Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **35** (2009), 1–14

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

UNIFORM G-CONVEXITY FOR VECTOR-VALUED L_p SPACES*

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Communicated by S. L. Troyanski

Abstract. Uniform G-convexity of Banach spaces is a recently introduced natural generalization of uniform convexity and of complex uniform convexity. We study conditions under which uniform G-convexity of X passes to the space of X-valued functions $L_p(\mu, X)$.

1. Introduction.

In the text below X and Y are Banach spaces; $L(X, Y)$ is the space of bounded linear operators from X into Y, $L(X) := L(X, X)$; S_X is the unit sphere of X. For a finite set G the number of its elements is denoted by $|G|$.

In the article [8] a generalization of unconditional convergence (G-unconditional convergence) for series in Banach spaces was studied. In that generalization instead of putting plus-minus coefficients to the summands, the operator

2000 *Mathematics Subject Classification:* 46B20.

Key words: Uniform convexity, complex uniform convexity, uniform G-convexity.

*The second named author was supported by Junta de Andalucía and FEDER grant P06-FQM-01438.

coefficients from a fixed bounded subset $G \subset L(X)$ were used. If G satisfies some natural "regularity" condition, then G-unconditional convergence implies the usual one, but the inverse implication although is true for finite-dimensional $G \subset L(X)$, is not true in general even for norm-compact $G \subset L(X)$. An analogue of cotype theory for G-unconditional convergence was build. This G-cotype theory works properly if G is a group or at least a semigroup, but does not work for arbitrary G. On this way in particular the following notions were introduced:

An operator family $G \subset L(X,Y)$ is called *regular*, if for every $\varepsilon > 0$ and for every element $x \in X$ there is a $T \in \mathrm{conv} G$ with $\|Tx\| < \varepsilon$.

Let $G \subset L(X)$. Let us call the following function the *modulus of G-convexity* of X:

$$\delta_X^G(t) = \inf_{\|x\|=\|y\|=1} \{\sup\{\|x + tTy\| : T \in G\} - 1\}.$$

A space X is called *uniformly G-convex*, if $\delta_X^G(t) > 0$ for all $t > 0$. A particular case of uniform G-convexity is the well-known uniform convexity (when $G = \{I, -I\}$; see, for example, [3] for extensive study of this concept) and complex uniform convexity (when $G = \{e^{i\theta} I : \theta \in (0, 2\pi]\}$ or, which is equivalent, when $G = \{I, -I, iI, -iI\}$; see [4], [6] and [1]).

The modulus of G-convexity has several properties of the usual modulus of convexity, and also an analogue of M.I. Kadets theorem about unconditionally convergent series in uniformly convex spaces was proved.

In the present article we study the inheritance of the uniform G-convexity of the space X by the space of $X-$valued functions $L_p(\mu, X)$, $p \in [1, \infty)$ in the (most interesting for us) case when G is a regular finite group. We show that uniform G-convexity of all $L_p(\mu, X)$ is equivalent to a bit stronger property of X, i.e. to the uniform G-convexity in terms of p-average, and that for $p \in (1, \infty)$ the latter property is equivalent to the uniform G-convexity of X. The case $p = 1$ remains somehow a miracle for us, because for some finite groups of operators uniform G-convexity of X implies uniform G-convexity of $L_1(\mu, X)$ and for others – does not, and we have no satisfactory classification of groups with respect to this property.

Some results of the paper can be generalized to wider classes of groups (say, to norm compact groups of operators), but we restricted ourselves to the finite groups study by three reasons: first of all the classical uniform and complex uniform spaces fit into finite groups scheme, next, in this case G-unconditional convergence implies unconditional convergence in ordinary sense, so uniform G-convexity implies finite cotype, and finally, even for finite groups we are not able to solve some of very natural questions. One of these problems we have mentioned just a few lines above. Another one is to characterize those regular finite groups of operators G, for which uniform G-convexity of X implies reflexivity of X.

Let (Ω, Σ, μ) be a measure space. A function $f : \Omega \to X$ is said to be strongly measurable, if there is a sequence of simple measurable functions, which converges to f almost everywhere. As usual $L_p(\mu, X)$ denotes the space of equivalence classes of strongly measurable functions with

$$\|f\|_p = \left(\int_\Omega \|f(t)\|^p d\mu(t) \right)^{1/p} < \infty,$$

where two functions are said to be equivalent if they are equal almost everywhere.

When G is a family of operators on the space X, we define a counterpart of this family on the space $L_p(\mu, X)$ in the following natural way: $\widetilde{G} = \{\widetilde{T} : (\widetilde{T}f)(\tau) = T(f(\tau)), T \in G\}$. Below we do not distinguish between the families G and $\widetilde{G}$, whenever it does not lead to confusion.

2. Main results. Let us remark at first, that in the case of finite operator group the regularity condition can be written substantially simplified:

Lemma 2.1. *Let $G = \{T_1, T_2, \ldots, T_n\} \subset L(X)$ be a finite group. If the group G is regular then $\sum\limits_{T \in G} T = 0$.*

P r o o f. By the definition, a group is regular if for every $\varepsilon > 0$ and every element $x \in X$ there is a set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \lambda_k > 0, \sum\limits_{k=1}^{n} \lambda_k = 1$, such that $\left\| \sum\limits_{k=1}^{n} \lambda_k T_k x \right\| < \varepsilon$. Denote $M = \max \|T_k\|$. Then for every $1 \le j \le n$, $\left\| \sum\limits_{k=1}^{n} \lambda_k T_j T_k x \right\| < M\varepsilon$. Hence, also the inequality $\dfrac{1}{n} \sum\limits_{j=1}^{n} \left\| \sum\limits_{k=1}^{n} \lambda_k T_j T_k x \right\| < M\varepsilon$ holds true. The left-hand side of the inequality can be estimated from bellow as follows:

$$\frac{1}{n} \sum_{j=1}^{n} \left\| \sum_{k=1}^{n} \lambda_k T_j T_k x \right\| \ge \left\| \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_k T_j T_k x \right\|$$

$$= \left\| \sum_{s=1}^{n} T_s x \left(\frac{1}{n} \sum_{j,k : T_j T_k = T_s} \lambda_k \right) \right\| = \left\| \frac{1}{n} \sum_{s=1}^{n} T_s x \right\|,$$

and this expression does not depend on ε. $\square$

Another remark explains why the case of isometries is the most attractive for us below. Namely, let $G = \{T_1, T_2, \dots, T_n\} \subset L(X)$ be a finite group, $M := \max\limits_{g \in G} \|g\|$. Then the expression

$$|||x||| = \max_{g \in G} \|g(x)\|$$

defines an equivalent norm on X in which all elements of G are isometries. Moreover, it is easy to check directly, that if $(X, \|\cdot\|)$ is uniformly G-convex, then $(X, |||\cdot|||)$ is uniformly G-convex as well with

$$\delta^G_{(X,|||\cdot|||)}(t) \geq M^{-1} \delta^G_{(X,\|\cdot\|)}(M^{-1}t).$$

Definition 2.2. *Define the modulus of G-convexity in terms of p-average of a space X where $G = \{T_1, T_2, \dots, T_n\} \subset L(X)$ as*

$$\gamma_X^{G,p}(t) = \inf_{x,y \in S_X} \left\{ \left(\frac{1}{n} \sum_{i=1}^{n} ||x + tT_i y||^p \right)^{1/p} - 1 \right\}.$$

The space X is said to be uniformly G-convex in terms of p-average (uniformly (G,p)-convex, to say it shorter) if $\gamma_X^{G,p}(t) > 0$ for $t > 0$.

Lemma 2.3. *The function $F_{x,\{y_k\}}(t) = \left(\frac{1}{n} \sum\limits_{k=1}^{n} \|x + ty_k\|^p \right)^{1/p}$ is convex.*

P r o o f. By the triangle inequality

$$F_{x,\{y_k\}}(\lambda t_1 + (1-\lambda)t_2) = \left(\frac{1}{n} \sum_{k=1}^{n} \|x + (\lambda t_1 + (1-\lambda)t_2) y_k\|^p \right)^{1/p} \leq$$

$$\leq \left(\frac{1}{n} \sum_{k=1}^{n} (\lambda \|x + t_1 y_k\| + (1-\lambda)\|x + t_2 y_k\|)^p \right)^{1/p}.$$

Let us consider auxiliary vectors $u = (\|x + t_1 T_1 y\|, \|x + t_1 T_2 y\|, \dots, \|x + t_1 T_n y\|)$ and $v = (\|x + t_2 T_1 y\|, \|x + t_2 T_2 y\|, \dots, \|x + t_2 T_n y\|)$ in the space $l_p^{(n)}$ and write the triangle inequality for them: $\|\lambda u + (1-\lambda)v\|_p \leq \lambda \|u\|_p + (1-\lambda)\|v\|_p$. Since the right part of the previous inequality is equal to $n^{-1/p} \|\lambda u + (1-\lambda)v\|_p$, then we have:

$$F_{x,\{y_k\}}(\lambda t_1 + (1-\lambda)t_2) \leq n^{-1/p}(\lambda \|u\|_p + (1-\lambda)\|v\|_p) =$$

$$= \lambda F_{x,\{y_k\}}(t_1) + (1-\lambda) F_{x,\{y_k\}}(t_2). \qquad \square$$

Lemma 2.4. *Let the function $f(s)$ be defined as follows:*

$$f(s) = \inf_{x,y\in S_X} \left(\left(\frac{1}{n} \sum_{k=1}^{n} \|x + s^{1/p}\, T_k y\|^p \right)^{1/p} - 1 \right)^p,$$

then $f(s)/s$ is nondecreasing.

Proof. Denote $\tau = s^{1/p}$. Then

$$\frac{f(s)}{s} = \frac{\inf\limits_{x,y\in S_X} \left(\left(\frac{1}{n} \sum\limits_{k=1}^{n} \|x + s^{1/p} T_k y\|^p \right)^{1/p} - 1 \right)^p}{s}$$

$$= \left(\frac{\inf\limits_{x,y\in S_X} \left(\frac{1}{n} \sum\limits_{k=1}^{n} \|x + \tau T_k y\|^p \right)^{1/p} - 1}{\tau} \right)^p.$$

Thus, to prove the lemma we need to show that for all $0 < \tau_1 < \tau_2$ and arbitrary elements $x, y \in S_X$ the following inequality holds:

$$\begin{aligned} \frac{\left(\frac{1}{n} \sum\limits_{k=1}^{n} \|x + \tau_1 T_k y\|^p \right)^{1/p} - 1}{\tau_1} &= \frac{F_{x,\{T_k y\}}(\tau_1) - 1}{\tau_1} \\ &\le \frac{\left(\frac{1}{n} \sum\limits_{k=1}^{n} \|x + \tau_2 T_k y\|^p \right)^{1/p} - 1}{\tau_2} \\ &= \frac{F_{x,\{T_k y\}}(\tau_2) - 1}{\tau_2}. \end{aligned}$$

By Lemma 2.3 the function $F_{x,\{T_k y\}}(\tau)$ is convex and equals 1 when $\tau = 0$, consequently, the function $F_{x,\{T_k y\}}(\tau) - 1$ is also convex and equals 0 when $\tau = 0$. Hence, the ratio $(F_{x,\{T_k y\}}(\tau) - 1)/\tau$, which is equal to the slope of segment between the origin and the point $(\tau, F_{x,\{T_k y\}}(\tau) - 1)$, does not decrease. $\square$

Remark. By the Figiel's theorem [5], if $f(s) > 0$ when $s > 0$ and $f(s)/s$ is a non-decreasing function, then there is a convex function $0 \le f_1(s) \le f(s)$

such that $f_1(s) > 0$ when $s > 0$. This remark will be of use in the proof of the following theorem.

Theorem 2.5. *Let $1 \le p < \infty$, $G = \{T_1, T_2, \dots, T_n\} \subset L(X)$ be a finite group. Then the following two conditions are equivalent:*

1. *The space $L_p(\mu, X)$ is G-convex for every measure space (Ω, Σ, μ).*
2. *The space X is uniformly (G, p)-convex.*
3. *The space $L_p(\mu, X)$ is uniformly (G, p)-convex for every measure space (Ω, Σ, μ).*

Proof. 1. $\Rightarrow$ 2. Denote $M = \max\limits_k \|T_k\|$. Since the space $L_p(\mu, X)$ is G-convex for every measure space (Ω, Σ, μ), it is also G-convex in the case of $\Omega = \{1, 2, \dots, n\}$, where the measure of every point equals $\dfrac{1}{n}$. For arbitrary $x, y \in S_X$ consider the functions $f, g : \Omega \to X :\ f(k) = x$ and $g(k) = T_k y$ for all k. Let us denote $C = 1/\|g\|$. It is obvious that $\dfrac{1}{M} \le C \le M$. Functions f and Cg are from the unit ball of the space $L_p(\mu, X)$. Thus, we can write the following chain of inequalities:

$$1 + \delta^G_{L_p(\mu,X)}(t) \le \sup_{T\in G} \left(\int \|f + tCTg\|^p d\mu \right)^{1/p} = \sup_{T\in G} \left(\frac{1}{n} \sum_i \|x + tCTT_i y\|^p \right)^{1/p}$$

$$(1) \qquad = \left(\frac{1}{n} \sum_i \|x + tCT_i y\|^p \right)^{1/p} = F_{x,\{T_i y\}}(Ct) =: F(C).$$

Inequality (1) means that $F(C) \ge F(0) = 1$ for every C. By convexity of F (Lemma 2.3) we obtain that F does not decrease when C grows. From this and from (1) it follows that

$$1 + \delta^G_{L_p(\mu,X)}(t) \le \frac{1}{n^{1/p}} \left(\sum_i \|x + tMT_i y\|^p \right)^{1/p}$$

Consequently,

$$(2) \qquad \delta^G_{L_p}(t) \le \gamma^{G,p}_X(Mt).$$

Q.E.D.

2. $\Rightarrow$ 3. Let x, y be arbitrary elements from the unit ball of the space $L_p(\mu, X)$. By small perturbation argument we can assume $x(\tau) \neq 0$ for all τ. This enables us to write the following chain of inequalities

$$
\begin{aligned}
&\left(\int_\Omega \frac{1}{n}\sum_k \|x(\tau)+tT_k y(\tau)\|^p d\mu(\tau)\right)^{1/p} \\
&\geq \left(\int_\Omega \|x(\tau)\|^p \left(1+\gamma_X^{G,p}\left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)\right)^p d\mu(\tau)\right)^{1/p} \\
&\geq \left(1+\int_\Omega \|x(\tau)\|^p \left(\gamma_X^{G,p}\left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)\right)^p d\mu(\tau)\right)^{1/p} \\
&= \left(1+\int_\Omega \|x(\tau)\|^p f\left(\frac{t^p\|y(\tau)\|^p}{\|x(\tau)\|^p}\right) d\mu(\tau)\right)^{1/p},
\end{aligned} \tag{3}
$$

where f is the function from Lemma 2.4. By the remark after Lemma 2.4 using convex minorante f_1 of f we can continue the chain of inequalities in the following way

$$
\begin{aligned}
&\geq \left(1+\int_\Omega \|x\|^p f_1\left(\frac{t^p\|y(\tau)\|^p}{\|x(\tau)\|^p}\right) d\mu(\tau)\right)^{1/p} \\
&\geq \left(1+f_1\left(\int_\Omega t^p\|y(\tau)\|^p d\mu(\tau)\right)\right)^{1/p} = (1+f_1(t^p))^{1/p} > 1.
\end{aligned}
$$

Hence, $\delta^G_{L_p(\mu,X)}(t) > 0$ for all t.

The remaining implication 3. $\Rightarrow$ 1. is evident. $\square$

Remark 1. If G is a finite group of isometries, then $M = 1$, and we obtain an improved estimate $\delta^G_{L_p(\mu,X)}(t) \leq \gamma_X^{G,p}(t)$ instead of (2).

Remark 2. Although our argument in the implication 2. $\Rightarrow$ 1. is short and uses simple convexity inequalities, it does not give the optimal estimate for the modulus of convexity. The reason is that at some point we use a very bad estimate

$$(1+a)^p \geq 1+a^p \tag{4}$$

which enables us to apply the convexity argument, but gives no chance to obtain the best possible inequality. Say, if $\gamma_X^{G,p}(t)$ has for small t a power-type estimate from below ($\gamma_X^{G,p}(t) \geq \alpha t^\beta$ for some $\beta \geq 1$), then from our estimates one has $\delta^G_{L_p(\mu,X)}(t) \geq \alpha_1 t^{\beta_1}$ with $\beta_1 = p\beta$ for small t. One can easily see that this estimate is far from being optimal. Say, in the case of $X = \mathbb{R}$, $G = \{I, -I\}$ and $p = 2$ one has $\beta = 2$ and $\delta^G_{L_2(\mu,X)}(t) \geq \alpha_1 t^2$ for small t, which is much better than $\delta^G_{L_2(\mu,X)}(t) \geq \alpha_1 t^4$ for small t which we have from our estimate.

But fortunately in the case of $\gamma_X^{G,p}(t) \geq \alpha t^\beta$ for small t one can work with (3) in a different way. Divide Ω into two parts: $\Omega_1 = \left\{\tau : \dfrac{t\|y(\tau)\|}{\|x(\tau)\|} \leq 1\right\}$ and $\Omega_2 = \left\{\tau : \dfrac{t\|y(\tau)\|}{\|x(\tau)\|} > 1\right\}$. On Ω_2 apply (4) and a linear estimate for $\gamma_X^{G,p}(t)$ but on Ω_1 apply $(1+a)^p \geq 1 + pa$ and $\gamma_X^{G,p}(t) \geq \alpha t^\beta$. Then one gets estimate of the form

$$\geq \left(1 + \alpha_1 \int_{\Omega_1} \|x(\tau)\|^p \left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)^\beta d\mu(\tau) + \alpha_1 \int_{\Omega_2} \|x(\tau)\|^p \left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)^p d\mu(\tau)\right)^{1/p}$$

$$=: A$$

In the case of $\beta \geq p$ in fact one can get the optimal estimate $\beta_1 = \beta$ along the same convexity lines as before: define $g(s)$ as the biggest convex minorante for $\min(s, s^{\beta/p})$, then

$$\begin{aligned} A &\geq \left(1 + \alpha_1 \int_\Omega \|x(\tau)\|^p \min\left\{\left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)^\beta, \left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)^p\right\} d\mu(\tau)\right)^{1/p} \\ &\geq \left(1 + \alpha_1 \int_\Omega \|x(\tau)\|^p g\left(\frac{t^p\|y(\tau)\|^p}{\|x(\tau)\|^p}\right) d\mu(\tau)\right)^{1/p} \\ &\geq \left(1 + \alpha_1 g\left(\int_\Omega t^p \|y(\tau)\|^p d\mu(\tau)\right)\right)^{1/p} = (1 + \alpha_1 g(t^p))^{1/p} > 1 + \alpha_2 t^\beta \end{aligned}$$

for small t. In the case of $\beta < p$ the optimal estimate is $\beta_1 = p$, and this can be done as follows:

$$A \geq \left(1 + \alpha_1 \int_{\Omega_1} \|x(\tau)\|^p \left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)^p d\mu(\tau) + \alpha_1 \int_{\Omega_2} \|x(\tau)\|^p \left(\frac{t\|y(\tau)\|}{\|x(\tau)\|}\right)^p d\mu(\tau)\right)^{1/p}$$

$$= (1 + \alpha_1 t^p)^{1/p} > 1 + \alpha_2 t^p$$

for small t. $\square$

Theorem 2.6. *Let G be a finite group and $\sum_{T\in G} T = 0$. Then for every $p \in (1, +\infty)$ uniform G-convexity of X is equivalent to its uniform G-convexity in terms of $p-$average.*

P r o o f. Sufficiency of the condition of space G-convexity in terms of $p-$average for its G-convexity for arbitrary p follows from the inequality between maximum of positive numbers and their $p-$average. Let us prove the necessity of this condition. Since X is G-convex, for any $t > 0$ there is a $\delta = \delta_X^G(t) > 0$ such that for given $x, y \in S_X$ there is a $T \in G$ such that

$$\|x + tTy\| \geq 1 + \delta.$$

The condition on the group G gives us the following inequality for arbitrary elements of the unit ball of the space X:

$$\frac{1}{n}\sum_k \|x + tT_k y\| \geq \|x\| = 1.$$

To prove the theorem it is enough to show that

$$\inf_{x,y\in S_X}\left(\frac{1}{n}\sum_k \|x + tT_k y\|^p\right)^{1/p} > 1.$$

Denoting $a_k := \|x + tT_k y\|$ we need to estimate from below the expression $\left(\sum\limits_k a_k^p/n\right)^{1/p}$. But we know that $\sum\limits_k a_k/n \geq 1$ and that at least one of a_k exceeds $1+\delta$. By symmetry this leads to the problem of calculating the minimum of $\left(\sum\limits_k a_k^p/n\right)^{1/p}$ for non-negative a_k under conditions $a_1 \geq 1+\delta$, and $\sum\limits_k a_k/n \geq 1$. Apparently, the necessary minimum is obtained when these inequalities become equalities. Therefore, we need to find the minimum of the expression $\left(\sum\limits_k a_k^p/n\right)^{1/p}$, when $a_1 = 1+\delta$ and $\sum\limits_k a_k = n$. It can be easily shown that under such conditions the desired minimum is reached when all the numbers a_k, $k > 1$ are equal, i.e. $a_k = (1-\delta)/(n-1)$, $k > 1$. Thus we obtain

$$\inf_{x,y\in S_X}\left(\frac{1}{n}\sum_k \|x + tT_k y\|^p\right)^{1/p} \geq \left(\frac{(1+\delta)^p + (n-1)\left(1 - \frac{\delta}{n-1}\right)^p}{n}\right)^{1/p} \tag{5}$$

$$> \left(\frac{1 + p\delta + (n-1)\left(1 - \frac{p\delta}{(n-1)}\right)}{n} \right)^{1/p} = 1. \qquad \square$$

Remark 3. For fixed n and for small values of t using inequality $(1+s)^p \geq 1 + ps + \frac{1}{2}p(p-1)s^2 + o(s^2)$ from (5) one can get estimate of the form $\delta_X^{G,p}(t) \geq c(p)(\delta_X^G(t))^2$, but $c(p) \to 0$ as $p \to 1$, so this estimate is of no use for $p = 1$. $\square$

From Theorem 2.5 and Theorem 2.6 we obtain the following statement:

Corollary 2.7. *For $1 < p < \infty$ and for a finite group G, satisfying the condition $\sum\limits_{T \in G} T = 0$, uniform G-convexity of X implies uniform G-convexity of the space $L_p(\mu, X)$ for every measure space (Ω, Σ, μ).*

This corollary in particular generalizes (and simplifies the proofs of) two well-known statements [2], [1]: that uniform convexity and complex uniform convexity pass from X to $L_p(\mu, X)$ for all $1 < p < \infty$.

Let us note that for $p = \infty$ the space $L_p(\mu, X)$ cannot be uniformly G-convex, with exception of the trivial case when Ω is an atom of μ and consequently $L_p(\mu, X) = X$. In fact, divide Ω into two disjoint sets A and B with nonzero measures and consider the expression $\|\chi_A x + t\chi_B G_k x\|$, where $\|x\| = 1$. When t is sufficiently small the second summand is less then 1 for every G_k, hence, the norm of the whole sum equals the norm of the first summand, i.e. equals one.

3. Some speculations around the $p = 1$ case. The case of $p = 1$ is essentially difficult. It is evident that $X = \mathbb{R}$ with $G = \{-I, +I\}$ gives an example of uniformly G-convex space which is not uniformly $(G, 1)$-convex, which explains the well-known fact that $L_1(\Omega, \Sigma, \mu)$ (with exception of the trivial case when Ω is an atom of μ) is not uniformly convex (in fact, is not strictly convex). But on the other hand, $X = \mathbb{C}$ with $G = \{\pm I, \pm iI\}$ is an example of uniformly $(G, 1)$-convex, which is the reason why Globevnik's theorem [4] on complex uniform convexity of $L_1(\Omega, \Sigma, \mu)$ is valid. Moreover, [6], [1] for $G = \{\pm I, \pm iI\}$ every uniformly G-convex space is uniformly $(G, 1)$-convex, and consequently $L_1(\mu, X)$ inherits uniform G-convexity of the space X.

In connection with this facts the following question naturally arises: what difference between the groups $\{-I, +I\}$ and $\{\pm I, \pm iI\}$ causes such a distinction in the inheritance of uniform G-convexity. For the present we do not have a satisfactory answer for this question. Below we collect some remarks, relevant to this problem.

Proposition 3.1. *Let X be a real Banach space with $\dim X = n$, $G = \{T_1, T_2, \dots, T_m\} \subset L(X)$ and $\sum_{T\in G} T = 0$. Suppose S_X has a face F with non-empty relative $(n-1)$-dimensional interior. Then X is not uniformly $(G,1)$-convex (in fact it is not even strictly $(G,1)$-convex in the natural sense).*

Proof. Let $x_0 \in F$ be the relative $(n-1)$-dimensional interior point of F, and let x^* be the real functional generating F (i.e. such that $F = \{x \in B_X : x^*(x) = 1\}$). According to our assumption there is an $\varepsilon > 0$, such that for every $z \in \varepsilon B_X$

$$\|x_0 + z\| = F(x_0 + z).$$

Then for $t > 0$ sufficiently small and for every $y \in S_X$ (for example, for $y = x_0$) we have

$$\frac{1}{n}\sum_{T\in G} \|x_0 + tTy\| = \frac{1}{n}\sum_{T\in G} F(x_0 + tTy) = 1,$$

which means that $\delta_X^{G,1}(t) = 0$. $\square$

Proposition 3.2. *Let G be a finite algebraic group, $|G| = n$. Then there is a complex Banach space X, and an injective representation $\phi : G \to L(X)$ such that $\phi(G)$ is regular, X is uniformly $\phi(G)$-convex with $\delta_X^{\phi(G)}(t) \ge t^2/(2n + 2) + 0(t^2)$ for small $t > 0$, but X is not uniformly $(\phi(G),1)$-convex.*

Proof. Let $\ell_\infty(G)$ be the space of all complex-valued functions on G, equipped with sup-norm, and take $X \subset \ell_\infty(G)$ consisting of those functions $x : G \to \mathbb{C}$, that $\sum\limits_{h\in G} x(h) = 0$. Define $\phi : G \to L(X)$ in the standard way: $((\phi(g))x)(h) := x(gh)$. Then the real subspace Y of X consisting of all real-valued $x \in X$ is $\phi(G)$-invariant, and B_Y is a polyhedra. So by the previous proposition Y is not uniformly $(\phi(G),1)$-convex, and hence Y is not uniformly $(\phi(G),1)$-convex as well. Now let us estimate $\delta_X^{\phi(G)}(t)$. For arbitrary $x, y \in S_X$ and $t > 0$

$$\max_{g\in G} \|x + t\phi(g)y\| = \max_{h\in G}\max_{g\in G} |x(h) + ty(gh)| =: A$$

Let $h_0 \in G$ be the point at which $|x(h_0)| = \|x\| = 1$. Without loss of generality we may assume that $x(h_0) = 1$ (otherwise multiply x by a suitable modulus-one constant). So we can continue the estimate as follows:

$$A \ge \max_{g\in G} |x(h_0) + ty(gh_0)| = \max_{g\in G} |1 + ty(g)| =: r.$$

If $\max_{g\in G} \operatorname{Re} y(g) > t/(2n+2)$ then evidently $r > 1 + t^2/(2n+2)$, which implies the estimate of $\delta_X^{\phi(G)}(t)$ we need. In the opposite case since $\sum_{g\in G} \operatorname{Re} y(g) = 0$ all $\operatorname{Re} y(g)$ must be not too "big negative":

$$\operatorname{Re} y(g) \geq -nt/(2n+2), \ \forall g \in G$$

So we have the following:

$$r^2 = \max_{g\in G} |1+ty(g)|^2 = \max_{g\in G}(1+2t \operatorname{Re} y(g)+t^2|y(g)|^2) \geq 1-\frac{2nt^2}{2n+2}+t^2 = 1+\frac{2t^2}{2n+1}.$$

Consequently, for small t

$$r \geq 1 + \frac{t^2}{2n+2} + 0(t^2). \qquad \square$$

Remark, that the same construction with real $\ell_\infty(G)$ will give even linear estimate from below for $\delta_X^{\phi(G)}(t)$.

Since G and $\phi(G)$ have the same structure as algebraic groups, Proposition 3.2 has the following meaning: there are no group-theoretic properties of a group $G \subset L(X)$ which can ensure implication (uniform G-convexity) $\Rightarrow$ (uniform $(G,1)$-convexity). So if we are looking for a property of $G \subset L(X)$, that is sufficient for the above implication, this property must involve the actions of operators $T \in G$ on elements of X. Something in this direction can be done if one looks at the proof (uniform G-convexity) $\Rightarrow$ (uniform $(G,1)$-convexity) in the case of $G = \{I, -I, iI, -iI\}$ given in [6]. The proposition below is a generalization of such kind. Remark, that it is applicable to both real and complex spaces.

Proposition 3.3. *Let X be a Banach space, $G = \{T_1, T_2, \dots, T_n\} \subset L(X)$ be a group of isometries, and $\sum_{T\in G} T = 0$. Suppose the following two conditions are fulfilled:*

(a) For every $x \in S_X$ the subspace $\operatorname{Lin} Gx := \operatorname{Lin}\{Tx\}_{T\in G}$ is 1-complemented in X, i.e. there is a linear projector $P_x : X \to \operatorname{Lin} Gx$ with $\|P_x\| = 1$.

(b) The family $\{\operatorname{Lin} Gx\}_{x\in S_X}$ is a uniformly $(G,1)$-convex family of subspaces, *i.e. $\delta(t) := \inf_{x\in X} \delta^{G,1}_{\operatorname{Lin} Gx}(t) > 0$ for all $t > 0$.*

Then if X is uniformly G-convex, then X is uniformly $(G,1)$-convex.

Proof. Remark that P_x in condition (a) can be taken in such a way, that $PT = TP$ for every $T \in G$. In fact, for arbitrary P_x from (a) the operator

$\hat{P}_x := \dfrac{1}{n}\sum\limits_{T\in G} T^{-1}P_xT$ is a projector onto $\operatorname{Lin} Gx$ with $\|\hat{P}_x\| = 1$ and $\hat{P}_xT = T\hat{P}_x$ for all $T \in G$. So $\hat{P}_x$ can be taken instead of P_x.

Fix arbitrary $x, y \in S_X$ and $t > 0$. We have to estimate from below the value of

$$A := \frac{1}{n}\sum_{T\in G} \|x + tTy\|.$$

Denote $\alpha = \dfrac{1}{2t(n-1)}\delta_X^G(t)$. Consider two cases. The first one is $\|P_xy\| \geq \alpha$. In this case

$$A \geq \frac{1}{n}\sum_{T\in G} \|P_x(x + tTy)\| = \frac{1}{n}\sum_{T\in G} \|x + tTP_xy\| \geq 1 + \delta(\alpha t).$$

In the opposite case of $\|P_xy\| < \alpha$ for every $T \in G$ we have $\|x + tTy\| \geq \|x + tTP_xy\| \geq 1 - \alpha t$. So

$$A \geq \frac{1}{n}(\max_{T\in G} \|x + tTy\| + (n-1)\alpha t) \geq 1 + \frac{1}{n}(\delta_X^G(t) + (n-1)\alpha t) = 1 + \frac{1}{2n}\delta_X^G(t).$$

So,

$$\delta_X^{G,1}(t) \geq \min\left\{\delta\left(\frac{1}{2(n-1)}\delta_X^G(t)\right), \frac{1}{2n}\delta_X^G(t)\right\} > 0. \qquad \square$$

Surely, for $G = \{I, -I, iI, -iI\}$ both conditions (a) and (b) are fulfilled, because $\operatorname{Lin} Gx = \mathbb{C}x$ is up to isometry the same space for all $x \in S_X$, and by complex Hahn-Banach theorem $\mathbb{C}x$ is 1-complemented in X. Remark also, that (b) is a necessary condition for $(G, 1)$-convexity of X. But (a) seems to be a too strong restriction.

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Received March 23, 2008