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PROBABILISTIC APPROACH TO THE NEUMANN PROBLEM FOR A SYMMETRIC OPERATOR

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Abstract. We give a probabilistic formula for the solution of a non-homogeneous Neumann problem for a symmetric nondegenerate operator of second order in a bounded domain. We begin with a γ-Hölder matrix and a $\mathcal{C}^{1,\gamma}$ domain, $\gamma > 0$, and then consider extensions. The solutions are expressed as a *double* layer potential instead of a single layer potential; in particular a new boundary function is discovered and boundary random walk methods can be used for simulations. We use tools from harmonic analysis and probability theory.

1. Introduction. Let $a = a(x)$, $x \in R^d, d \geq 1$, be a $d \times d$ real matrix, a_0, a_1 positive constants, D a bounded domain in R^d, f and g interior and boundary functions, $n(\cdot)$ the unit outward pointing normal to ∂D and ∂_{n_a} the

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co-normal derivative, i.e. $\partial_{n_a}(\cdot) = \dfrac{1}{2} a\nabla(\cdot)\cdot n$. We assume that for all $\xi \in R^d$ we have uniform ellipticity and boundedness

$$a_0\|\xi\|^2 \le a(x)\xi\cdot\xi \le a_1\|\xi\|^2. \tag{1}$$

Consider the following non homogeneous Neumann problem

$$\begin{aligned} \frac{1}{2}\nabla\cdot a\nabla u &\overset{D}{=} -f, \\ \partial_{n_a} u &\overset{\partial D}{=} g. \end{aligned} \tag{2}$$

The interior operator in (2) will be denoted by A. It is well known, see e.g. [15], that the above problem has a solution (unique modulo additive constants) whenever the following compatibility, or centering, condition holds

$$\int_D f(x)dx + \int_{\partial D} g(\alpha)d\alpha = 0, \tag{3}$$

in which case solutions u are understood in the weak sense

$$1/2\int_D a\nabla u\cdot\nabla\varphi dx = \int_D f\varphi dx + \int_{\partial D} g\varphi d\alpha, \tag{4}$$

for all φ in a suitable space.

1.1. Notations. We adopt standard notations especially when function spaces are involved, e.g. the subscripts c and 0 stand respectively for "compact support" and "vanish at infinity", the superscript $*$ for "dual" and $\mathcal{D}(\cdot)$ for "domain of". Our a.e.'s are understood to be with respect to the Lebesgue measure of the underlying space and a.s. stands for almost surely. Moreover, the letters x, y, ... are reserved for interior variables, whereas α, β, ... stand for boundary variables. The volume of D, respectively the area of ∂D, is denoted by $|D|$, respectively $|\partial D|$. The normalised Lebesgue measure $d\alpha/|\partial D|$ will be sometimes noted μ_0. The scalar product in $L^2(\partial D, \mu_0)$ is denoted by $(\cdot,\cdot)_\partial$. Unimportant constants will be denoted by c, c' ... and they may vary from line to line.

1.2. Motivation and results. There exist already probabilistic representations for the solution u. The first probabilistic formula for u in (2) where $f = 0$ is due to [14] in the case $A = \Delta/2$ in balls where dealing with the local time $L(t)$, see Section 2.5, is made easy thanks to the presence of spherical symmetries.

Then, exploiting the machinery of probabilistic potential theory and especially the Shur-Meyer representation theorem for additive functionals, [5] generalized this result to more general smooth domains D. This work has been improved, still in the case of the Laplacian, to the situation of Lipschitz domains in [4] using techniques from PDE theory and analysing the fundamental solution of the parabolic PDE with a reflecting boundary condition which describes the transition density of reflecting Brownian motion.

In our approach it seemed to us more natural to work directly on the boundary stochastic processes in an intrinsic way and consider non differentiable a. We take advantage of an ergodicity phenomenon in the boundary. Theorem 4 below shows that for an appropriate boundary function N, see (18), which we call the Neumann boundary function in honor of Carl Gottried Neumann, the Neumann problem (2) can be reduced to a Dirichlet problem (7) whereby expressing the solution u in terms of a *double* layer potential instead of a single layer potential which is the standard theory for the Neumann problem,

$$u(x) = Gf(x) - \int_{\partial D} \partial_{n_a(\alpha)} G(x, \alpha) N(\alpha) d\alpha, \tag{5}$$

where $G(\cdot, \cdot)$ is the Green function for D, under sufficient regularity on the data; see (22) and (23) for the general case. We emphasize that the function N is *not* the trace on the boundary of the ordinary Neumann kernel, see e.g. [15], which plays the role of a Green function for the Neumann problem. To our knowledge our theorems are new, see e.g. [25] for a recent account of the layer potential theory in a Lipschitz domain and its applications. We expect that our representation formulas will also yield new results in the aforementioned field because the study of the boundary stochastic processes works at the level of the sample paths, see e.g. the classical work of [21] in Potential Theory. Probability solutions to some PDEs and problems in Analysis is not new and solutions can sometimes be given before analytic ones. Concerning rough data, loosely speaking, our work on the λ-semigroup suggests a unified intuitive probabilistic counterpart to recent analytic advances in the study of the Neumann problem. Indeed new variants of function spaces are now being frequently introduced (it is not possible to list them all here in a comprehensive way) and added to the wealth of function spaces already in existence, see e.g. [1]. This reconfirmation of the deep interplay between Analysis and Probability Theory is a contribution of our paper, in addition to the novelty of our representations.

We begin in section 4 with classical data: a γ-Hölder matrix and a $\mathcal{C}^{1,\gamma}$-domain, $\gamma = 1 - d/p$ for some large p; under these classical conditions the solution of the A-Dirichlet problem admits a proper gradient on the boundary, i.e. at least bounded. Clearly, the results of this section are still valid in the situations where a bounded gradient on the boundary exists, see Section 4.3 for a variant. On the other hand, when $f = 0$ we can take a continuous and D Lipschitz; our representation formula below (23) is of some interest since it provides an alternative expression for the integral representation of [4] in which $A = (1/2)\Delta$ and where specific properties of the reflecting Brownian motion are used. However, the Neumann boundary function $\bar{N}$ in (23) is associated with a special boundary semigroup which we call the λ-semigroup. A Hunt process associated with the λ-semigroup can be firmly identified when ∂D is essentially uniformly $\mathcal{C}^1$, see [1, Section 7.51.] We give in Section 3 a more explicit discussion about these boundary stochastic processes. In Section 5.2 we study a class of problems where the function $\bar{N}$ can be expressed in terms of the trace process itself and give an interior representation which generalises the probabilistic formula of [5].

Note that as far as practical simulations are concerned, the boundary random walk method in [23] may serve as a model for future implementations.

1.3. Assumptions on the data. In this paper $p > d$ unless explicitly stated.

1.3.1. When f is non-trivial.

Condition 1. *The boundary ∂D is a finite disjoint union of closed and bounded surfaces in $W^{2,p}$.*

When $p > d$, the boundary is given locally by $\mathcal{C}^{1,\gamma}(R^{d-1})$ functions, $\gamma = 1 - d/p$. As far as our coefficients are concerned, we assume that

Condition 2. *$a \in W^{1,p}(D)$, $f \in L^p(D)$ and $g \in L^\infty(\partial D)$ with*

$$\sum_{i,j,k}^{d} \|\partial_{x_k} a_{ij}\|_{L^p(D)} + \|f\|_{L^p(D)} \leq a_1.$$

1.3.2. The case $f = 0$.

Condition 3. *The matrix a is continuous and ∂D is a finite disjoint union of closed and bounded surfaces in $\mathcal{C}^{0,1}$.*

1.4. An equivalent PDE formulation. We shall adopt a boundary treatment by elliminating the interior function f. In this section we assume that $f \neq 0$, (1) and conditions 1 and 2 hold.

Set $g_0 = g - \partial_{n_a} Gf$ and consider the homogeneous Neumann problem $Au_0 \overset{D}{=} 0, \partial_{n_a} u_0 \overset{\partial D}{=} g_0$. By the Green formula we have $\int_{\partial D} g_0 d\alpha = 0$. This shows that the homogeneous Neumann problem is compatible and that $u = Gf + u_0$ gives the solution (modulo additive constants) of the system (2). The rest of the paper is devoted to establishing a probabilistic representation for both Gf and u_0.

2. Preliminaries. In this section we gather for the convenience of the reader some insights and facts that will be needed below. We have endeavoured to underline somewhat deep intuitive facts and to keep to the minimum the various definitions which are to be found in the references at the end of the paper. At times, an issue is settled by means of an example. Our paper can be understood on intuitive grounds.

2.1. Sobolev spaces and multipliers. Let $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(R^d)$ consists, see e.g. [1], of all $u \in L^p(R^d)$ s.t. for some $v \in L^p(R^d)$,

$$\int_{R^d} u(x)\nabla\varphi(x)dx = -\int_{R^d} v(x)\varphi(x)dx, \forall\varphi \in \mathcal{C}_c^\infty(R^d). \tag{6}$$

The space $W^{2,p}(R^d)$ is defined in the obvious way. The functions in these spaces are defined within sets of Lebesgue measure zero. For sub-domains D of R^d there are several ways to define the Sobolev spaces, however for a bounded Lipschitz one, they all turn out to be equivalent to (6) with D instead of R^d. When D has bounded $\mathcal{C}^1$ boundary and $p > d$ we have $W^{2,p}(D) \subset \mathcal{C}^1(\bar{D})$.

The fractional Sobolev spaces $W^{s,p}(R^d)$, $s \in R$, are defined thanks to the Fourier transform. Manifold fractional Sobolev spaces are defined in a similar way thanks to a system of local coordinates and reduction to (a sub-domain of) a Euclidean space. When ∂D is locally given by R^{d-1}-Lipschitz functions we have

$$W^{1/2,2}(\partial D) = \{\varphi \in L^2(\partial D) / \int_{\partial D\times\partial D} \frac{|\varphi(\alpha) - \varphi(\beta)|^2}{\|\alpha - \beta\|^d} d\alpha d\beta < \infty\}.$$

The set of traces of $W^{1,2}(D)$ functions on ∂D is the space $W^{1/2,2}(\partial D)$ and the trace operator $Tr : W^{1,2}(D) \to W^{1/2,2}(\partial D)$ is onto. It is compact when taking values in the larger space $L^2(\partial D)$, see [3, Section 2]. We shall write φ^∂ for $Tr(\varphi)$ and suppress the superscript ∂ when no ambiguity arises.

2.1.1. Sobolev multipliers. According to classical results, see e.g. the comments following Lemmas 8.1 and 8.2 of [16] and Theorem 6.31 of [13], some regularity of the domain, e.g. $\mathcal{C}^{2,\alpha}$, $\alpha > 0$, is needed to derive at least $\mathcal{C}^2$-regularity results up to the boundary for the various elliptic boundary value problems. However, by means of the theory of Sobolev multipliers it is possible to relax the regularity conditions on the boundary and still have $W^{2,p}(D)$-regularity. Loosely speaking, the multiplier theory bears some resemblance with the theory of Dirichlet forms (see Section 2.3) in that the technics act at the level of the variational formulas without trying to explicitly tackle the quantities that appear individually in these formulas, see the example given on p. 1 of [18] and [2] for an insight. In the classical Sobolev treatment of PDEs, variational solutions are usually shown to have more regularity depending on the regularity of the data. The multiplier space $M(W^{m,p}(D) \to W^{l,p}(D))$ is the class of functions φ s.t. the pointwise product $\varphi u \in W^{l,p}(D)$ for all $u \in W^{m,p}(D)$. Let us consider the problem $Au \overset{D}{=} -f, Bu \overset{\partial D}{=} g$ where $A(\cdot) = \sum_{|\nu|\leq 2} a_\nu(x)D^\nu(\cdot)$ and $B(\cdot) = \sum_{|\nu|\leq 1} b_\nu(\alpha)D^\nu(\cdot)$ where $a_\nu \in \mathcal{C}(\bar{D}), b_\nu \in \mathcal{C}^1(\bar{D})$ and ν is a multi-index. If $p > d$, $f \in L^p(D)$, $g \in W^{1-(1/p),p}(\partial D)$ and the boundary is $W^{2-(1/p),p}$ then $u \in W^{2,p}(D)$ by Theorem 7.3.2 in [18].

2.2. The Green and Harmonic operators. Let us consider the Dirichlet problem $Aw \overset{D}{=} -f, w \overset{\partial D}{=} \varphi$. Given the inequalities (1) and conditions 1, 2, it follows from Theorem 15.1 in [16] that the function w belongs to $W^{2,p}(D) \cap \mathcal{C}^{1,\gamma}(\bar{D})$. The norms of w in both $W^{2,p}(D)$ and $\mathcal{C}^{1,\gamma}(\bar{D})$ are bounded by a constant which depends only on d, p, a_0, a_1, $\|w\|_{L^2(D)}, \|\varphi\|_{W^{2,p}(D)}$ and D, where φ stands for an extension to D of our boundary values.

The solution when $\varphi = 0$ is given by $w = Gf$ where G is the Green operator. On the other hand, the solution of the Dirichlet problem

$$\begin{aligned} Aw &\overset{D}{=} 0, \\ w &\overset{\partial D}{=} \varphi, \end{aligned} \tag{7}$$

where $\varphi \in W^{1/2,2}(\partial D)$ is said to be A-harmonic and noted $H\varphi$; H is called the harmonic operator applied to the boundary function φ. The harmonic measure at x, i.e. the integral kernel of H, is denoted by $H(x, d\alpha)$ and represents the exit measure from D for our reflecting diffusion $X(t)$ below. See [15] sections 1.2 and 1.3 for some properties of the operators G, H in a more general setting.

2.3. Dirichlet forms and their potential theory. The Gauss approach to the solution of (7) (historically with $A = \Delta$) is to minimise the quantity $\mathcal{E}(u,u)$ over a suitable domain $\mathcal{D}(\mathcal{E})$ of the following symmetric form

$$\mathcal{E}(u,v) = 1/2 \int_D a\nabla u \cdot \nabla v dx. \tag{8}$$

See e.g. [19] for a comprehensive historical account. The number $\mathcal{E}(u,u)$ is called the Dirichlet integral or energy form and does indeed have the corresponding Gauss's physical interpretation. The probabilistic counterpart of the theory inaugurated by Gauss concerns symmetric Markov processes. The form $\mathcal{E}$ is a quadratic form which is essentially a kinetic energy and therefore represents a *movement*. The Lebesgue measure is unsufficient for a natural treatment as we are constantly dealing with a hidden concept of derivation and its difficulties linked to manipulating higher order infinitesimals in Lebesgue means over various sets. Gradually relaxing the conditions on the data the following concept of capacity has appeared for an open subset E of D

$$\mathrm{cap}(E) = \inf\{\|u\|^2_{L^2(D)} + \mathcal{E}(u,u)/u \geq 1 \text{ a.e. on } E\}, \tag{9}$$

and we set $\mathrm{cap}(E) = \infty$ if the set within the braces is empty (for fundamentals see [11]). The set function $\mathrm{cap}(\cdot)$ is only countably sub-additive and the concept of integration with respect to $\mathrm{cap}(\cdot)$ is a special one (called the Choquet integral and extensively used in Fuzzy Analysis for example) but nevertheless provides a finer description of small sets than does the Lebesgue measure, see e.g. [6]. Indeed, a subset $E \subset R^d$ with $\mathrm{cap}(E) = 0$ has zero Lebesgue measure whereas $\mathrm{cap}(E)$ may be infinite at an E with zero Lebesgue measure. Thanks to $\mathrm{cap}(\cdot)$ accurate representatives of elements of $\mathcal{D}(\mathcal{E})$ can be defined. A function u is quasi-continuous if it is continuous outside open sets of arbitrarily small capacity. A function φ in $\mathcal{D}(\mathcal{E})$ has always a quasi-continuous representative $\tilde{\varphi}$, unique modulo sets of zero capacity. A statement depending on $x \in E$ is said to hold quasi-everywhere (q.e.) on E if there is a set $E_0 \subset E$ with zero capacity s.t. the statement is true $\forall x \in E \backslash E_0$.

2.4. The Steklov problem. We now outline the main results (as far as we are concerned here) from [3]. The following PDE where λ is a real number

$$\begin{aligned} Au &\overset{D}{=} 0, \\ \partial_{n_a} u &\overset{\partial D}{=} \frac{\lambda}{|\partial D|} u, \end{aligned} \tag{10}$$

is called the (normalized) Steklov problem. In [3] the domain D is bounded Lipschitz and the matrix a continuous. The variational formulation for the problem (10) (i.e. for example establishing thanks to a Lagrange multiplier argument a Euler-Lagrange equation then solving it) shows that the spectrum is non negative. The eigenspace corresponding to $\lambda = 0$ is shown to be of dimension one, it is generated by the function $S_0 = 1$. Thanks to a variational formulation for the first non zero eigenvalue it is possible to derive a boundary trace inequality which shows that the following inner product in $W^{1,2}(D)$

$$(\varphi, \psi)_a = \frac{1}{2} \int_D a\nabla\varphi \cdot \nabla\psi dx + (\varphi^\partial, \psi^\partial)_\partial,$$

turns out to be equivalent to the standard norm of $W^{1,2}(D)$. Let $\mathcal{B}_a$ be the closed unit ball of $W^{1,2}(D)$ relative to $\|\cdot\|_a$ and $\mathcal{B}_a^{\partial,1}$ be the bounded closed and convex set of elements $\varphi \in \mathcal{B}_a$ with the constraint $\int_{\partial D} \varphi^\partial d\alpha = 0$. Put

$$\delta_1 = \sup\{\|\varphi^\partial\|_\partial / \varphi \in \mathcal{B}_a^{\partial,1}\}.$$

Then it is shown that the maximizers S_1 of this problem are Steklov eigenfunctions corresponding to the first non zero eigenvalue λ_1 and moreover we have $\delta_1 = \dfrac{1}{1+\lambda_1}$. Successive eigenvalues and eigenfunctions are found thanks to an iterative procedure. They satisfy $(S_j^\partial, S_k^\partial)_\partial = \dfrac{1}{1+\lambda_j}\delta_{jk}$, $j, k \geq 0$, where δ_{jk} is the Kronecker symbol. The spectrum is shown to be discrete, each λ_j has finite multiplicity and $\lambda_j \to \infty$ as $j \to \infty$. Moreover, the sequence of all the Steklov eigenfunctions $\{S_j, j \geq 0\}$ is an orthonormal basis for the subspace, say $W_a^{1,2}(D)$, of $W^{1,2}(D)$ which is $\|\cdot\|_a$-orthogonal to $W_0^{1,2}(D)$, i.e.

$$W^{1,2}(D) = W_0^{1,2}(D) \oplus W_a^{1,2}(D). \tag{11}$$

The Fourier-Steklov coefficients of a function $\varphi \in W_a^{1,2}(D)$ in the Steklov system is denoted by φ_j. It is also established that the system $\{\sqrt{1+\lambda_j}S_j^\partial, j \geq 0\}$ is an orthonormal basis for the space $L^2(\partial D, \mu_0)$. We shall subsequently write $\tilde{S}_j^\partial$ for $\sqrt{1+\lambda_j}S_j^\partial$ and φ_j^∂ for the coefficients of $\varphi \in L^2(\partial D)$ in this boundary Steklov system, i.e. $\varphi_j^\partial = \int_{\partial D} \varphi(\alpha)\tilde{S}_j^\partial(\alpha) d\mu_0$, $j \geq 0$. For example in the case of a ball in R^d, these are just the well known spherical harmonics, see e.g. [8]. In the unit disc of R^2, we have in polar coordinates $x = (r\cos\alpha, r\sin\alpha), r \leq 1, \alpha \in [0, 2\pi)$, for $j \geq 1$

$$
\begin{aligned}
S_{2j-1}(x) &= r^j \sin(j\alpha) \\
S_{2j}(x) &= r^j \cos(j\alpha) \qquad (12)\\
\lambda_{2j-1} &= \lambda_{2j} = j.
\end{aligned}
$$

These Steklov eigenfunctions allow us to give an explicit series representation for the solution w in $W^{1,2}(D)$ of the Dirichlet problem (7). We have for $\varphi \in W^{1/2,2}(\partial D)$,

$$
H\varphi = \sum_{j=0}^{\infty} \sqrt{1+\lambda_j}\varphi_j^{\partial} S_j, \tag{13}
$$

from which one deduces the interesting spectral characterization of $W^{1/2,2}(\partial D)$: a function $\varphi \in W^{1/2,2}(\partial D)$ iff

$$
\sum_{j=0}^{\infty}(1+\lambda_j)(\varphi_j^{\partial})^2 < \infty. \tag{14}
$$

2.5. Probability background. Analytic concepts are being increasingly translated into probabilistic ones and because the latter are couched in terms of the sample paths, one enjoys a great freedom to formulate problems (and solutions), at least on the intuitive level. The best known stochastic processes are the Feller ones. Let E be essentially a locally compact metric space. A semigroup P_t on E is said to be Feller if the class $\mathcal{C}_0(E)$ is left invariant under P_t. The important class of Markov chains is not Feller. A whole generation of mathematicians has endeavoured to unify the theory of Markov processes, the name of P. A. Meyer is strongly associated with the story and the field of Probabilistic Potential Theory in now on firm grounds. Surprisingly, all Markov processes are essentially obtained by a suitable compactification of the Feller semigroups (see the intuition based book of D. Williams [27]) and this leads to the concept of "Right Process". Constructing a Markov process corresponding to only L^2-semigroups is more involved since a Riesz representation theorem for the continuous linear functionals on classes of continuous functions is no more available. This is indeed the case of our λ-process below.

If a, D were more regular, then A would be a classical elliptic operator giving rise to an Itô reflecting diffusion, which is Feller, see [17]. When a is only measurable we can still build a diffusion process thanks to the theory of Dirichlet forms, see [11]. Regular Dirichlet forms are associated with the class of Hunt processes (which includes the Feller ones). In sharp contrast to the Feller processes, the Hunt processes are defined at the level of the sample paths. A Hunt process possesses useful properties which are essentially the right continuity of paths, their quasi-left continuity and the strong Markov property.

The appropriate Dirichlet form is here the symmetric bilinear form on $L^2(D)$ given by (8) with domain $\mathcal{D}(\mathcal{E}) = W^{1,2}(D)$. It is well known that under condition (1) the form $\mathcal{E}$ is closed, regular and strongly local in $\bar{D}$, see e.g [12]. There exists a continuous strong Markov process $X(t)$ in $\bar{D}$ associated with $\mathcal{E}$. The form $\mathcal{E}$ can be extended to the larger Dirichlet space $\mathcal{F}_e$ which is Hilbert when endowed with the inner product $\mathcal{E}$ iff $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient. By definition, $\varphi \in \mathcal{F}_e$ iff there exists an $\mathcal{E}$-Cauchy sequence h_n in $W^{1,2}(D)$ s.t. $h_n \to \varphi$ a.e. in D. We have $W^{1,2}(D) = \mathcal{F}_e \bigcap L^2(D)$. Starting from D, the exit properties of $X(t)$ from D are identical to those of the absorbing process $X_0(t)$ which is associated with $\mathcal{E}$ on the domain $W_0^{1,2}(D)$. Since the latter is transient (intuitively speaking, a process is transient if it wanders off to infinity a.s. whereas recurrents ones keep on coming back to already visited regions), it follows that when $\varphi \in W^{1,2}(D)$ we have $\mathcal{H}\varphi^\partial(x) = H\varphi^\partial(x) = E^x[\varphi^\partial(X_\tau)]$, $x \in D$, where the operator $\mathcal{H}(\cdot)$ is associated with the limit as $\lambda \to 0+$ of the λ-order hitting kernel $\mathcal{H}_F^\lambda(x, E) = E^x[\exp(-\lambda\tau_F)I_E(X_{\tau_F})]$, E being a Borel subset of $\bar{D}$, F a nearly Borel subset of ∂D and $\tau_F = \inf\{t > 0/X(t) \in F\}$. For example a Brownian particle $B(t)$ starting inside a Euclidean ball, $d \geq 1$, must exit the ball in finite time a.s.. For a regular boundary, the quantity

$$\mathcal{H}_{\partial D}^0(x, E) = P^x(B_{\tau_{\partial D}} \in E)$$

is given by the Poisson integral (at x) of the boundary function I_E, i.e. it solves the Dirichlet problem with boundary function I_E. Obviously, we can take more general boundary functions than indicators. The point is that since we work at the level of the sample paths we can go very far when choosing these boundary functions. A subset F is nearly Borel if $\exists B_1, B_2$ Borel sets s.t. $B_1 \subset F \subset B_2$ with $P(\exists t \geq 0/X(t) \in B_2 \backslash B_1) = 0$.

We emphasize that the operator $\mathcal{H}$ is applied to a larger boundary space than merely $W^{1/2,2}(\partial D)$ outside of which the Sobolev-PDE Dirichlet problem (7) is not solvable, see Sections 3.1 and 3.3.

The local time on the boundary, i.e. $L(t) = \displaystyle\int_0^t I_{\{\partial D\}}(X_s)ds$, is a continuous additive functional which is associated with the surface measure $d\alpha$

via the Revuz correspondence. Intuitively, $L(t)$ measures the time spent at ∂D up to t. In fact, we rigorously have $L(t) = \lim_{\delta\to 0}(1/\delta)\int_0^t I_{D^\delta}(X_s)ds$ where $D^\delta = \{x \in \bar{D}/d(x,\partial D) \le \delta\}$. In general additive functionals (still noted $L(t)$) are essentially adapted, right continuous, non decreasing and time homogeneous, i.e. $L(t+s,\omega) = L(s,\omega) + L(t,\theta_s(\omega)), t, s \ge 0$ where $\theta_.(\omega)$ is the shift operator. For a motivated account see [9, Chapter 15]. Here is a sum up. Take a smooth matrix a and $D = R^d$. For a bounded function h set $L(t) = \int_0^t h(X_s)ds$. A random Laplace transform relative to the measure $dL(t)$ is defined, averaged at $x \in R^d$ and shown to be represented as a suitable space integral of a certain kernel. The simplest case being that of $h(x) = 1$, $x \in R^d$, leading straightaway to the resolvent operator of the diffusion X. More subtle killings are then considered but the theory concerns essentially $L(t)$'s with a finite potential $E^x(L(\infty))$. In Revuz [22] a breakthrough to recurrent processes is achieved and shown to have links with the ergodic properties of X. The idea is to look for the "speed" of $L(t)$ which has a similarity with the search for the infinitesimal generator in semigroup theory.

The trace of X_t on the boundary is denoted by X_t^∂, i.e. $X_t^\partial = X(\tau_t)$, where τ_t is the right continuous inverse of $L(t)$, i.e. $\tau_t = \inf\{s/L(s) > t\}$. From the general theory of Markov processes, X_t^∂ is a strong Markov process with right continuous paths. We shall set $\mu_t(\alpha, d\beta)$, $t \ge 0$, for the law of X_t^∂ starting from α, i.e. $\mu_t(\alpha, E) = P^\alpha(X(t) \in E)$ where the probability operators P^α act on the space of right continuous boundary functions and are knitted together by the strong Markov property, see [27]. The process X_t^∂ is associated with the symmetric form on the boundary $\mathcal{E}^\partial(\varphi,\psi) = \mathcal{E}(\mathcal{H}\varphi,\mathcal{H}\psi)$ with domain $\mathcal{D}(\mathcal{E}^\partial)$ which consists of all $\varphi \in L^2(\partial D)$ which are equal a.e. on ∂D to the trace of some $\psi \in \mathcal{F}_e$, i.e. the trace space. We know from Theorem 6.2.1 of [11] that the semigroup P_t^∂ of the process X_t^∂ is a symmetric strongly continuous semigroup of bounded linear operators on $L^2(\partial D)$. Its non-positive selfadjoint generator A^∂ seems to have been first identified in [24].

3. A key boundary semigroup. This section is of independent interest. We suppose the condition in Section 1.3.2 holds. Let us introduce the λ-semigroup already mentioned in Section 1.2. Let $\varphi \in L^2(\partial D)$ and $\lambda > 0$, the following PDE

$$
\begin{aligned}
&Au_\lambda \overset{D}{=} 0, \\
&\partial_{n_a} u_\lambda + \frac{\lambda}{|\partial D|} u_\lambda \overset{\partial D}{=} \varphi,
\end{aligned} \tag{15}
$$

can be solved thanks to a standard variational argument, i.e. there exists a unique $u_\lambda \in W^{1,2}(D)$ s.t. $\forall \psi \in W^{1,2}(D)$ we have

$$
\frac{1}{2} \int_D a \nabla u_\lambda \cdot \nabla \psi dx = \int_{\partial D} \left(\varphi - \frac{\lambda}{|\partial D|} u_\lambda^\partial \right) \psi^\partial d\alpha.
$$

We shall also write $u_\lambda = u_\lambda(\varphi)$. The main result of this section is the

Theorem 1. *The family of boundary operators u_λ^∂, $\lambda > 0$, is a symmetric strongly continuous contraction resolvent on $L^2(\partial D, \mu_0)$ with $(\lambda/|\partial D|) u_\lambda^\partial(1) = 1$. Its generator is associated with the closed symmetric form $\bar{\mathcal{E}}^\partial(\varphi, \psi) = \frac{1}{2} \int_D a \nabla H \varphi(x) \cdot \nabla H \psi(x) dx$ with domain $\mathcal{D}(\bar{\mathcal{E}}^\partial) = W^{1/2,2}(\partial D)$. Its semigroup $\bar{P}_t^\partial$ admits the representation*

$$
\bar{P}_t^\partial \varphi = \int_{\partial D} \varphi(\alpha) d\mu_0(\alpha) + \sum_{j=1}^\infty \varphi_j^\partial \exp\left(-\frac{\lambda_j}{|\partial D|} t \right) \tilde{S}_j^\partial, \tag{16}
$$

for all $t \geq 0$. Moreover, we have the exponential bound

$$
\|\bar{P}_t^\partial \varphi\|_\partial \leq c \exp(-c' t) \|\varphi\|_\partial, \tag{17}
$$

where c and c' are two positive constants $\forall \varphi \in L^2(\partial D)$ with $\int_{\partial D} \varphi(\alpha) d\alpha = 0$; in particular, the Neumann boundary function

$$
\bar{N}(\alpha) = \int_0^\infty \bar{P}_t^\partial \varphi(\alpha) dt, \tag{18}
$$

is a well defined element of $W^{1/2,2}(\partial D)$.

P r o o f. Let $\varphi \in L^2(\partial D)$ and $\lambda, \mu > 0$. The variational formulation for (15) implies that for $j \geq 0$

$$
\frac{1}{2} \int_D a \nabla u_\lambda \cdot \nabla S_j dx = \int_{\partial D} (\varphi - \frac{\lambda}{|\partial D|} u_\lambda^\partial) S_j^\partial d\alpha,
$$

which clearly gives the interior Fourier-Steklov coefficient of $u_\lambda(\varphi)$ in the system $\{S_j, j \geq 0\}$

$$u_{\lambda,j}(\varphi) = (u_\lambda(\varphi), S_j)_a = |\partial D| \frac{\sqrt{1+\lambda_j}}{\lambda+\lambda_j} (\varphi, \tilde{S}_j^\partial)_\partial.$$

We have therefore

$$u_\lambda(u_\mu^\partial(\varphi)) = \sum_{j=0}^{\infty} |\partial D| \frac{\sqrt{1+\lambda_j}}{\lambda+\lambda_j} (u_\mu^\partial(\varphi), \tilde{S}_j^\partial)_\partial S_j,$$

hence as the trace operator is compact we have

$$u_\lambda^\partial(u_\mu^\partial(\varphi)) = \sum_{j=0}^{\infty} \frac{|\partial D|^2}{(\lambda+\lambda_j)(\mu+\lambda_j)} (\varphi, \tilde{S}_j^\partial)_\partial \tilde{S}_j^\partial.$$

On the other hand, a direct calculation yields

$$u_\lambda^\partial(\varphi) - u_\mu^\partial(\varphi) = \left(\frac{\mu}{|\partial D|} - \frac{\lambda}{|\partial D|}\right) \sum_{j=0}^{\infty} \frac{|\partial D|^2}{(\lambda+\lambda_j)(\mu+\lambda_j)} (\varphi, \tilde{S}_j^\partial)_\partial \tilde{S}_j^\partial,$$

showing that the resolvent equation holds for the family $u_\lambda^\partial(\varphi)$, $\lambda > 0$.

Let us show that it is strongly continuous. It suffices to prove (from the general theory) that the range $\{u_\lambda^\partial(\varphi)/\varphi \in L^2(\partial D)\}$ is dense in $L^2(\partial D)$. Indeed, for $\varphi = \tilde{S}_j^\partial$, $j \geq 0$, we have $u_\lambda^\partial(\varphi) = (|\partial D|/(\lambda+\lambda_j))\tilde{S}_j^\partial$ so that the whole family of the Steklov traces $\{\tilde{S}_j^\partial, j \geq 0\}$ lies in the range of u_λ^∂.

To show the contraction property it suffices to write

$$\frac{\lambda}{|\partial D|} u_\lambda^\partial(\varphi) = \sum_{j \geq 0} \frac{1}{1+(\lambda_j/\lambda)} \varphi_j^\partial \tilde{S}_j^\partial,$$

from which the result follows by calculating the norm in $L^2(\partial D, \mu_0)$ of both sides and noticing that $\forall j \geq 1$, $(\lambda_j/\lambda) > 0$.

It now remains to identify the domain of its associated closed form $\bar{\mathcal{E}}^\partial$ on $L^2(\partial D, \mu_0)$. We use the criterion in Lemma 1.3.4 of [11], i.e. we shall prove that $\lim\limits_{\lambda\to\infty} (\lambda/|\partial D|) \int_{\partial D} [\varphi - (\lambda/|\partial D|) u_\lambda^\partial(\varphi)]\varphi d\alpha < \infty$ iff $\varphi \in W^{1/2,2}(\partial D)$. We have by straightforward calculations,

$$\frac{\lambda}{|\partial D|} \int_{\partial D} \left(\varphi - \frac{\lambda}{|\partial D|} u_\lambda^\partial\right) \varphi d\alpha = \sum_{j=0}^{\infty} \frac{\lambda \cdot \lambda_j}{\lambda+\lambda_j} (\varphi_j^\partial)^2.$$

Hence, our limit (which is a supremum) is finite iff

$$\sum_{j=0}^{\infty} \lambda_j (\varphi_j^\partial)^2 < \infty,$$

which clearly is equivalent to criterion (14) and our claim is proved.

Set $\bar{A}^\partial$ for the generator of the resolvent u_λ^∂, i.e. $\bar{A}^\partial = (\lambda/|\partial D|) - (u_\lambda^\partial)^{-1}$. When studying the strong continuity point we saw that $\forall j \geq 0$ we have

$$(u_\lambda^\partial)^{-1}(\tilde{S}_j^\partial) = \frac{\lambda + \lambda_j}{|\partial D|} \tilde{S}_j^\partial$$

from which follows the differential equation

$$d(\bar{P}_t^\partial \tilde{S}_j^\partial)/dt = -(\lambda_j/|\partial D|) \bar{P}_t^\partial \tilde{S}_j^\partial$$

which gives $\bar{P}_t^\partial \tilde{S}_j^\partial = \tilde{S}_j^\partial \exp(-t\lambda_j/|\partial D|)$. Let $\varphi \in L^2(\partial D)$, since the operator $\bar{P}_t^\partial$ is bounded, we have

$$\bar{P}_t^\partial \varphi = \int_{\partial D} \varphi(\alpha) d\mu_0(\alpha) + \sum_{j=1}^{\infty} \varphi_j^\partial \exp\left(-\frac{\lambda_j}{|\partial D|} t\right) \tilde{S}_j^\partial.$$

Next, let $\varphi \in L^2(\partial D)$ be centered, i.e. $\displaystyle\int_{\partial D} \varphi d\alpha = 0$ and let $t > 0$. Define the boundary function $\bar{N}(t, \alpha) = \displaystyle\int_0^t \bar{P}_s^\partial \varphi(\alpha) ds$ which belongs to the domain of $\bar{A}^\partial$ by the general theory of semigroups. We also have $\bar{A}^\partial \bar{N}(t, \alpha) = \bar{P}_t^\partial \varphi(\alpha) - \varphi(\alpha)$. The semigroup $\bar{P}_t^\partial$ clearly satisfies the exponential bound (17) at φ. This not only implies that $\bar{P}_t^\partial \varphi \to 0$ as $t \to \infty$ but also that $\displaystyle\int_0^t \bar{P}_s^\partial \varphi ds \to \bar{N}$, strongly in $L^2(\partial D)$. As $\bar{A}^\partial$ is closed it follows that $\bar{N} \in \mathcal{D}(\bar{A}^\partial) \subset W^{1/2,2}(\partial D)$ and that $\bar{A}^\partial \bar{N}(t)$ converges strongly to $\bar{A}^\partial \bar{N} = -\varphi$ as $t \to \infty$. $\square$

3.1. Relation between the $\boldsymbol{\lambda}$ and trace semigroups. Since the forms $\bar{\mathcal{E}}^\partial$ and $\mathcal{E}^\partial$ are equal on $W^{1/2,2}(\partial D)$, it follows that the domain of $\bar{\mathcal{E}}^\partial$ is closed in that of $\mathcal{E}^\partial$. In fact it is a closed ideal, i.e. when $0 \leq \varphi \leq \psi$ where $\varphi \in \mathcal{D}(\mathcal{E}^\partial)$ and $\psi \in W^{1/2,2}(\partial D)$ then $\varphi \in W^{1/2,2}(\partial D)$. We have the

Theorem 2. *Suppose the condition in Section* 1.3.2 *is satisfied. The space* $W^{1/2,2}(\partial D)$ *is an ideal of* $\mathcal{D}(\mathcal{E}^\partial)$. *There exists a non negative* $\mathcal{E}^\partial$*-quasicontinuous function* F *s.t.* $\mathcal{D}(\bar{\mathcal{E}}^\partial) = \{\varphi \in \mathcal{D}(\mathcal{E}^\partial) / \tilde{\varphi} = 0 \, q.e. \, on \, \{F = 0\}\}$. *Moreover,*

$\forall t \geq 0$ and $\varphi \in L^2(\partial D)$ we have the semigroup domination

$$|\bar{P}_t^{\partial}\varphi| \leq P_t^{\partial}|\varphi|. \tag{19}$$

P r o o f. The existence of the Stollmann function F comes from [26]. Next, we apply the criterion of [20]. When $0 \leq \varphi \leq \psi$ then we know from the definition of the operator $\mathcal{H}$ that $\mathcal{H}\varphi \leq H\psi$ which implies that $\mathcal{H}\varphi \in L^2(D)$.

On the other hand, there exists an $\mathcal{E}$-Cauchy sequence h_n in $W^{1,2}(D)$ s.t. $h_n \to \mathcal{H}\varphi$ a.e. in D as $n \to \infty$. Let $\epsilon > 0$, by Egorov's theorem $\exists E_\epsilon \subset D$, with $|E_\epsilon| < \epsilon$ and outside of which the convergence of $h_n \to \mathcal{H}\varphi$ is uniform. It clearly follows that $\sup_n \|I_{D\setminus E_\epsilon} h_n\|_{W^{1,2}(D)} < \infty$, so that $I_{D\setminus E_\epsilon} h_n$ converges weakly in $W^{1,2}(D)$, perhaps through a subsequence of n. The limit is obviously $I_{D\setminus E_\epsilon}\mathcal{H}\varphi \in W^{1,2}(D)$.

Note that ∇h_n converge a.e. in D, perhaps through a subsequence, still noted n. Since $I_{D\setminus E_\epsilon}\mathcal{H}\varphi \in W^{1,2}(D)$ then $\|\nabla I_{D\setminus E_\epsilon} h_n\|^2_{L^2(D)} \to \|\nabla I_{D\setminus E_\epsilon}\mathcal{H}\varphi\|^2_{L^2(D)}$. It follows $\exists n(\epsilon)$ s.t. $\forall n \geq n(\epsilon)$ we have

$$\|\nabla I_{D\setminus E_\epsilon}\mathcal{H}\varphi\|^2_{L^2(D)} \leq \|\nabla I_{D\setminus E_\epsilon} h_n\|^2_{L^2(D)} + 1.$$

Our result follows by applying Fatou's lemma because $\sup_n \|\nabla h_n\|_{L^2(D)} < \infty$. □

3.2. The $\boldsymbol{\lambda}$-process. Here we impose a minor regularity condition on D, we have the

Theorem 3. *Let a be continuous and $D \in W^{2-(1/p),p}$, then to $\bar{\mathcal{E}}^{\partial}$ is associated a Hunt process $\bar{X}_t^{\partial}$ perhaps evolving isometrically in another space, the isometry being explained in appendix A of* [11], *see also the first remark in* 3.3.

P r o o f. By the Sobolev embedding $D \in \mathcal{C}^1$. Let us regularize a, a_n say, thanks to a standard mollifier. The sequence $a_n \in \mathcal{C}^\infty(\bar{D})$. Take $\varphi \in W^{1-(1/p),p}(\partial D)$. From Section 2.1.1 we know that the gradient of $u_\lambda^n(\varphi)$ admits a continuous extension to ∂D. This shows that the $\lambda^{(n)}$-semigroup is identical to the A^n-trace semigroup which is Markovian by [11]. It is also well known that $u_\lambda^n(\varphi) \to u_\lambda(\varphi)$ as $n \to \infty$ strongly in $W^{1,2}(D)$, $\lambda > 0$; it follows that $u_\lambda^{\partial,n}(\varphi) \to u_\lambda^{\partial}(\varphi)$ strongly in $L^2(\partial D)$, $\lambda > 0$. Let $t, \epsilon > 0$, we have $-\epsilon \leq \bar{P}_t^{\partial,n}\varphi \leq 1+\epsilon$ where $0 \leq \varphi \leq 1$. This inequality still holds for $\varphi \in L^2(\partial D)$ thanks to an elementary density argument. Letting $n \to \infty$ and then $\epsilon \to 0$ and using the Trotter-Kato theorem we deduce that $\bar{P}_t^\partial$ is Markovian. Next, apply the results of appendix A in [11] to construct the associated Hunt process. □

3.3. Remarks. When D is uniformly $\mathcal{C}^1$ and $W^{2-(1/p),p}$ the λ-process can be defined on ∂D itself. Indeed it follows from [1, Section 7.51] that $\mathcal{C}^\infty(\partial D)$ is dense in $W^{1/2,2}(\partial D)$, since it is also dense in $\mathcal{C}(\partial D)$ endowed with the sup-norm, the Dirichlet form $\bar{\mathcal{E}}^\partial$ is regular. A Hunt process can be associated with $\bar{\mathcal{E}}^\partial$ as described in [11, Chapter 4] and starting from every α outside a fixed properly exceptional set $E_0 \subset \partial D$.

Since $W^{1/2,2}(\partial D)$-functions are defined only within sets E_0 with $|E_0| = 0$, it follows that E_0 may be taken as the zero-set of the Stollmann function F. In other words, the λ-process is insensitive to a whole boundary region (in E_0) where the trace process may evolve in a non trivial way. It would be interesting to find relations between h-Hausdorff measures of E_0 and the trace process.

The transition density $\bar{p}_t(\alpha, \beta)$, if it exists, should be given at least formally by the series

$$\bar{p}_t(\alpha, \beta) = \sum_{j=0}^{\infty} \exp(-\frac{\lambda_j}{|\partial D|} t) \tilde{S}_j^\partial(\alpha) \tilde{S}_j^\partial(\beta),$$

for all $t \geq 0$. Indeed, suppose that we have the following spectral

Condition 4. *Suppose that the Steklov eigenfunctions* $\tilde{S}_j^\partial, j \geq 1$, *are bounded with* $\|\tilde{S}_j^\partial\|_\infty = c(j)$ *and* $\forall t > 0$

$$\sum_{j=0}^{\infty} c(j)^2 \exp(-\lambda_j t) < \infty. \tag{20}$$

Then for $\varphi, \psi \in L^2(\partial D)$ and $t \geq 0$, we have by the dominated convergence theorem

$$\begin{aligned}
(\bar{P}_t^\partial \varphi, \psi)_\partial &= \sum_{j=0}^{\infty} \exp\left(-\frac{\lambda_j}{|\partial D|} t\right) (\varphi, \tilde{S}_j^\partial)_\partial (\psi, \tilde{S}_j^\partial)_\partial \\
&= \sum_{j=0}^{\infty} \exp\left(-\frac{\lambda_j}{|\partial D|} t\right) \int_{\partial D \times \partial D} \varphi(\alpha)\psi(\beta) \tilde{S}_j^\partial(\alpha) \tilde{S}_j^\partial(\beta) d\mu_0 d\mu_0 \\
&= \left(\int_{\partial D} \varphi(\alpha) \bar{p}_t(\alpha, \cdot) d\mu_0(\alpha), \psi \right)_\partial ,
\end{aligned}$$

taking smooth ψ's yields our claim.

4. The probabilistic representation.

4.1. The trace process generator. We have

Lemma 1. *Assume that* (1) *and Conditions* 1, 2 *hold for some* $p > d$. *Let* $\varphi \in W^{2,p}(D)$, *the* L^2*-generator of the process* X_t^∂ *is the Dirichlet-to-Neumann map applied to* φ^∂

$$A^\partial \varphi^\partial = -\partial_{n_a} H\varphi^\partial.$$

Proof. Let $\alpha \in \partial D$ and $\varphi \in W^{2,p}(\partial D)$. Note that the gradient of $H\varphi$ is bounded on $\bar{D}$, see Section 2.2. By the Itô-Fukushima formula the additive functional $H\varphi(X_t) - H\varphi(\alpha)$ is the sum of a martingale additive functional M_t^φ of finite energy and a continuous additive functional N_t^φ of zero energy. Since $X(t)$ has continuous paths, M_t^φ is a continuous martingale whose increasing process is given by $\int_0^t \|\sigma \nabla H\varphi\|^2(X_s)ds$, where $\sigma\sigma^* = a$. Therefore M_t^φ is a Brownian martingale of the form $\int_0^t \nabla H\varphi\sigma(X_s)dB_s$ where B_t is Brownian motion in R^d starting at the origin. It remains to identify the process of zero energy N_t^φ. By the Green formula we have

$$\frac{1}{2}\int_D a\nabla H\varphi \cdot \nabla\psi dx = \int_{\partial D} \partial_{n_a} H\varphi\, \psi^\partial d\alpha, \tag{21}$$

$\forall \psi \in W^{1,2}(D)$. Taking moreover $\psi \in \mathcal{C}(\bar{D})$ we see that

$$|\mathcal{E}(H\varphi, \psi)| \le c \sup_{\alpha \in \partial D} |\psi(\alpha)| \le c \sup_{x \in \bar{D}} |\psi(x)|.$$

Theorem 2.1 of [12] shows that the condition of Corollary 5.4.2 of [11] is satisfied. It follows that N_t^φ is of bounded variation and that its Revuz measure is equal to $-(1/2)a\nabla H\varphi.n(\alpha)d\alpha$ so that $N_t^\varphi = -1/2 \int_0^t a\nabla H\varphi.n(X_s)dL(s)$. Making the time substitution corresponding to τ_t and then taking an expectation we see that $P_t^\partial \varphi(\alpha) - \varphi(\alpha) = \int_0^t P_s^\partial \psi(\alpha)ds$ where $\psi = -\partial_{n_a} H\varphi$. By the Cauchy-Schwarz inequality

$$\begin{aligned}\left\| \frac{P_t^\partial \varphi - \varphi}{t} - \psi \right\|_{L^2(\partial D)}^2 &= \int_{\partial D} \left[\frac{1}{t} \int_0^t (P_s^\partial \psi - \psi)(\alpha)ds \right]^2 d\alpha \\ &\le \frac{1}{t} \int_0^t \|P_s^\partial \psi - \psi\|_{L^2(\partial D)}^2 ds,\end{aligned}$$

which converges to zero as $t \to 0$, showing our result. $\square$

4.2. The Neumann boundary function N.

Corollary 1. *Under the same conditions as in Lemma* 1*, the* λ*-process and the trace process are identical.*

Proof. Let $\varphi \in W^{2,p}(D)$. The $W^{2,p}(D)$ solution $H\varphi^\partial$ of the Dirichlet problem (7) satisfies the variational relation (21). Let $\psi \in W^{1,2}(D)$; solving if need be a Dirichlet problem with boundary data ψ^∂, we can assume ψ to be A-harmonic. We can write

$$\bar{\mathcal{E}}^\partial(\varphi^\partial, \psi^\partial) = -\int_{\partial D} (A^\partial \varphi^\partial)\, \psi^\partial d\alpha.$$

Since this relation is valid for all smooth ψ we see that $\bar{A}^\partial \varphi^\partial = A^\partial \varphi^\partial$. A density argument shows that the λ-process and the trace process are identical. $\square$

The corollary shows that the Neumann boundary function (18) is also given by $N = \int_0^\infty P_t^\partial \varphi dt$ with $\int_{\partial D} \varphi(\alpha) d\alpha = 0$. We are now in the position to state the

Theorem 4. *Assume that uniform ellipticity and boundedness* (1)*, and Conditions* 1 *and* 2*. The Neumann boundary function can be defined in terms of the trace semigroup. A weak solution of the Neumann problem* (2) *is given by*

$$\begin{aligned} u(x) &= E^x \left[\int_0^\tau f(X_t)dt + N(X_\tau) \right] \\ &= Gf(x) + \int_{\partial D} H(x,\alpha) N(\alpha) d\alpha, \end{aligned} \tag{22}$$

where $\tau = \tau_{\partial D}$.

Proof. It is well known that we have $Gf(x) = E^x \int_0^\tau f(X_t)dt$.

Let us turn to u_0, see Section 1.4. Let $\varphi \in W^{1,2}(D)$. As $\int_{\partial D} g_0 d\alpha = 0$, the argument of the last point of the proof of Theorem 1 applies, i.e. we have $\lim_{t\to\infty} A^\partial N(t) = A^\partial N = -g_0$, where $N(t) = \int_0^t P_s^\partial g_0 ds$. Setting $u_0 = HN$ we have by definition of the form $\mathcal{E}^\partial$

$$\frac{1}{2} \int_D a \nabla u_0 \cdot \nabla H \varphi^\partial dx = \int_{\partial D} g_0 . \varphi^\partial d\alpha.$$

On the other hand $\varphi = \varphi_0 + \varphi_a$ where $\varphi_0 \in W_0^{1,2}(D)$ and $\varphi_a \in W_a^{1,2}(D)$, see (11). We clearly have $H\varphi^\partial = H\varphi_a^\partial$ and therefore by (13) we deduce that

$$(HN, H\varphi^\partial)_a = (HN, H\varphi_a^\partial)_a = \sum_{j=0}^{\infty}(1+\lambda_j)N_j^\partial \varphi_{a,j}^\partial,$$

$$(HN, \varphi)_a = (HN, \varphi_a)_a = \sum_{j=0}^{\infty}(1+\lambda_j)N_j^\partial \varphi_{a,j}^\partial.$$

Therefore $\forall \varphi \in W^{1,2}(D)$ we have

$$\frac{1}{2}\int_D a\nabla u_0 \cdot \nabla \varphi dx = \int_{\partial D} g_0 \varphi d\alpha,$$

and our theorem is proved. □

4.3. A $W^{2-(1/p),p}$-domain. Assume moreover that D is uniformly $\mathcal{C}^1$ and $a \in \mathcal{C}^1$. Passing to the adjoint semigroup $P_t^{\partial,*}$, it is well known that the L^2-spectral gap (17) also takes place in the sense of the total variation norm

$$\|P_t^{\partial,*}\mu(\cdot) - P_t^{\partial,*}\mu_0(\cdot)\|_{TV} = 2 \sup_{E\in\mathcal{B}(\partial D)} |P_t^{\partial,*}\mu(E) - P_t^{\partial,*}\mu_0(E)|,$$

for all measures μ with densities in $L^2(\partial D, \mu_0)$. If our process is $d\alpha$-irreducible, i.e. $\forall E \subset \partial D\backslash E_0$ with $|E| > 0$ one has $E^\alpha \int_0^\infty I_E(X_t^\partial)dt > 0$, $\alpha \in \partial D\backslash E_0$, and aperiodic, i.e. for some $E \subset \partial D\backslash E_0$ of positive area there is a t_0 s.t. $\forall t \geq t_0$ and $\forall \alpha \in \partial D\backslash E_0$ one has $P_t^\partial(\alpha, E) > 0$ we can show that Dirac point masses also converge exponentially to equilibrium in $\|\cdot\|_{TV}$ thanks to a Lyapounov function argument. We have

Theorem 5. *Under the above conditions, the convergence in* (17) *takes place (regarding the adjoint semigroup) in the total variation norm.*

P r o o f. The Steklov eigenfunction $\tilde{S}_1^\partial$ is continuous on the compact ∂D, by Section 2.1.1. For some large constant c define the Lyapounov function $W(\alpha) = \tilde{S}_1^\partial(\alpha) + c$ and write

$$\begin{aligned} A^\partial W &= -\lambda_1 W + \lambda_1 c \\ &= -\epsilon\lambda_1 W + [\lambda_1 c - (1-\epsilon)\lambda_1 W]. \end{aligned}$$

Enough room is left to exhibit inside the brackets a petite set (a well established french word). Indeed, define for a suitable ϵ the non empty compact set $K = \{\tilde{S}_1^\partial \leq c\epsilon/(1-\epsilon)\}$. It follows that $A^\partial W \leq -\epsilon\lambda_1 W + c\lambda_1 I_K$. It then suffices to apply Theorem 5.2 of [10]. $\square$

5. Examples and extensions. We now illustrate the above theory.

5.1. The representation when $f = 0$. Without the constraint of interior regularity, we have the following theorem which is proved as in Theorem 4,

Theorem 6. *Assume that* (1) *and the condition in Section* 1.3.2 *hold. A weak solution of the Neumann problem* (2) *is given by*

$$u(x) = E^x \bar{N}(X_\tau) = \int_{\partial D} H(x,\alpha)\bar{N}(\alpha)d\alpha, \tag{23}$$

where $\tau = \tau_{\partial D}$.

5.2. A class of problems. In view of the last remark of Section 3.3 and of the fact that the λ-process transition semigroup is anyway absolutely continuous a.e. (this follows immediately from equation (16) by taking indicators whose sets are of Lebesgue measure zero) it is not too restrictive to assume in this section that

Condition 5. *Suppose that* $D \in W^{2-(1/p),p}$ *and uniformly* $\mathcal{C}^1$ *and that the* λ*-process has a transition density* $\bar{p}_t^\partial(\alpha,\beta)$ *that is jointly continuous in* ∂D*,* $\forall t > 0$.

The set of couples a, D which satisfy Condition 5 is not empty. Indeed, take e.g. planar Brownian motion reflecting in the unit disc. From (12) it follows by elementary analysis that the series in Section 3.3 is uniformly convergent. We have

Theorem 7. *Under Condition* 5*, Theorem* 6 *is valid with the trace process Neumann boundary function. Moreover, the latter is bounded and the convergence in the time integral takes place in* $L^\infty(\partial D)$.

P r o o f. Let $t > 0$. Applying Lemma 2.3 of [7] we have $\bar{p}_t(\alpha,\beta) > 0$ in ∂D. In particular, $\exists c(t) > 0$ s.t. $\forall \alpha, \beta \in \partial D$

$$c(t) \leq \bar{p}_t(\alpha,\beta). \tag{24}$$

Exponential convergence for the trace process can now be proved.

Lemma 2. *Under the above notations, the process X_t^∂ possesses a unique invariant probability measure, i.e. μ_0, and there exist two positive constants c, c' s.t. for all $t \geq 0$,*

$$\sup_{\alpha \in \partial D} \|P_t^\partial(\alpha, \cdot) - \mu_0(\cdot)\|_{TV} \leq c \exp(-c't), \tag{25}$$

where $\|\cdot\|_{TV}$ stands for the total variation norm.

P r o o f. We use a Doeblin argument. It follows from (24) and (19) that there exists a non trivial probability measure ν s.t. for any $t \geq 0$, we have for some $c > 0$, $c\nu(d\beta) \leq P_t^\partial(\alpha, d\beta)$ for all $\alpha \in \partial D$. Hence, for any two probability measures μ_1 and μ_2 on ∂D we have

$$\|P_t^{\partial,*}\mu_1 - P_t^{\partial,*}\mu_2\|_{TV} \leq c\|\mu_1 - \mu_2\|_{TV}. \tag{26}$$

This inequality is easily established when the measures μ_1 and μ_2 are mutually singular as in this case $\|\mu_1 - \mu_2\|_{TV} = 2$. The general case follows thanks to the Hahn decomposition since there is a covering of total mass 1/2 between the measures $P_t^{\partial,*}\mu_1$ and $P_t^{\partial,*}\mu_2$. Upon iterating the inequality (25) with $\mu_2 = \mu$ the Lemma is established. $\square$

We also have the

Lemma 3. *The function $P_t^\partial g_0$ is bounded and we have for some positive c, c' and all $t \geq 0$,*

$$\|P_t^\partial g_0\|_{L^\infty(\partial D)} \leq c \exp(-c't).$$

P r o o f. Let $g_0^n = \sum_i c_i^n I_{F_i^n}$, where the c_i^n's are constants and the F_i^n's are Borel subsets of ∂D, be a sequence of step functions converging uniformly to g_0. We have by a well known inequality,

$$
\begin{aligned}
&[P_t^{\partial} g_0^n(\alpha)]^2 \\
&= \left[\int_{\partial D} g_0^n(\beta)\mu_t(\alpha, d\beta)\right]^2 \\
&\leq 2\left[\int_{\partial D} g_0^n(\beta)(\mu_t(\alpha, d\beta) - \mu_0(d\beta))\right]^2 + 2\left[\int_{\partial D} g_0^n(\beta)\mu_0(d\beta)\right]^2 \\
&\leq 2\sup_i (c_i^n)^2 \left(\sum_i |\mu_t(\alpha, F_i^n) - \mu_0(F_i^n)|\right)^2 + 2\left[\int_{\partial D} g_0^n(\beta)\mu_0(d\beta)\right]^2 \\
&\leq c\exp(-c't)\|g_0^n\|^2_{L^\infty(\partial D)} + c''\left[\int_{\partial D} g_0^n(\beta)\mu_0(d\beta)\right]^2 .
\end{aligned}
$$

It now suffices to use the fact that g_0 is centered and let n tend to ∞. □

The theorem immediately follows from the last two lemmas. □

5.3. Interior representation. For simplicity assume that the domain D and the functions a, f, g are smooth. It is well known that the process $X(t)$ defines a strongly continuous semigroup of bounded linear operators P_t on $L^2(D)$ with

$$
\begin{aligned}
\partial_t P_t\varphi(x) &\overset{D}{=} AP_t\varphi(x), \\
\partial_{n_a} P_t\varphi(\alpha) &\overset{\partial D}{=} 0,
\end{aligned}
$$

where $\varphi \in L^2(D)$. It is also well known that an interior spectral gap in $L^2(D)$ for P_t

$$\|P_t\varphi\|_{2,D} \leq c\exp(-c't)\|\varphi\|_{2,D}, \tag{27}$$

provided $\int_D \varphi dx = 0$, follows directly from the Poincaré inequality

$$\int_D \varphi^2 dx \leq c\int_D \|\nabla\varphi\|^2 dx,$$

where $\varphi \in W^{1,2}(D)$ with $\int_D \varphi d\alpha = 0$. The function $\int_0^\infty P_t\varphi dt$ is then well defined and belongs to the domain of the L^2-generator $\mathcal{A}$ of the process X.

Let $\tilde{g}$ be any smooth function which satisfies $\partial_{n_a}\tilde{g} = g$ on ∂D. Rewrite the system (2) as

$$\begin{aligned} A\tilde{u} &\overset{D}{=} -\tilde{f}, \\ \partial_{n_a}\tilde{u} &\overset{\partial D}{=} 0, \end{aligned} \tag{28}$$

where $\tilde{f} = f + A\tilde{g}$ and $\tilde{u} = u - \tilde{g}$. By taking the function $\tilde{g} - (1/|\partial D|)\int_D \tilde{g}dx$ if necessary, we can assume that $\int_D \tilde{g}dx = 0$. The solution of the Dirichlet problem $Aw \overset{D}{=} -\tilde{f}$, $w^\partial = 0$, is given by $w = G\tilde{f}$ where G is the Green operator. The boundary function $\partial_{n_a}G\tilde{f}$ is in $L^\infty(\partial D)$. It is clear that the function $\tilde{u} = G\tilde{f} + v$, where v is the solution of the (compatible) Neumann problem $Av \overset{D}{=} 0$, $\partial_{n_a}v \overset{\partial D}{=} -\partial_{n_a}G\tilde{f}$, gives the solution (modulo additive constants) of the system (28).

Let $x \in D$. Theorem 4 gives with the obvious notations $u(x) = \tilde{g}(x) + G\tilde{f}(x) + H\tilde{N}(x)$. We have by the dominated convergence theorem

$$\begin{aligned} H\tilde{N}(x) &= -E^x \int_0^\infty P_t^\partial(\partial_{n_a}G\tilde{f})(X_\tau)dt \\ &= -\lim_{T\to\infty}\int_{\partial D} d\alpha H(x,\alpha)\int_0^T dt P_t^\partial(\partial_{n_a}G\tilde{f})(\alpha) \\ &= -\lim_{T\to\infty}\int_{\partial D} d\alpha H(x,\alpha)E^\alpha\int_0^T \partial_{n_a}G\tilde{f}(X_{\tau_t})dt \\ &= -\lim_{T\to\infty}\int_{\partial D} d\alpha H(x,\alpha)E^\alpha\int_0^{\tau_T} \partial_{n_a}G\tilde{f}(X_s)dL(s) \\ &= \lim_{T\to\infty} E^x[E^{X_\tau}\int_0^{\tau_T}\tilde{f}(X_s)ds], \end{aligned}$$

where we used in the last but one line the Itô formula for $G\tilde{f}$ on the process X, i.e. we have for any $\alpha \in \partial D$

$$-E^\alpha\int_0^{\tau_T}\partial_{n_a}G\tilde{f}(X_s)dL(s) = E^\alpha\int_0^{\tau_T}\tilde{f}(X_s)ds.$$

We deduce by the strong Markov property that

$$H\tilde{N}(x) = \lim_{T\to\infty}\left[E^x\int_0^{\tau_T}\tilde{f}(X_s)ds\right] - G\tilde{f}(x),$$

where the limit exists since $\tilde{f}$ has spatial mean zero. Note also that $\tau_T \to \infty$ as $T \to \infty$ almost surely by uniform ellipticity and we have

$$\begin{aligned} G\tilde{f}(x) + H\tilde{N}(x) &= \lim_{T\to\infty} E^x \left[\int_0^{\tau_T} \tilde{f}(X_s)ds \right] \\ &= E^x \left[\int_0^{\infty} \tilde{f}(X_s)ds \right] \\ &= \lim_{T\to\infty} E^x \left[\int_0^{T} \tilde{f}(X_s)ds \right]. \end{aligned}$$

A further application of the Itô formula yields

$$E^x \tilde{g}(X_T) = \tilde{g}(x) + E^x \int_0^T A\tilde{g}(X_s)ds - E^x \int_0^T g(X_s)dL(s),$$

which implies by the spectral gap (27) and by the fact that $\tilde{g}$ is centered

$$u(x) = \lim_{T\to\infty} \left[E^x \int_0^T f(X_s)ds + E^x \int_0^T g(X_s)dL(s) \right].$$

The Brosamler relation, see [5], follows by taking the particular case $A = (1/2)\Delta$ and $f = 0$.

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