Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica
Mathematical Journal
Сердика
Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **39** (2013), 215–232

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ASYMPTOTIC BEHAVIOR IN SLIDING MODE CONTROL SYSTEMS

Tullio Zolezzi

Communicated by M. I. Krastanov

Dedicated to A. Dontchev and V. Veliov

Abstract. Practical stability of real states of nonlinear sliding mode control systems is related to asymptotic vanishing of the corresponding sliding errors. Conditions are found such that, if the equivalent control achieves exponential stability, then real states are practically stable. In special cases, their exponential stability is obtained. A link between convergence of regularization procedures and metric regularity is pointed out.

1. Introduction. We consider nonlinear sliding mode control systems described by ordinary differential equations, operating on the unbounded time horizon. A significant issue, particularly relevant for uncertain systems, deals with asymptotic stabilization of the system by choosing a suitable sliding manifold. A theoretically and practically important problem is to predict the asymptotic behavior of states of the system under any type of small imperfections preventing ideal sliding, which guarantees asymptotic stability, as such imperfections disappear. This problem has been considered in [14], [15] and [17]. Let us assume

2010 *Mathematics Subject Classification:* 93B12, 93D15.
Key words: sliding mode control, practical stability, asymptotic convergence.

some form of asymptotic stability of the ideal states, namely those which fulfill exactly the sliding condition. A reasonable and useful property we expect from real states, namely those which approximately fulfill the sliding condition, is some form of practical stability. In a different context, the related property of ultimate boundedness, sometimes obtained by smoothing a stabilizing discontinuous control, has been proved e.g. in [1, 4, 12]. Moreover, it is shown in [17] that if the equivalent control achieves exponential stability, then real states corresponding to smooth control laws are practically stable provided the sliding error vanishes uniformly. (In [11] ultimate boundedness criteria are obtained for sliding mode uncertain systems.)

In this paper we assume in some results existence of the (appropriately defined) equivalent control, see [2], and consider real states corresponding to control laws in the pointwise sense, assuming their exponential stability. We observe that practical stability is very intimately related to an elementary form of (rather weak) convergence, we call asymptotic convergence, of the real states toward ideal states and of the sliding error to zero. Moreover, for uniformly bounded real states, their practical stability implies asymptotic vanishing of the sliding error. Conversely, we show that asymptotic vanishing of the sliding error suffices to obtain practical stability, under suitable assumptions, provided the dynamics are linear and time-invariant. This improves the (few) known results on this topic, weakening the assumptions of uniform convergence of the sliding error. For some control systems with affine dependence on the control variable, we show that real states are exponentially stable as the ideal ones. Uniform convergence of the real states on the whole unbounded time horizon is also obtained under suitable assumptions.

In Section 2 we collect the basic notions and properties of the control systems we consider. In Section 3, practical stability and asymptotic convergence are related to each other. The main results are collected in Section 4, as follows. For nonlinear control systems, practical stability is deduced from uniform vanishing of the sliding error. In the linear time-invariant case, the same conclusion is obtained from asymptotic vanishing of the sliding error. For some classes of affine (in the control variable) systems, exponential stability of real states is proved. Sufficient conditions for uniform convergence of the real states are obtained. An example, showing the role of the boundedness assumptions we impose on real states, is presented in Section 5. In Section 6 we show that, in the setting of differential inclusions, the convergence toward the set of sliding states of the real states is controlled by a fixed multiple of the sliding error via global metric regularity of the sliding mapping, provided the mapping defining the sliding manifold has maximum rank. The proofs are collected in Section 7.

2. Problem statement. We consider sliding mode control systems on the unbounded time horizon,

$$\dot{x} = f(x, u), t \geq 0, \tag{2.1}$$

with sliding manifold

$$s(x) = 0, \tag{2.2}$$

state variable $x \in R^N$, control variable $u \in R^P$ and $s(x) \in R^M$. We fix a constant $c_0 > 0$ and assume that for every initial state x_0 with $s(x_0) = 0$ and $|x_0| \leq c_0$ there exists some (possibly discontinuous) feedback control law such that a corresponding solution x of (2.1) exists on $[0, +\infty)$ (possibly in Filippov's sense, see [8]) and fulfill the sliding condition

$$s[x(t)] = 0, t \geq 0. \tag{2.3}$$

The reaching phase is not at issue here, and we assume that the sliding phase, which is the relevant behavior here, occurs starting at time $t = 0$.

We shall employ the following assumptions (see below the precise statements). First, there exists the equivalent control for the system (2.1), (2.2) as defined in [2]. Second, we assume that the origin is an exponentially stable equilibrium for the dynamics corresponding to the equivalent control (thus taking into account the sliding condition). Third, we consider real states of the control system, i.e. those which satisfy the sliding constraint (2.3) only approximately, corresponding to mildly smooth control laws. More precisely we posit the following assumptions A1, A2:

A1. The mappings

$$f : R^N \times R^P \to R^N \text{ and } s : R^N \to R^M$$

are of class C^1 and $s(0) = 0$.

A2. For every $x \in R^N$ and $\gamma \in R^M$ there exists a unique solution $u = u^*(x, \gamma)$ of

$$s_x(x) f(x, u) = \gamma,$$

where s_x denotes the Jacobian matrix of s, with $u^*(x, 0)$ of class C^1, and $f[0, u^*(0, 0)] = 0$.

We call $u^* = u^*(x, 0)$ the *equivalent control* of the sliding mode control system (2.1), (2.2). Any solution $x = x(t), t \geq 0$ of (2.1) which does not necessarily fulfill the sliding condition (2.3), such that the *sliding error* $|s[x(t)]|$ is small (in a sense which will be made precise in the sequel) will be called a *real state* of the sliding mode control system (2.1), (2.2). An *ideal state* of (2.1), (2.2) is any solution $x = x(t)$ of (2.1), $t \geq 0$, which fulfills (exactly) the sliding condition (2.3).

The purpose of this paper is to find conditions on the vanishing of the sliding error on $[0, +\infty)$ which are sufficient to guarantee some form of practical stability of the real states, assuming that the ideal states (driven by the equivalent control) fulfill exponential stability. The problem is intimately related to suitable forms of convergence on the unbounded time horizon of the real states toward the ideal one as the sliding error vanishes. For bounded time horizon, this problem has been considered in sliding mode control theory under the names of regularization procedure (see [14, 15]), or approximability property (see [2] and [3]). If conditions hold such that some form of asymptotically stable behavior of the ideal states imply practical stability of the real states as the sliding error vanishes, then the regularization procedure can be considered as satisfactorily performed. In this case, any nonideality leading to small violations of the sliding condition allows some practically significant stability properties of the real states, as required in applications. The last result of the paper shows that, if the Jacobian of the mapping defining the sliding manifold has maximum rank, then metric regularity is present, thus if the sliding error vanishes, then the distance to sliding goes to zero bounded above by the sliding error.

3. Practical stability and asymptotic convergence. A form of practical stability we shall consider is defined as follows. A set S of real states of (2.1), (2.2) is *practically stable* (with respect to a given notion of vanishing of the sliding error) if for every $\varepsilon > 0$ there exists $T > 0$ such that, for every $z \in S$ with sliding error sufficiently small, we have

$$|z(t)| \leq \varepsilon \quad \text{if} \quad t \geq T.$$

Thus practical stability here means that real states enter any given ball centered at the origin at sufficiently large times provided the sliding errors are sufficiently small (compare with the notion of ultimate boundedness with arbitrarily small bound, see [16]). A related notion we shall consider is the following. A sequence z_n of real states of (2.1), (2.2) will be called *asymptotically practically stable, APS* for short, if for every $\varepsilon > 0$ there exist $T > 0, p > 0$ such that

$$|z_n(t)| \leq \varepsilon \quad \text{if} \quad t \geq T \quad \text{and} \quad n \geq p.$$

If $s[z_n(t)]$ converges to 0 uniformly on $[0, +\infty)$, then (with respect to such form of vanishing of the sliding error) practical stability implies APS.

The above definition of APS is related to the following notion. Given a sequence of functions

$$y_n, y : [0, +\infty) \to R^N$$

we write $y_n \to y$ *asymptotically* if

$$y_n(t) - y(t) \to 0 \quad \text{as} \quad t \to +\infty, \ n \to +\infty;$$

i.e. for every $\varepsilon > 0$ there exist positive numbers T and p such that

$$|y_n(t) - y(t)| \le \varepsilon \quad \text{if} \quad t \ge T \text{ and } n \ge p.$$

Strictly speaking, asymptotic convergence is not a true convergence, since uniqueness of the limit can fail. Consider e.g.

$$y_n(t) = e^{-nt}, \quad y(t) = \frac{A}{1+t}, \quad t \ge 0;$$

then $y_n \to y$ asymptotically for every $A \in R$. Of course, if $y_n \to y$ uniformly on $[0, +\infty)$, then $y_n \to y$ asymptotically. However uniform convergence on every bounded interval (as required by the definition of approximability of sliding mode control systems in [2]) does not imply asymptotic convergence, as it is shown by $y_n(t) = [t/(t+1)]^n, t \ge 0$. The (obvious and) basic link between the two notions is the following:

Proposition 3.1. *The sequence $y_n \to y$ asymptotically if and only if $y_n - y$ is APS.*

Connections between asymptotic convergence and stability properties are also considered in the following results:

Proposition 3.2. *A sequence z_n of real states is APS if there exists an ideal state y such that $z_n \to y$ asymptotically and $y(t) \to 0$ as $t \to +\infty$.*

Remark. If $z_n(0) \to \bar{x}$ as $n \to +\infty$, a reasonable choice of the ideal state y in Proposition 3.2 is the one issued from $(0, \bar{x})$ (compare with the definition of approximability in [2]).

Moreover, asymptotic vanishing of the sliding error is a necessary condition for APS, as follows.

Proposition 3.3. *Suppose that $s(0) = 0$. If a sequence of real states of* (2.1), (2.2) *is APS, then the corresponding sliding error vanishes asymptotically.*

4. Convergence of real states and practical stability. A consequence of the results in [17] is that uniform vanishing on $[0, +\infty)$ of the sliding error and exponential stability of the ideal states corresponding to the equivalent control yield asymptotic convergence of the real states and their practical stability. Following [17] we therefore consider the condition

A3. There exist positive constants M and c such that for every $\bar{t} \geq 0, \bar{x} \in R^N$ with $s(\bar{x}) = 0$, a (unique) solution $y = y(t)$ of

$$\dot{x} = f[x, u^*(x, 0)], x(\bar{t}) = \bar{x}$$

exists on $[0, +\infty)$ and

$$|y(t)| \leq M e^{-c(t-\bar{t})} |\bar{x}| \text{ if } t \geq \bar{t}.$$

Every such $y = y(t)$ will be called the *ideal state* issued from $(\bar{t}, \bar{x})$. The following conditions deal with the class of control laws which correspond to real states. Let us call *smooth* those control laws

$$u : [0, +\infty) \times R^N \to R^P$$

of class C^1, with locally Lipschitz continuous gradient, such that $f[0, u(t, 0)] = 0, t \geq 0$. We fix the positive constants Δ_0 and H and consider the set $\Sigma = \Sigma(\Delta_0, H)$ of all smooth control laws such that the following property holds. For any $u \in \Sigma, t \geq 0, \Delta \leq \Delta_0, x \in R^N$ such that $|s(x)| \leq \Delta$, the unique solution $z = z(\theta)$ of (2.1) with $z(t) = x$ fulfill $|s(z(\theta)| \leq \Delta$ and $|z(\theta)| \leq H, \theta \geq 0$. We denote by $S(\Delta_0, H)$ the set of all (real) states corresponding to control laws in $\Sigma(\Delta_0, H)$.

Theorem 4.1. *Assume* A1, A2, *and* A3. *Then for every* $H > 0, \bar{x} \in R^N$ *and every sequence* z_n *of real states corresponding to smooth control laws such that*

$$s(\bar{x}) = 0, |z_n(t)| \leq H \quad \textit{for every} \quad n \quad \textit{and} \quad t \geq 0, z_n(0) \to \bar{x} \quad \textit{as} \quad n \to +\infty,$$

$$s[z_n(t)] \to 0 \quad \textit{uniformly on} \; [0, +\infty),$$

we have

$$z_n \to y \textit{ asymptotically}$$

for the ideal state y *with* $y(0) = \bar{x}$, *and* z_n *is APS. Moreover* $S(\Delta_0, H)$ *is practically stable for every* $\Delta_0 > 0$.

By Theorem 4.1, if the equivalent control provides exponential stability, then those real states which correspond to control laws in any class Σ are practically stable provided the sliding error vanishes uniformly on $[0, +\infty)$. However, essentially weaker conditions suffice to obtain the same conclusions for particular classes of sliding mode control systems. In the next two theorems we consider real states which are absolutely continuous on every bounded interval of $[0, +\infty)$, corresponding in the almost everywhere sense to the control laws. In the linear time-invariant case, we show that asymptotic convergence to 0 of the sliding error suffices. Thus we consider

$$f(x, u) = Ax + Bu, s(x) = Cx \tag{4.1}$$

where A, B, C are constant matrices of the appropriate dimension.

Theorem 4.2. *Suppose that* A2, A3 *and* (4.1) *hold. Let x_n be any sequence of real states such that*

$$Cx_n \to 0 \quad \textit{asymptotically}, \tag{4.2}$$

$$x_n(0) \to y(0) \quad \textit{as} \quad n \to +\infty, \tag{4.3}$$

with y an ideal state. Suppose that x_n are uniformly bounded on every bounded interval of $[0, +\infty)$. *Then*

$$x_n \to y \textit{ asymptotically in } [0, +\infty)$$

and x_n is APS.

Remark. The conclusion can be obtained in an obvious way provided C is nonsingular, without assuming A2, A3.

The conclusion of Theorem 4.2 can be improved for nonlinear systems (2.1), (2.2) with affine dependence on the control variable under special assumptions, as follows. Consider

$$f(x, u) = A(x) + B(x)u, \tag{4.4}$$

and suppose that A2 holds. Then $M = P$ and

$$K(x) = \{I - B(x)[s_x(x)B(x)]^{-1}s_x(x)\}A(x) \tag{4.5}$$

is well defined, where I denotes the $N \times N$ identity matrix. We shall need the following strengthening of the uniqueness part of A3, namely

A4. For every $x_0 \in R^N$ in some neighborhood of the sliding manifold (2.2) there exists a (unique) solution on $[0, +\infty)$ of

$$\dot{y} = A(y) + B(y)u^*(y, 0), y(0) = x_0, \tag{4.6}$$

where u^* is given by A2;

so that $\dot{y} = K(y), y(0) = x_0, K$ given by (4.5). We obtain exponential stability assuming uniform boundedness of the real trajectories, even if the sliding error is not assumed to vanish asymptotically (however see Proposition 3.3).

Theorem 4.3. *Let f be given by* (4.4) *and assume* A2, A3, A4. *Suppose that $A, B \in C^2(R^N), s \in C^3(R^N)$ and*

$$s(0) = 0, A(0) = 0, A_x(0) = 0. \tag{4.7}$$

Let S be any set of uniformly bounded real states, fulfilling (2.1) *almost everywhere in* $[0, +\infty)$. *Then S is exponentially stable.*

The last result of this section deals with conditions assuring uniform convergence on the whole $[0, +\infty)$ of the real states towards an ideal one, as required in some problems, e.g. asymptotic tracking, where asymptotic convergence may be considered too weak. We deal with sliding mode control systems, assuming that the dynamics are time-dependent and of the form

$$f = f(t, x, u) = A(t, x) + B(t, x)u. \tag{4.8}$$

We shall need the existence of the equivalent control u^*, namely (as in A2) the following condition

A5. For every $t \geq 0, x \in R^N$ and $\gamma \in R^M$ there exists a unique solution $u = u^*(t, x, \gamma)$ of class C^1 of $s_x(x)f(t, x, u) = \gamma$ with $f[t, 0, u^*(t, 0, 0)] = 0, t \geq 0$.

Moreover we require that the ideal states (corresponding to the equivalent control) vanish asymptotically (not necessarily with exponential stability), namely

A6. For every sliding state of (2.1), (2.2), f as in (4.8), corresponding to the equivalent control, we have $y(t) \to 0$ as $t \to +\infty$.

We shall also employ the definition of approximability of the sliding mode control system on every fixed bounded time interval as given in [2]. In the next theorem we assume that the regularization procedure works on every bounded interval. We obtain uniform convergence on $[0, +\infty)$ of the real states and their APS by suitable assumptions of uniform boundedness and uniform integrability, provided the ideal states vanish at infinity. With f given by (4.8) let us consider (2.1) in the form $\dot{x} = f(t, x, u)$.

Theorem 4.4. *Let f be given by* (4.8). *For every sequence x_n of real states and for every ideal state y of* (2.1), (2.2), *such that $x_n(0) \to y(0)$ as $n \to +\infty$, we have for a subsequence that $x_n \to y$ uniformly on $[0, +\infty)$ and x_n are APS, provided* A5, A6 *hold and the following conditions are met:*

• *The states x_n and the corresponding (in the almost everywhere sense) control laws are uniformly bounded on* $[0, +\infty)$.

• *For every $L > 0$ there exists $K > 0$ such that*

$$|B(t,x)[s_x(x)B(t,x)]^{-1}| \le K \text{ if } t \ge 0, |x| \le L.$$

• *The system* (2.1), (2.2) *fulfills approximability on every bounded time interval.*

• *For every $L > 0$ there exists φ, measurable on $[0, +\infty)$, such that $\int_0^{+\infty} \varphi dt$ converges and*

$$|A(t,x)| + |B(t,x)| \le \varphi(t) \text{ if } t \ge 0, |x| \le L. \tag{4.9}$$

Remark. The whole sequence $x_n \to y$ uniformly on $[0, +\infty)$ if $y(0)$ uniquely determines the ideal state y (as guaranteed by A4).

5. Example. Previous results guarantee practical stability assuming existence in the large and uniform boundedness of the real states. The following example, inspired by [13], shows that under conditions close to those required in the previous results, uniform boundedness, even existence in the large, of the real states we consider is not to be expected. Let the sliding mode control system be defined by

$$\dot{x}_1 = x_2, \dot{x}_2 = u, \dot{x}_3 = -(1 + x_2)x_3^3, s = x_1 + x_2.$$

The dynamics are as in (4.8), and the equivalent control $u^*(t, x, \gamma) = -x_2 + \gamma$ fulfills condition A5. Every ideal state y is given by

$$y_1(t) = -y_2(0)e^{-t}, y_2(t) = y_2(0)e^{-t}, y_3^2(t) = \frac{y_3^2(0)}{1 + y_3^2(0)[2t - 2y_2(0)e^{-t} + 2y_2(0)]}.$$

Thus $y_3(t)$ is well defined on the whole $[0, +\infty)$ with $y_3(t) \to 0$ as $t \to +\infty$, as required by A6, provided $y_2(0) \ge -1$. In this case

$$|y_k(t)| \le |y_2(0)|e^{-t}, k = 1, 2; |y_3(t)| \le |y_3(0)|, t \ge 0$$

hence asymptotic stability of the ideal states. Now consider

$$u = -2nx_2 - n^2 x_1, \quad n = 1, 2, 3, \ldots$$

and the initial conditions $x_1(0) = 1, x_2(0) = 0$. Then we get the corresponding sequence x_n of real states with

$$x_{1,n}(t) = e^{-nt} + nte^{-nt}, \quad x_{2,n}(t) = -n^2 te^{-nt}$$

so that, as $n \to +\infty, s(x_n) \to 0$ uniformly on every $[t_0, +\infty), t_0 > 0$. Moreover if $x_{3,n}(0) \neq 0$ we have

$$2x_{3,n}^2(t) = \frac{1}{t + nte^{-nt} + e^{-nt} - 1 + 1/[2x_{3,n}^2(0)]}. \tag{5.1}$$

Fix any $t_0 \in (0,1)$, let $x_{3,n}(0)$ be independent of n and let $x_{3,n}^2(0) > 1/[2(1-t_0)]$, then the denominator of (5.1) is negative at t_0 for every n sufficiently large. Hence the sequence x_n is not uniformly bounded in $[t_0, +\infty)$ (even worse, is not well defined there).

6. Regularization and metric regularity. In this section we point out a relation between the distance of real states from the set of the ideal ones and the corresponding sliding error, by showing that metric regularity of the sliding mapping is available under the maximum rank condition. We consider a more general setting that before, as follows. We are given a multifunction

$$F : [0, +\infty) \times R^N \rightrightarrows R^N,$$

a mapping $s \in C^1(R^N)$ and a positive number H. We posit the following assumptions:

A7. The set $F(t,x)$ is nonempty, compact and convex for every $t, x; F(\cdot, x)$ is measurable for every x, and $F(t, \cdot)$ is upper semicontinuous for every $t; s$ is of class C^1 and $N \geq M$.

A8. For every $x_0 \in R^N$ such that $|x_0| \leq H$ there exist locally absolutely continuous solutions x to

$$\dot{x}(t) \in F[t, x(t)] \text{ and } s[x(t)] = 0, t \geq 0, x(0) = x_0. \tag{6.1}$$

The setting of the previous section is now generalized to control systems modeled by the differential inclusion (6.1) with the viability constraint given by the sliding condition (2.3). Explicit sufficient conditions for A8 are well known, see [5]. For any fixed $x_0 \in R^N$ with $|x_0| \leq H$ we consider the following sets and mapping.

$A = A(x_0)$ is the set of all solutions x of

$$\dot{x}(t) \in F[t, x(t)], t \geq 0, x(0) = x_0;$$

$\Sigma = \Sigma(x_0)$ is the set of all solutions of (6.1);

$G = G(x_0) : A(x_0) \to C^0([0, +\infty), R^M)$ is defined by

$$G(x) = s[x(\cdot)].$$

Thus A is the set of all real states of the sliding mode control system (6.1), and Σ the one of all ideal states. We consider $A(x_0), \Sigma(x_0)$ as subsets of the Banach space $C^0([0,+\infty), R^N)$ equipped with the uniform norm. Given $\bar{x} \in A(x_0)$, the mapping $G = G(x_0)$ is *metrically regular* at $\bar{x}$, see [6], if there exists $M > 0$ such that

$$\text{dist}[x, G^{-1}(y)] \leq M \, \text{dist}[y, G(x)] \tag{6.2}$$

for every (x, y) sufficiently close to $[\bar{x}, G(\bar{x})]$; here

$$\text{dist}(a, Q) = \inf \{\|a - q\| : q \in Q\}.$$

We obtain an estimate of the distance of any real state from the set of all sliding states of (6.1) making use of the Lyusternik-Graves theorem and the compactness of $A(x_0)$.

Theorem 6.1. *Assume* A7, A8 *and suppose that*

$$\sup\{|w| : w \in F(t, x)\} \leq \varphi(t)(1 + |x|) \tag{6.3}$$

for a.e.t ≥ 0, *every* x *and some* $\varphi \in L^1([0,+\infty))$*;*

(6.4) *the Jacobian matrix of s is everywhere of (maximum) rank M.*

Then there exists a constant $D > 0$ such that for every x_0 with $|x_0| \leq H$ and every $x \in A(x_0)$ we have

$$\text{dist}(x, \Sigma) \leq D\|G(x)\|. \tag{6.5}$$

Thus the distance of any real state from the sliding, measured in the uniform norm over $[0, +\infty)$, is controlled by the sliding error. If a regularization procedure (as mentioned in section 2) is employed, and any sequence x_n of real states fulfills $G(x_n) \to 0$ uniformly on $[0, +\infty)$, then some subsequence of x_n converges uniformly to some ideal state of the system, due to (6.5) and compactness of Σ, with rate not slower than the uniform vanishing of the sliding error. If the control system fulfills further uniqueness conditions, as the existence of the equivalent control guaranteed by A2, then the original sequence x_n converges to the ideal state issued from x_0 corresponding to the equivalent control: see [2, 3]. A quite similar result to (6.5) holds for systems on the bounded horizon $[0, T]$.

7. Proofs.

Proof of Proposition 3.2. We have

$$|z_n(t)| \leq |z_n(t) - y(t)| + |y(t)|, t \geq 0.$$

By asymptotic convergence, given $\varepsilon > 0$ we have $|z_n(t) - y(t)| \leq \varepsilon$ if n and t are sufficiently large. Moreover $|y(t)| \leq \varepsilon$ if t is sufficiently large, hence the conclusion. $\square$

Proof of Proposition 3.3. By continuity of s at $x = 0$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|s(x)| \leq \varepsilon$ if $|x| \leq \delta$. If x_n is APS, then $|x_n(t)| \leq \delta$ provided n and t are sufficiently large, hence $|s[x_n(t)]| \leq \varepsilon$ for the same n and t, whence the conclusion. $\square$

Proof of Theorem 4.1. By uniform convergence of $s(z_n)$ there exists $k > 0$ such that $z_n \in S(\Delta_0, H)$ for some $\Delta_0 > 0$ and every $n \geq k$. By Theorem 3 of [17] and the remark following its statement, given $\varepsilon > 0$ there exist positive constants $\Delta \leq \Delta_0$ and T such that if z is a real state and

$$|s[z(t)]| \leq \Delta, t \geq 0 \tag{7.1}$$

then $|z(t)| \leq \varepsilon, t \geq T$. By uniform convergence of $s(z_n)$ we have (7.1) for every $t \geq 0$ and $z = z_n$ with every sufficiently large n, whence APS of z_n. Let y be the ideal state issued from $(0, \bar{x})$. Then by practical stability and A3 we have

$$|z_n(t) - y(t)| \ \leq |z_n(t)| + |y(t)| \leq 2\varepsilon$$

if n and t are sufficiently large, whence asymptotic convergence of z_n. Practical stability of $S(\Delta_0, H)$ comes again from Theorem 3 of [17]. $\square$

Proof of Theorem 4.2. By A2 we have $M = P$ and CB is non singular. By A3 the matrix

$$K = [I - B(CB)^{-1}C]A,$$

I denoting the $N \times N$ identity matrix, is Hurwitz. Write

$$U = B(CB)^{-1}C.$$

For every n and $t \geq 0$, integrating by parts we find

$$x_n(t) - y(t) = e^{Kt}[x_n(0) - y(0)] + Ux_n(t) - e^{Kt}Ux_n(0) + \int_0^t Ke^{K(t-\theta)}Ux_n(\theta)d\theta.$$

We analyze the behavior of each of the four terms in the previous equality, as far as asymptotic convergence is concerned. By (4.3) and the Hurwitz property of $K, e^{Kt}[x_n(0) - y(0)] \to 0$ asymptotically. The same is true for Ux_n by (4.2), and for $e^{Kt}Ux_n(0)$ by (4.3). Now we show that the last term

$$\int_0^t Ke^{K(t-\theta)}Ux_n(\theta)d\theta \to 0 \text{ asymptotically.} \tag{7.2}$$

Given $\varepsilon > 0$, by (4.2) there exist T, p such that

$$|Cx_n(t)| \leq \varepsilon \text{ if } t \geq T \text{ and } n \geq p,$$

then

$$\left| \int_T^t Ke^{K(t-\theta)} Ux_n(\theta) d\theta \right| \leq \text{ (constant) } \varepsilon \int_0^t |Ke^{K(t-\theta)}| d\theta \leq \text{ (constant) } \varepsilon \tag{7.3}$$

since $\int_0^t |Ke^{K(t-\theta)}| d\theta$ is bounded in $[0, +\infty)$ because

$$|e^{K(t-\theta)}| \leq \text{(constant)} e^{-\alpha(t-\theta)}, t \geq \theta \geq 0$$

for some positive constant α, due to the Hurwitzian property of K. The proof will be ended by showing that an estimate analogous to (7.3) is true for $\int_0^T Ke^{K(t-\theta)} Ux_n(\theta) d\theta$. By uniform boundedness of x_n in $[0, T]$ there exists a constant $L > 0$ such that

$$|x_n(\theta)| \leq L \text{ if } 0 \leq \theta \leq T.$$

Moreover there exists a positive constant α such that

$$|e^{Kt}| \leq \text{ (constant) } e^{-\alpha t}, |e^{-K\theta}| \leq \text{ (constant) } e^{\alpha\theta}$$

for every $t, \theta \geq 0$. Then

$$\left| \int_0^T Ke^{K(t-\theta)} Ux_n(\theta) d\theta \right| \leq \text{ (constant) } Le^{-\alpha t} e^{\alpha T} \leq \varepsilon$$

if t is sufficiently large. So (7.2) holds and this completes the proof. $\square$

Proof of Theorem 4.3. Given $x \in S$, let $s(t) = s[x(t)]$. Then, by A2, K given by (4.5) is well defined. Moreover for every $x \in S$ and almost every $t \geq 0$ we have

$$\dot{s} = s_x(x)(A(x) + B(x)u),$$

then

$$u = [s_x(x)B(x)]^{-1}[\dot{s} - s_x(x)A(x)]$$

whence

$$\dot{x} = K(x) + B(x)[s_x(x)B(x)]^{-1} \dot{s}.$$

We apply a slight extension of Theorem 2.2.2 of [10], by considering an arbitrarily fixed ball of initial points. To this aim, given x_0 in such a ball, let $y = y(t, x_0)$

denote the solution of (4.6) and consider the nonsingular matrix $\Phi = \partial y/\partial x_0$ for which, as well known,

$$\dot{\Phi} = K_x[y(t,x_0)]\Phi, \Phi(0) = I.$$

Due to the assumed smoothness of A, B and s, K_x is Lipschitz continuous on bounded sets. By (4.7) we have $K_x(0) = 0$, hence if $s(x_0) = 0$, by A3,

$$|K_x[y(t,x_0)]| \leq M_1|x_0|e^{-ct}, c > 0, t \geq 0, \tag{7.4}$$

for a suitable constant M_1. Then by (7.4)

$$|\Phi^{-1}(t)| = \left| I - \int_0^t \Phi^{-1}(\theta)K_x[y(\theta,x_0)]d\theta \right| \leq C_1 + C_2 \int_0^t |\Phi^{-1}(\theta)|e^{-c\theta}d\theta$$

for suitable constants C_1, C_2. Therefore by Gronwall's lemma

$$|\Phi^{-1}(t)| \leq C_1 e^{C_2 \int_0^t e^{-c\theta}d\theta} \leq C_3, t \geq 0 \tag{7.5}$$

for a suitable constant C_3. Write

$$R = B(s_x B)^{-1}s_x(A + Bu).$$

Then by (7.5), if $t \geq 0$ we have

$$\begin{aligned}
&|\Phi^{-1}(t)R[y(t)]| \leq \\
&\quad \leq C_3|B[y(t)](s_x[y(t)]B[y(t)])^{-1}s_x[y(t)][A[y(t)] + B[y(t)]u^*(y(t),0)]| \leq \\
&\quad \leq C_4|A[y(t)]| + C_5|u^*(y(t),0)|
\end{aligned}$$

for suitable constants C_4, C_5, since $|y(t)|$ is uniformly bounded by A3. Since A is Lipschitz continuous on bounded sets, by (4.7)

$$|A[y(t)]| \leq \text{ (constant) } |y(t)|, t \geq 0.$$

Due to A2 and $u^*(0,0) = 0$ we have that $u^*(\cdot,0)$ is Lipschitz continuous on bounded sets, hence

$$|u^*(y(t),0)| \leq \text{ (constant) } |y(t)|, t \geq 0.$$

Thus

$$|\Phi^{-1}(t)R[y(t)]| \leq \text{ (constant) } |y(t)|, t \geq 0.$$

Then remembering A3 we obtain (notations of [10, Theorem 2.2.2])

$$g(t,w) = \text{ (constant) } we^{-ct}, r(t,0,u_0) = u_0 e^{\text{ (constant) } (1-e^{-ct})}.$$

Thus by (2.2.9) of [10] we get

$$|x(t)| \leq M^*|x_0|e^{-ct}, x \in S, t \geq 0,$$

where M^* is a suitable constant independent of $x \in S$, and this completes the proof. $\square$

P r o o f o f T h e o r e m 4.4. We prove uniform convergence on $[0, +\infty)$. Approximability on every bounded time interval implies that, as $n \to +\infty, x_n(t) \to y(t)$ uniformly on every $[0, T], T > 0$, see [2]. Then uniform convergence on $[0, +\infty)$ will be proved, taking into account Ascoli-Arzelà's theorem for bounded intervals, if the following holds: for every $\varepsilon > 0$ there exists $T > 0$ such that

$$|x_n(t'') - x_n(t')| \leq \varepsilon \text{ for every } n \text{ and } t', t'' > T \tag{7.6}$$

(see [7, Theorem IV.6.5, p. 266]). Write

$$s_n = s(x_n), A_n = A(\cdot, x_n), B_n = B(\cdot, x_n), U_n = B_n(s_x(x_n)B_n)^{-1}.$$

Then for every n

$$\dot{s}_n = s_x(x_n)(A_n + B_n u_n)$$

where x_n corresponds to the control law u_n, so that

$$u_n = (s_x(x_n)B_n)^{-1}(\dot{s}_n - s_x(x_n)A_n)$$

whence

$$\dot{x}_n = A_n + U_n(\dot{s}_n - s_x(x_n)A_n).$$

Let now $t'' > t' \geq 0$. Then for every n

$$x_n(t'') - x_n(t') = \int_{t'}^{t''} A_n(t)dt + \\ + \int_{t'}^{t''} U_n(t)\dot{s}_n(t)dt - \int_{t'}^{t''} U_n(t)s_x(x_n(t))A_n(t)dt. \tag{7.7}$$

Given $\varepsilon > 0$, by (4.9) and uniform boundedness of U_n there exists $T_1 > 0$ such that, for every n,

$$\left|\int_{t'}^{t''} A_n(t)dt\right| + \left|\int_{t'}^{t''} U_n(t)s_x(x_n(t))A_n(t)dt\right| \leq \\ \leq \text{ (constant) } \int_{t'}^{t''} \varphi(t)dt \leq \text{ (constant) } \varepsilon$$

provided $t', t'' > T_1$. Moreover, by uniform boundedness of x_n and the corresponding control laws, we have again by (4.9)

$$|\dot{s}_n(t)| \leq \text{ (constant) } \varphi(t).$$

Hence there exists $T_2 > 0$ such that

$$\left| \int_{t'}^{t''} U_n(t)\dot{s}_n(t)dt \right| \leq \text{ (constant) } \varepsilon$$

provided $t', t'' \geq T_2$ and every n. Remembering (7.7), the above estimates imply (7.6), hence uniform convergence of a subsequence of x_n towards y. Since $y(t) \to 0$ as $t \to +\infty$, this gives APS of x_n as required. □

Proof of Theorem 6.1. We fix any x_0 with $|x_0| \leq H$ and apply the Lyusternik-Graves theorem, see [9, p. 504]. Since $s = s(x)$ is smooth, we have that the Gâteaux derivative of G is given by

$$G'(\bar{x})(y) = \frac{\partial s}{\partial x}[\bar{x}(\cdot)]y(\cdot)$$

for every $\bar{x}, y$; hence G is everywhere strictly differentiable and $G'(\bar{x})$ is onto by (6.4). By the Lyusternik-Graves theorem, for every $\bar{x} \in \Sigma = G^{-1}(0)$ there exist $D > 0$ and $\varepsilon > 0$ such that

$$\text{dist}(x, \Sigma) \leq D\|G(x)\| \tag{7.8}$$

if $x \in A = A(x_0)$ and $\|x - \bar{x}\| < \varepsilon$. By A7 and (6.3) the set of all solutions on $[0, T]$ of

$$\dot{x}(t) \in F[t, x(t)], 0 \leq t \leq T, x(0) = x_0 \tag{7.9}$$

is compact in $C^0([0,T], R^N)$ for every $T > 0$, by [5, Corollary 7.3]. Moreover by (6.3) Gronwall's lemma implies that there exists $K > 0$ such that

$$|x(t)| \leq K, t \geq 0$$

for every solution x of (7.9). Then for any $t > T$ we have

$$|x(t) - x(T)| \leq \int_T^t \varphi(s)[1 + |x(s)|]ds \leq (1 + K)\int_T^t \varphi(s)ds$$

again by (6.3). We apply [7, Theorem IV.6.5, p. 266] and this implies compactness of $A = A(x_0)$ in $C^0([0, +\infty), R^N)$. By compactness and (7.8), there exists a finite number of points $\bar{x}_1, \ldots, \bar{x}_p \in A$ and positive numbers $\varepsilon_1, \ldots, \varepsilon_p$ such that (7.8) holds true if $\|x - \bar{x}_k\| < \varepsilon_k, k = 1, \ldots, p$, and for some fixed $D > 0$, moreover A is covered by the union of the balls of center $\bar{x}_k$ and radius $\varepsilon_k, k = 1, \ldots, p$. It follows that (7.8) holds for every $x \in A$ as required.

Acknowledgements. Work partially supported by MIUR, progetto PRIN, "Sistemi distribuiti su larga scala: stima, ottimizzazione e controllo, con applicazioni".

REFERENCES

[1] B. R. Barmish, G. Leitmann. On ultimate boundedness control of uncertain systems in the absence of matching assumptions. *IEEE Trans. Autom. Control* **27** (1982), 153–158.

[2] G. Bartolini, T. Zolezzi. Control of nonlinear variable structure systems. *J. Math. Anal. Appl.* **118** (1986), 42–62.

[3] G. Bartolini, T. Zolezzi. Behavior of variable-structure control systems near the sliding manifold. *Systems Control Lett.* **21** (1993), 43–48.

[4] M. Corless, G. Leitmann. Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems. *IEEE Trans. Autom. Control* **26** (1981), 1139–1144.

[5] K. Deimling. Multivalued differential equations. De Gruyter Series in Nonlinear Analysis and Applications, vol. **1**. Berlin, Walter de Gruyter & Co., 1992.

[6] A. L. Dontchev, R. T. Rockafellar. Implicit functions and solution mappings. A view from variational analysis. Springer Monographs in Mathematics. Dordrecht, Springer, 2009.

[7] N. Dunford, J. T. Schwartz. Linear Operators. I. General Theory. Pure and Applied Mathematics, vol. **7**. New York, Interscience Publishers Inc; London, Interscience Publishers Ltd., 1958.

[8] A. F. Filippov. Differential equations with discontinuous righthand sides. Translated from the Russian. Mathematics and its Applications (Soviet Series), vol. **18**. Dordrecht, Kluwer Academic Publishers Group, 1988.

[9] A. D. Ioffe. Metric regularity and subdifferential calculus. *Usp. Mat. Nauk* **55**, *3* (2000), 103–162 (in Russian); English translation in: *Russian Math. Surv.* **55**, *3* (2000), 501–558.

[10] V. Lakshmikantham, S. Leela, A. A. Martinyuk. Stability analysis of nonlinear systems. Monographs and Textbooks in Pure and Applied Mathematics, vol. **125**. New York, Marcel Dekker Inc., 1989.

[11] X. Y. Lu, S. K. Spurgeon. Robust sliding mode control of uncertain non linear systems. *Systems Control Lett.* **32** (1997), 75–90.

[12] E. P. Ryan, M. Corless. Ultimate boundedness and asymptotic stability of a class of uncertain dynamical systems via continuous and discontinuous feedback control. *IMA J. Math. Control Inf.* **1** (1984), 223–242.

[13] H. J. Sussmann, P. V. Kokotovic. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Trans. Autom. Control* **36** (1991), 424–440.

[14] V. I. Utkin. Sliding modes and their applications in variable structure systems. Mir, 1978.

[15] V. I. Utkin. Sliding modes in control and optimization. Translated and revised from the 1981 Russian original. Communications and Control Engineering Series. Berlin, Springer-Verlag, 1992.

[16] T. Yoshizawa. Stability theory by Liapunov's second method. Publications of the Mathematical Society of Japan, No. 9. Tokyo, The Mathematical Society of Japan, 1966.

[17] T. Zolezzi. Real states of stable sliding mode control systems. *Systems Control Lett.* **57** (2008), 778–783.

DIMA
University of Genova
via Dodecaneso 35
16146 Genova, Italy
e-mail: zolezzi@dima.unige.it

Received April 8, 2013