$ admits at least one non trivial solution y. According to Lemma 2, it means that

$$A^*(y) = \begin{pmatrix} \overline{z_1}.y \\ \overline{z_2}.y \end{pmatrix}.$$

Thus, for every $(x_1, x_2) \in \mathcal{H}^2$, one has

$$\begin{aligned} 0 &= \left\langle A^*(y) - \begin{pmatrix} \overline{z_1}.y \\ \overline{z_2}.y \end{pmatrix}, \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\rangle_{\mathcal{H}^2} \\ &= \left\langle \begin{pmatrix} (A_1^* - \overline{z_1}).y \\ (A_2^* - \overline{z_2}).y \end{pmatrix}, \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\rangle_{\mathcal{H}^2} \\ &= \langle (A_1^* - \overline{z_1}).y, x_2 \rangle_{\mathcal{H}} + \langle (A_2^* - \overline{z_2}).y, -x_1 \rangle_{\mathcal{H}} \\ &= \langle y, (A_1 - z_1)x_2 \rangle_{\mathcal{H}} - \langle .y, (A_2 - z_2)x_1 \rangle_{\mathcal{H}} \\ &= \langle y, (A_1 - z_1)x_2 - (A_2 - z_2)x_1 \rangle_{\mathcal{H}}. \end{aligned}$$

Thus,

$$y \notin Ran\, D^1_{(A_1 - z_1, A_2 - z_2)}$$

and finally

$$(z_1, z_2) \in \sigma_T^{(3)}(A_1, A_2).$$

□

4. Taylor spectrum and involutive automorphisms of unit ball. In this section we invetigate the Taylor spectrum under action of involutive automorphisms of unit ball $\mathbb{D}^2$. Such automorphisms are defined in [8] by

$$\Phi_\lambda(z) = \lambda - \frac{\sqrt{1-\|\lambda\|^2}}{1-\langle z,\lambda\rangle}\left(z - \left(1-\sqrt{1-\|\lambda\|^2}\right)\frac{\langle z,\lambda\rangle}{\|\lambda\|^2}.\lambda\right),$$
$$\lambda \in \mathbb{D}^2,\ \lambda = (\lambda_1,\lambda_2) \neq 0.$$

Connection between automorphisms of unit ball and multicontractions has been made in [2] and [4] where some very interesting properties have been established. In particular, if Φ_λ is an involutive automorphism of unit ball and $A = (A_1, A_2)$ is a commutative 2-contraction then, (see [2], sections 4 and 5) one can define operator

$$\Phi_\lambda(A) = \Lambda - D_{\Lambda^*}(1_H - A\Lambda^*)^{-1} A D_\Lambda$$

where the operator $\Lambda = (\lambda_1.1_H, \lambda_2.1_H)$ is defined from H^2 into $\mathcal{H}$ by :

$$\Lambda(x_1,x_2) = \lambda_1.x_1 + \lambda_2.x_2.$$

Proposition 4 and Theorem 1 below summarise important for us results obtained in [2] and [4].

Proposition 4. *Let Φ_λ an involutive automorphism of unit ball and $A = (A_1, A_2)$ a commutative 2-contraction. Then, $\Phi_\lambda(A)$ is a commutative 2-contraction such that,*

$$(4.1)\ I - \Phi_\lambda(A)^*\Phi_\lambda(A) = D_\Lambda(1_\mathcal{H} - A^*\Lambda)^{-1}(I - A^*A)(1_\mathcal{H} - \Lambda^*A)^{-1}D_\Lambda,$$
$$(4.2)\ I - \Phi_\lambda(A)\Phi_\lambda(A)^* = D_{\Lambda^*}(1_\mathcal{H} - A\Lambda^*)^{-1}(I - AA^*)(1_\mathcal{H} - \Lambda A^*)^{-1}D_{\Lambda^*}.$$

Theorem 1. *Let Φ_λ be an involutive automorphism of unit ball and $A = (A_1, A_2)$ a commutative 2-contraction. Then,*

1. *Operators $\Omega : \mathcal{D}_{\Phi_\lambda(A)} \to \mathcal{D}_A$ and $\Omega_* : \mathcal{D}_{\Phi_\lambda(A)^*} \to \mathcal{D}_{A^*}$ defined by*

$$\Omega\left(D_{\Phi_\lambda(A)}(X)\right) = D_A(1_\mathcal{H} - \Lambda^*A)^{-1}D_\Lambda(X),$$
$$\Omega_*\left(D_{\Phi_\lambda(A)^*}(X)\right) = D_{A^*}(1_\mathcal{H} - \Lambda A^*)^{-1}D_{\Lambda^*}(X)$$

are unitaries.

2. $\theta_{\Phi_\lambda(A)}$ *and* θ_A *are connected by the relation*

$$\Omega_*\theta_{\Phi_\lambda(A)}(z_1,z_2)=\theta_A(\Phi_\lambda(z_1,z_2))\,\Omega.$$

It can be shown that:

$$(1_{\mathcal{H}}-A\Lambda^*)^{-1}=(1_{\mathcal{H}}-\overline{z_1}A_1-\overline{z_2}A_2)^{-1}, \tag{4.3}$$

$$(1_{\mathcal{H}^2}-\Lambda^*A)^{-1}= \tag{4.4}$$
$$\begin{bmatrix}(1_{\mathcal{H}}-\overline{\lambda_1}A_1-\overline{\lambda_2}A_2)^{-1}(1_{\mathcal{H}}-\overline{\lambda_2}A_2) & \overline{\lambda_2}A_1.(1_{\mathcal{H}}-\overline{\lambda_1}A_1-\overline{\lambda_2}A_2)^{-1}\\ \overline{\lambda_1}A_2.(1_{\mathcal{H}}-\overline{\lambda_1}A_1-\overline{\lambda_2}A_2)^{-1} & (1_{\mathcal{H}}-\overline{\lambda_1}A_1-\overline{\lambda_2}A_2)^{-1}(1_{\mathcal{H}}-\overline{\lambda_1}A_1)\end{bmatrix}$$

$$D_\Lambda=\frac{1}{|\lambda_1|^2+|\lambda_2|^2} \tag{4.5}$$
$$\times\begin{bmatrix}|\lambda_2|^2+|\lambda_1|^2\sqrt{1-|\lambda_2|^2-|\lambda_1|^2} & \overline{\lambda_1}\lambda_2\left(\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}-1\right)\\ \lambda_1\overline{\lambda_2}\left(\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}-1\right) & |\lambda_1|^2+|\lambda_2|^2\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}\end{bmatrix}.$$

Using (4.3), one can show that

$$\Phi_\lambda(A)=(B_1(\lambda),B_2(\lambda)) \tag{4.6}$$

where

$$\begin{aligned}B_1(\lambda)\ &=\ \lambda_1.1_{\mathcal{H}}-\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}\,(1_H-\overline{\lambda_1}A_1-\overline{\lambda_2}A_2)^{-1}\\ &\quad\times\Big\{A_1\left(|\lambda_2|^2+|\lambda_1|^2\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}\right)\\ &\quad+\lambda_1\overline{\lambda_2}A_2\left(\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}-1\right)\Big\},\end{aligned}$$

$$\begin{aligned}B_2(\lambda)\ &=\ \lambda_2.1_{\mathcal{H}}-\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}\,(1_H-\overline{\lambda_1}A_1-\overline{\lambda_2}A_2)^{-1}\\ &\quad\times\Big\{\overline{\lambda_1}\lambda_2A_1\left(\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}-1\right)\\ &\quad+A_2\left(|\lambda_1|^2+|\lambda_2|^2\sqrt{1-|\lambda_2|^2-|\lambda_1|^2}\right)\Big\}.\end{aligned}$$

Formulas (4.3), (4.4), and (4.5) allow us to find the explicit forms of operators $\Phi_\lambda(A)$, Ω and Ω_*. On the other hand, from (4.2) follows that if D_{A^*} is one to one, then $D_{\Phi_\lambda(A)^*}$ is also one to one. Using Theorem 1, one can obtain the following caracterization for Taylor spectrum of $\Phi_\lambda(A)$ in terms of solutions of equations

$$\theta_A(z_1,z_2)\,D_A(X)=0 \quad \text{and} \quad (\theta_A(z_1,z_2))^*\,D_{A^*}(y)=0.$$

Proposition 5. $\Phi_\lambda(z_1,z_2)\in\sigma_T^{(1)}(\Phi_\lambda(A))$ *if and only if equation* $\theta_A(z_1,z_2)\,D_A(X)=0$ *admits at least two nontrivial solutions* X_1 *and* X_2 *such that,*

$$X_1=(1_{\mathcal{H}}-\Lambda^*A)^{-1}D_\Lambda\begin{pmatrix} y\\ 0\end{pmatrix},\quad X_2=(1_{\mathcal{H}}-\Lambda^*A)^{-1}D_\Lambda\begin{pmatrix} 0\\ y\end{pmatrix},\quad y\in\mathcal{H}.$$

P r o o f. According to Proposition 3, $\Phi_\lambda(z_1,z_2)\in\sigma_T^{(1)}(\Phi_\lambda(A))$ if and only if there exists a nonnul vector $y\in\mathcal{H}$ such that,

$$Y_1=\begin{pmatrix} y\\ 0\end{pmatrix},\quad Y_2=\begin{pmatrix} 0\\ y\end{pmatrix}$$

are solutions of equation

$$\theta_{\Phi_\lambda(A)}(\Phi_\lambda(z_1,z_2))\,D_{\Phi_\lambda(A)}(Y)=0.$$

Using Theorema 1 and the fact that Φ_λ is involutive, it is equivalent to the existence of a nonnul vector $y\in\mathcal{H}$ such that,

$$Y_1=\begin{pmatrix} y\\ 0\end{pmatrix},\quad Y_2=\begin{pmatrix} 0\\ y\end{pmatrix}$$

are solutions of equation

$$\Omega_*^{-1}\theta_A(z_1,z_2)\,\Omega D_{\Phi_\lambda(A)}(Y)=\Omega_*^{-1}\theta_A(z_1,z_2)\,D_A\Big((1_{\mathcal{H}}-\Lambda^*A)^{-1}D_\Lambda(Y)\Big)=0$$

which is equivalent to the equation

$$\theta_A(z_1,z_2)\,D_A\Big((1_{\mathcal{H}}-\Lambda^*A)^{-1}D_\Lambda(Y)\Big)=0.$$

□

Corollary 1. *Let* $(z_1, z_2) \in \mathbb{C}^2$*. Then,* $(z_1, z_2) \in \sigma_T^{(1)}(A)$ *and* $\Phi_\lambda(z_1, z_2) \in \sigma_T^{(1)}(\Phi_\lambda(A))$ *if and only if there exists two nonnul vectors* x *and* y *in* $\mathcal{H}$ *such that vectors:*

$$X_1 = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

$$Y_1 = (1_{\mathcal{H}} - \Lambda^* A)^{-1} D_\Lambda \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{and} \quad Y_2 = (1_{\mathcal{H}} - \Lambda^* A)^{-1} D_\Lambda \begin{pmatrix} 0 \\ y \end{pmatrix}$$

are both solutions of equation

$$\theta_A(z_1, z_2) D_A(X) = 0.$$

Proposition 6. *Let* $(z_1, z_2) \in \mathbb{C}^2$*. Assume that equation* $(\theta_A(z_1, z_2))^* D_{A^*}(y) = 0$ *admits at least one non trivial solution. Then,*

$$\Phi_\lambda(z_1, z_2) \in \sigma_T^{(3)}(\Phi_\lambda(A)).$$

P r o o f. Note at first that operator D_{Λ^*} is invertible. Since Φ_λ is involutive then, according Theorem 1,

$$
\begin{aligned}
&(\theta_A(z_1, z_2))^* D_{A^*}(y) = 0 \\
&\Leftrightarrow \quad (\theta_A(\Phi_\lambda(\Phi_\lambda(z_1, z_2))))^* D_{A^*}(y) = 0 \\
&\Leftrightarrow \quad \left(\Omega_* \theta_{\Phi_\lambda(A)}(\Phi_\lambda(z_1, z_2)) \Omega^{-1}\right)^* D_{A^*}(y) = 0 \\
&\Leftrightarrow \quad \Omega \left(\theta_{\Phi_\lambda(A)} \Phi_\lambda(z_1, z_2)\right)^* \Omega_*^{-1} D_{A^*}(y) = 0 \\
&\Leftrightarrow \quad \left(\theta_{\Phi_\lambda(A)} \Phi_\lambda(z_1, z_2)\right)^* \Omega_*^{-1} D_{A^*}(y) = 0 \\
&\Leftrightarrow \quad \left(\theta_{\Phi_\lambda(A)} \Phi_\lambda(z_1, z_2)\right)^* \Omega_*^{-1} D_{A^*}\left((1_{\mathcal{H}} - \Lambda A^*)^{-1} D_{\Lambda^*} D_{\Lambda^*}^{-1}(1_{\mathcal{H}} - \Lambda A^*) y\right) = 0 \\
&\Leftrightarrow \quad \left(\theta_{\Phi_\lambda(A)} \Phi_\lambda(z_1, z_2)\right)^* D_{\Phi_\lambda(A)^*}\left(D_{\Lambda^*}^{-1}(1_{\mathcal{H}} - \Lambda A^*) y\right) = 0 \\
&\Leftrightarrow \quad \left(\theta_{\Phi_\lambda(A)} \Phi_\lambda(z_1, z_2)\right)^* D_{\Phi_\lambda(A)^*}(X) = 0,
\end{aligned}
$$

where

$$X = D_{\Lambda^*}^{-1}(1_{\mathcal{H}} - \Lambda A^*) y.$$

Since y is nonnul then, $X = D_{\Lambda^*}^{-1}(1_H - \Lambda A^*) y$ is also nonnul and according Proposition 3, it follows that

$$\Phi_\lambda(z_1, z_2) \in \sigma_T^{(3)}(\Phi_\lambda(A_1, A_2)).$$

□

Proposition 7. $\Phi_\lambda(z_1, z_2) \in \sigma_T^{(2)}(A_1, A_2)$ *if and only if the equation* $\theta_A(z_1, z_2) D_A(X) = 0$ *admits at least one solution* X *such that*

$$X \neq \begin{pmatrix} B_1(\lambda) h - w_1.h \\ B_2(\lambda) h - w_2.h \end{pmatrix}, \quad \forall h \in \mathcal{H}$$

where $(w_1, w_2) = \Phi_\lambda(z_1, z_2)$.

Proof. It follows immeditely from Proposition 2. □

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Received December 23, 2006