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NEWTON-SECANT METHOD FOR FUNCTIONS WITH VALUES IN A CONE

Alain Pietrus, Célia Jean-Alexis

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This paper is dedicated to A. L. Dontchev and V. Veliov at the occasion of their 65th and 60th birthday respectively

Abstract. This paper deals with variational inclusions of the form $0 \in K - f(x) - g(x)$ where f is a smooth function from a reflexive Banach space X into a Banach space Y, g is a function from X into Y admitting divided differences and K is a nonempty closed convex cone in the space Y. We show that the previous problem can be solved by a combination of two methods: the Newton and the Secant methods. We show that the order of the semilocal method obtained is equal to $\dfrac{1+\sqrt{5}}{2}$. Numerical results are also given to illustrate the convergence at the end of the paper.

1. Introduction. The variational inclusions were introduced by Robinson [21, 22] as an abstract model for various problems encountered in different fields such as engineering, economy, transport theory, etc.

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Several studies dealing with the variational inclusions of the form

$$0 \in f(x) + F(x). \tag{1}$$

have been carried out during the last decades where $f : X \to Y$ is a function, $F : X \rightrightarrows Y$ is a set-valued map and X, Y are Banach spaces.

In the smooth case (f is smooth), Dontchev [5, 6] gave an interesting contribution to approximate a solution x^* of (1). For this, he introduced a sequence obtained from a partial linearization of the single-valued part. More precisely, he associated to (1) the Newton-type sequence

$$\begin{cases} x_0 \text{ is a given starting point} \\ 0 \in f(x_k) + f'(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \end{cases} \tag{2}$$

and established the quadratic convergence when f' (the Fréchet derivative of f) is locally Lipchitz around the solution x^* under a pseudo-Lipschitz property for the set-valued map $(f + F)^{-1}$. For more details on the Lipschitz property, also called Aubin property or Lipschitz like property, the reader could refer to [1, 2, 7, 8, 15, 16, 23, 24]. Following Dontchev's works, we can find in the literature various papers about the resolution of this type of variational inclusions in which the authors use metric regularity.

It seems natural to study the mild differentiable case: we suppose that the single-valued part of equation (1) can be written as a sum of a smooth function with a mild differentiable perturbation and we study the following variational inclusion

$$0 \in f(x) + g(x) + F(x) \tag{3}$$

where $f : X \to Y$ is a smooth function, $g : X \to Y$ is a function which admits divided differences on a neighborhood $\Omega \subset X$ of x^* solution of (3) and $F : X \rightrightarrows Y$ is a set-valued map.

The authors in [9] associated to (3) the inclusion

$$\begin{cases} x_0 \text{ is a given starting point} \\ 0 \in f(x_k) + g(x_k) + \big(f'(x_k) + [x_{k-1}, x_k; g]\big)(x_{k+1} - x_k) + F(x_{k+1}) \end{cases} \tag{4}$$

where $[x_{k-1}, x_k; g]$ is an operator called divided difference of g at the points x_{k-1} and x_k, see Definition 2.1. In the case where F is the set $\{0\}$, the previous inclusion is equivalent to the iterative method introduced by Cãtinas [3], who proves that the order of convergence is the same as the method of chords $\left(p = \dfrac{1+\sqrt{5}}{2}\right)$. In [9] as in [5], the authors obtained existence and convergence results respectively for (1) and (3) under some metric regularity condition and by using the fixed point theorem given in [4] .

Let us note that although metric regularity is a very interesting tool for obtaining theoretical results (existence of sequences, rate of convergence, etc.) in the field of variational inclusions, it becomes a concept which is sometimes not easy to be verified in practice. In addition, the methods resulting from the use of metric regularity assumptions generally only furnish local convergence of sequences.

Our aim in this paper is to approximate a solution x^* of (3), in the case where $F(x) = -K$ for all $x \in X$, where $K \subset Y$ is a nonempty closed convex cone. For this, we introduce a new algorithm, we start with two points x_0, x_1; if x_k is computed, the new iterate x_{k+1} is a solution of a minimization problem in which appears the first order divided difference of g.

The method of approximation proposed has a semilocal convergence and the generated sequence converges to some x^* which is solution of (3). In addition, we show that the order of convergence of this algorithm is $p = \dfrac{1+\sqrt{5}}{2}$ under some classical conditions. We note that the metric regularity concept is not used here.

Let us underline that in [17], the inclusion (3) has been solved in the same spirit. Nevertheless, the mapping g was Lipschitz and was not supposed to possess divided differences, besides the author obtained a linear convergence through a Zincensko's type method.

In the next section, we recall some preliminaries on different concepts as divided differences, normed convex processes and majorizing sequences. In Section 3, we introduce and describe the new algorithm for solving (3). Section 4 is devoted to our main theorem and its proof. Finally, in Section 5 we give some numerical results and make a comparison with the method introduced in [17]. We denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r.

2. Preliminaries.

In this section, we collect some results that we will need to introduce our algorithm and to prove our main result.

2.1. Divided differences.

Definition 2.1. *Let X and Y be two Banach spaces. A linear operator acting from X into Y is called a first order divided difference of the operator*

$g : X \longrightarrow Y$ on the points x_0, y_0, denoted by $[x_0, y_0; g]$, if the following property holds:

$$[x_0, y_0; g](y_0 - x_0) = g(y_0) - g(x_0), \quad \textit{for } x_0 \neq y_0.$$

In addition, if g is Fréchet differentiable at $x_0 \in X$ then $[x_0, x_0; g] = g'(x_0)$.

Definition 2.2. *Let X and Y be two Banach spaces. A linear operator acting from X into $\mathcal{L}(X, Y)$ is called a second order divided difference of the operator $g : X \longrightarrow Y$ on the points x_0, y_0 and z_0 denoted by $[x_0, y_0, z_0; g]$, if the following property holds:*

$$[x_0, y_0, z_0; g](z_0 - x_0) = [y_0, z_0; g] - [x_0, y_0; g], \quad \textit{for } x_0, y_0, z_0, \textit{distinct.}$$

In addition, if g is twice Fréchet differentiable at $x_0 \in X$ then $[x_0, x_0, x_0; g] = \dfrac{1}{2} g''(x_0)$.

These operators have been used in various works, for instance in [3, 10].

2.2. Normed convex processes.

Definition 2.3. *A mapping T from the real linear space X to the real linear space Y is a convex process if it satisfies*

a) $T(x) + T(z) \subset T(x + z)$ for all $x, z \in X$

b) $T(\lambda x) = \lambda T(x)$ for every $\lambda > 0$ and every $x \in X$

c) $0 \in T(0)$.

In fact, a convex process from X to Y is a mapping defined on X into subsets of Y, whose graph is a convex cone in $X \times Y$ containing the origin. If the graph is closed, then we refer to a closed convex process.

The idea of convex processes has been introduced by Rockafellar [25, 26] and this concept has been clearly formalized and studied by Robinson [19]. Lewis [13, 14] shows that the concept of convex processes is an interesting tool to study ill-conditioned linear systems and inclusions. The following definitions and results come from Robinson's paper. For a convex process T, we define respectively the domain, the range and the inverse by: dom T is the set of points x for which $T(x) \neq \emptyset$, range T is $\bigcup\{T(x), x \in \operatorname{dom} T\}$ and the inverse, noted T^{-1}, is a mapping from range T onto dom T with $T^{-1}(y) = \{x \mid y \in T(x)\}$.

Note that dom T and range T are both convex cones containing 0 and the inverse which always exists, is itself a convex process. Finally, if X and Y are normed spaces, we can define the norm of T by

$$\|T\| := \sup\{\inf\{\|y\| \mid y \in T(x)\} \mid \|x\| \leq 1,\ x \in \operatorname{dom} T\}$$

and we shall call that a convex process is normed if its norm is finite.

Now we are going to give a theorem which is useful for the proof of our main theorem.

Theorem 2.1 ([19, Theorem 5]). *Let X be a Banach space and Y a normed linear space. Let T and Δ be convex processes from X into Y. Assume that T, T^{-1} and Δ are normed and that $\|T^{-1}\|.\|\Delta\| < 1$. Suppose further that* $\operatorname{dom} T \subset \operatorname{dom} \Delta$, $\Delta(\operatorname{dom} T) \subset \operatorname{range} T$, $\operatorname{dom} T$ *is closed and* $(T - \Delta)(x)$ *is closed for each* $x \in \operatorname{dom} T$. *Then the convex process $T - \Delta$ has the following properties:*

- $\operatorname{range} T \subset \operatorname{range}(T - \Delta)$,
- $(T - \Delta)^{-1}_{\operatorname{range} T}$ *is a normed convex process and* $\|(T - \Delta)^{-1}_{\operatorname{range} T}\| \leq \dfrac{\|T^{-1}\|}{(1 - \|T^{-1}\|.\|\Delta\|)}$.

Let us quote that if Δ is just a linear transformation from X into Y, then it is a convex process and the conclusion of the theorem is again valid.

2.3. Majorizing sequences. The following definitions, properties and examples come from Rheinboldt's paper [18]. In fact, the so-called concept of majorizing sequences has been introduced by Kantorovich in [11] in which appears a new proof of semilocal convergence of Newton's method (for nonlinear equations) based on this concept.

Definition 2.4. *Let (x_k) be a sequence in the metric space (X, ρ). Then a real nonnegative sequence (t_n) is said to majorize (x_k) if*

$$\rho(x_{k+1}, x_k) \leq t_{k+1} - t_k, \qquad k = 0, 1, \ldots.$$

It is easy to observe that any majorizing sequence (t_k) of (x_k) is necessarily nondecreasing and for $m > k \geq 0$,

$$\rho(x_m, x_k) \leq \sum_{j=k}^{m-1} \rho(x_{j+1}, x_j) \leq \sum_{j=k}^{m-1} (t_{j+1} - t_j) = t_m - t_k.$$

Hence, if $\lim t_k = t^* < +\infty$ exists then (x_k) is a Cauchy sequence in X; therefore, if X is complete, $\lim x_k = x^*$ also exists and for $m \to +\infty$, the error estimate is immediate and $\rho(x^*, x_k) \leq t^* - t_k$ for $k = 0, 1, \ldots$.

For the majorizing principle to be useful, some estimations are necessary to obtain a majorizing sequence (t_k) for a given sequence (x_k). Thus, this principle requires appropriate assumptions, either about the generating mechanism of the sequence (x_k) or at least about the relation between the successive iterates of the sequence (x_k). For more results and specific studies, the reader could refer to Rheinboldt's paper [18].

3. Description of the algorithm. In the rest of the paper we assume that X is a reflexive Banach space.

Let us consider a subset $X_0 \subset X$. For any fixed v and $w \in X_0$, we define a set-valued mapping $T(v,w)$ from X to Y by

$$T(v,w)x := (f'(w) + [v,w;g])x - K, \qquad x \in X.$$

From the definition, it is easy to see that $T(v,w)$ is a normed convex process from X to Y. Its inverse, defined for any $y \in Y$ by

$$T^{-1}(v,w)y := \{z \in X \mid (f'(w) + [v,w;g])z \in y + K\},$$

is also a normed convex process.

Given two starting points x_0 and x_1 in X_0 such that $T^{-1}(x_0,x_1)[-f(x_1) - g(x_1)] \neq \emptyset$, we obtain $x_2 - x_1$ as a projection of the origin in X on $T^{-1}(x_0,x_1)[-f(x_1) - g(x_1)]$. Then we repeat the procedure by using x_1 and x_2 as starting points. At the k^{th} step, we have x_{k-1}, x_k and we define $x_{k+1} - x_k$ as a projection of the origin in X on $T^{-1}(x_{k-1},x_k)[-f(x_k) - g(x_k)]$.

It is easy to see that an equivalent way to write the algorithm is, if x_k is already computed, the point x_{k+1} appears to be any solution of the minimization problem

$$(5) \quad minimize\{\|x - x_k\| / f(x_k) + g(x_k) + (f'(x_k) + [x_{k-1},x_k;g])(x - x_k) \in K\}.$$

Algorithm. Newton-Secant-cone $(f, g, K, x_0, x_1, \varepsilon)$
1. If $T^{-1}(x_0,x_1)[-f(x_1) - g(x_1)] = \emptyset$, stop.

2. Do while $e > \varepsilon$
(a) Choose x as a solution of the problem

$$minimize\{\|x - x_1\| / f(x_1) + g(x_1) + (f'(x_1) + [x_0,x_1;g])(x - x_1) \in K\}.$$

(b) Evaluate $e := \|x - x_1\|$; $x_0 := x_1$; $x_1 := x$.

3. Return x.

Remark 1. If $K = \{0\}$, the procedure is exactly the same given in [3].

Remark 2. If $g(x) = 0$ for all $x \in X$, the procedure is exactly the Newton-type given in [20] and in this case one obtains quadratic convergence.

Remark 3. If instead of the cone K, we consider any set-valued mapping depending on x, we can observe that the inclusion given by the constraints in (5) is exactly the method introduced by Geoffroy and Piétrus in [9]. In this last case the numerical treatment is not easy and moreover one obtains only a local convergence result under a metric regularity assumption which depends strongly on the solution.

Remark 4. The continuity of the linear operator $f'(x_k)$ (the Fréchet derivative) and the fact that K is closed and convex, imply that the feasible set of (5) is a closed convex set for all $k \in \mathbb{N}^*$. Then the existence of a feasible point $\tilde{x}$ implies that any solution of (5) must lie in the intersection of the feasible set of (5) with the closed ball of center x_k and radius $\|\tilde{x} - x_k\|$. Since X is reflexive and the function $\|x - x_k\|$ is weakly lower semicontinuous, a solution of (5) exists (see [12]). Then it is clear that if (5) is feasible then it is solvable and its convexity implies that any local solution will be global.

4. Convergence analysis of the algorithm.

For the convergence analysis of our algorithm, we need to prove the following proposition.

Proposition 4.1. *Consider the sequence (t_n) defined by*

i) $\quad t_0 = 0, \quad t_1 = \alpha, \quad t_2 = \beta.$

ii) $$t_{n+1} - t_n = M\left(\frac{L}{2}(t_n - t_{n-1})^2 + 2K(t_n - t_{n-1})(t_n - t_{n-2})\right)$$

where α, β, M, L and K are positive constants.

If the conditions

iii) $\alpha \le M$, $\beta \le 2\alpha$,

iv) $$q = M^2\left(\frac{L}{2} + 2K\right) < 1$$

are fulfilled, then the sequence (t_n) *is such that all its terms belong to* $\mathbb{B}_s(t_1)$ *where* $s = \dfrac{M}{q}\sum_{l=1}^{\infty} q^{u_l}$, (u_l) *is the Fibonacci's sequence defined by* $u_0 = u_1 = 1$ *and* $u_{l+1} = u_l + u_{l-1}$ *with* $l \geq 1$. *Moreover, the sequence* (t_k) *is convergent and one has the following error estimate*

$$t^* - t_n \leq \frac{M}{q\left(1 - q^{\frac{p^n(p-1)}{\sqrt{5}}}\right)} q^{\frac{p^n}{\sqrt{5}}}$$

where $t^* = \lim t_n$ *and* $p = \dfrac{1+\sqrt{5}}{2}$.

Proof. The idea of the proof comes from [3], it consists to show first by induction (for any $n \geq 2$) that

$$t_n \in \mathbb{B}_s(t_1), \tag{6}$$

$$t_n - t_{n-1} \leq t_{n-1} - t_{n-2}, \tag{7}$$

$$t_n - t_{n-1} \leq q^{u_{n-1}-1} M. \tag{8}$$

It is clear that (6), (7) and (8) hold for $n = 2$.
For $k \geq 2$, let us suppose that (6), (7), (8) has been checked for $2 \leq n \leq k$.

Using the assumption (7) of the induction, one has

$$t_{k+1} - t_k \leq M\left(\frac{L}{2} + 2K\right)(t_k - t_{k-1})(t_{k-1} - t_{k-2}).$$

Thanks to assumption (8), one obtains

$$t_{k+1} - t_k \leq M\Big(\frac{L}{2} + 2K\Big) q^{u_{k-2}-1} M (t_k - t_{k-1})$$

which implies

$$t_{k+1} - t_k \leq q^{u_{k-2}}(t_k - t_{k-1}) \leq t_k - t_{k-1}$$

and proves (7) for $n = k+1$.

We also have

$$t_{k+1} - t_k \leq q^{u_{k-2}}(t_k - t_{k-1}) \leq q^{u_{k-2}} q^{u_{k-1}-1} M = q^{u_k - 1} M$$

and proves (8) for $n = k+1$.

Moreover, one has

$$t_{k+1} - t_1 = t_2 - t_1 + t_3 - t_2 + \cdots + t_k - t_{k-1} + t_{k+1} - t_k$$

and using (8), one obtains

$$t_{k+1} - t_1 \leq \frac{M}{q}(q^{u_1} + q^{u_2} + \cdots + q^{u_{k-1}} + q^{u_k}) < s$$

which corresponds to (6).

Now, let us show that (t_k) is a Cauchy sequence.

For any $k \geq 1$, $m \geq 1$, one has

$$t_{k+m} - t_k \leq \frac{M}{q}\left(q^{u_k} + q^{u_{k+1}} + \cdots + q^{u_{k+m-2}} + q^{u_{k+m-1}}\right).$$

It is well-know that $u_l \geq \dfrac{p^l}{\sqrt{5}}$ for $l \geq 1$ and this implies

$$t_{k+m} - t_k \leq \frac{M}{q} q^{\frac{p^k}{\sqrt{5}}}\left(1 + q^{\frac{p^{k+1}-p^k}{\sqrt{5}}} + q^{\frac{p^{k+2}-p^k}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-2}-p^k}{\sqrt{5}}} + q^{\frac{p^{k+m-1}-p^k}{\sqrt{5}}}\right).$$

Using the fact that $p^{k+s} - p^k = p^k(p^s - 1)$ and Bernoulli's inequality, one obtains

$$t_{k+m} - t_k \leq \frac{M}{q} q^{\frac{p^k}{\sqrt{5}}}\left(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k 2(p-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(m-2)(p-1)}{\sqrt{5}}} + q^{\frac{p^k(m-1)(p-1)}{\sqrt{5}}}\right)$$

$$\leq \frac{M q^{\frac{p^k}{\sqrt{5}}}\left(1 - q^{\frac{p^k(p-1)m}{\sqrt{5}}}\right)}{q\left(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}\right)}.$$

The last inequality gives the convergence of (t_n) and we set $t^* = \lim t_n$. The error estimate is obtained with $m \to \infty$. $\square$

We now give the main result of our paper.

Theorem 4.1. *Let X, X_0, Y, f, g and T be as previously defined.*

Suppose that there exist points $x_0, x_1 \in X_0$, such that $T(x_0, x_1)$ carries X onto Y and there are real positive numbers B, L, K, M_1 and α satisfying the following properties:

(*a*) $\|T^{-1}(x_0, x_1)\| \leq B$,

(*b*) *For each* $x, y \in X_0$, *we have*

$$\|f'(x) - f'(y)\| \leq L\|x - y\|,$$

(*c*) g *is continous on* X_0 *and admits divided differences of first and second order on* X_0,

(*d*) $BM_1 < 1$; $\alpha \leq \beta \leq 2\alpha$,

(*e*) $\|x_1 - x_0\| \leq \alpha \leq \dfrac{B}{1 - BM_1}$ *and* $\|x_2 - x_1\| \leq \beta - \alpha$, *where* x_2 *is obtained from* x_0 *and* x_1 *by the algorithm (such a point exists by Remark* 4 *in the previous section),*

(*f*) *For any distinct points* x, y *and* z *in* X_0, $\|[x, y, z; g]\| \leq K$, $\|f'(y) + [x, y; g] - f'(x_1) - [x_0, x_1; g]\| \leq M_1$ *and* $q = \dfrac{B^2}{(1 - BM_1)^2}\left(\dfrac{L}{2} + 2K\right) < 1$.

There exists t^* *such that if* $\mathbb{B}_{t^*}(x_0) \subset X_0$, *then the algorithm* (5) *generates at least a sequence* (x_k) *such that for all* $k \geq 0$, x_k *remains in* $\mathbb{B}_{t^*}(x_0)$ *and converges to some* x^* *such that* $f(x^*) + g(x^*) \in K$. *Moreover, one has the following error estimate*

$$\|x^* - x_k\| \leq \frac{M}{q(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}})} q^{\frac{p^k}{\sqrt{5}}}$$

where $p = \dfrac{1 + \sqrt{5}}{2}$.

Proof. Let us observe that if for any $k \in I\!N$, we set

$$\begin{aligned} A_k = {} & f(x_{k+1}) + g(x_{k+1}) - f(x^*) - g(x^*) - [f(x_{k+1}) - f(x_k) \\ & -(f'(x_k) + [x_{k-1}, x_k; g])(x_{k+1} - x_k) + g(x_{k+1}) - g(x_k)] \end{aligned}$$

the following inclusion holds

$$A_k \in K - f(x^*) - g(x^*) \quad \text{and by continuity} \quad (A_k) \to 0.$$

Since $K - f(x^*) - g(x^*)$ is closed, we obtain $f(x^*) + g(x^*) \in K$ and then x^* solves our problem.

If we can prove the existence of a convergent sequence (t_k) which majorizes (x_k), then (x_k) becomes a Cauchy sequence and converges to some $x^* \in X_0$.

We follow the rest of the proof by induction showing that the sequence (x_k) satisfies (5), remains in some ball $\mathbb{B}_{t^*}(x_0)$ and there exists a nondecreasing sequence (t_k) satisfying

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \qquad \forall k \in \mathbb{N}. \tag{9}$$

Assumption (e) gives the existence of a point x_2 which solves (5) for $k = 1$. Moreover by setting $t_2 = \beta$, $t_1 = \alpha$ and $t_0 = 0$, we obtain $\|x_1 - x_0\| \leq \alpha = t_1 - t_0$, $\|x_2 - x_1\| \leq \beta - \alpha = t_2 - t_1$ that means that our result is checked for $k = 0$ and $k = 1$.

Let us suppose now that we have obtained $x_1, \ldots, x_k$ from the algorithm given by (5) and $t_0, t_1, \ldots, t_k$ such that

$$\|x_j - x_{j-1}\| \leq t_j - t_{j-1}, \qquad \forall j = 3, \ldots k.$$

Thus, this implies that the points $x_0, x_1, \ldots, x_k$ belong to the set $\mathbb{B}_{t^*}(x_0)$.

We also have $T(x_{j-1}, x_j)x = (f'(x_j) + [x_{j-1}, x_j; g])x - K$ which could be rewritten

$$T(x_{j-1}, x_j)x = (f'(x_1)+[x_0, x_1; g])x - K + (f'(x_j) - f'(x_1) + [x_{j-1}, x_j; g] - [x_0, x_1; g])x.$$

This implies that $T(x_{j-1}, x_j)x = (T(x_0, x_1) - \Delta_j)x$ with

$$\Delta_j(x) = -(f'(x_j) - f'(x_1) + [x_{j-1}, x_j; g] - [x_0, x_1; g])x.$$

According to assumptions (a), (d) and (f), one has

$$\|T^{-1}(x_0, x_1)\|.\|\Delta_j\| < 1.$$

In this case, the application of Theorem 2.1 allows us to obtain that $T(x_{j-1}, x_j)$ carries X onto Y, $T^{-1}(x_{j-1}, x_j)$ is normed and

$$\|T^{-1}(x_{j-1}, x_j)\| \leq \frac{\|T^{-1}(x_0, x_1)\|}{1 - \|T^{-1}(x_0, x_1)\|.\|\Delta_j\|} \leq \frac{B}{1 - BM_1}, \quad \forall j = 2, \ldots, k.$$

The fact that $T(x_{j-1}, x_j)$ carries X onto Y implies that (5) is feasible and hence solvable for $j = k$ and we obtain the existence of x_{k+1} which solves (5).

Now let us consider the problem which consists to find a point x which is a solution of the problem

$$
\begin{aligned}
(10) \qquad f(x_k) + g(x_k) + (f'(x_k) + [x_{k-1}, x_k; g])(x - x_k) &\in f(x_{k-1}) + (f'(x_{k-1}) \\
&\quad + [x_{k-2}, x_{k-1}; g])(x_k - x_{k-1}) + g(x_{k-1}) + K.
\end{aligned}
$$

We can observe that since x_k solves (5) then the right-hand side of (10) is contained in the cone K and we can conclude that any x satisfying (10) is necessarily feasible for (5).

Rewriting (10), one can obtain x as the solution of the following inclusion

$$
\begin{aligned}
(11) \qquad x - x_k \in T^{-1}(x_{k-1}, x_k)\Big(&- f(x_k) - g(x_k) + f(x_{k-1}) + g(x_{k-1}) \\
&+ (f'(x_{k-1}) + [x_{k-2}, x_{k-1}; g])(x_k - x_{k-1})\Big).
\end{aligned}
$$

Since the right-hand of (11) contains an element of least norm, we can find some $\tilde{x}$ satisfying (11) (and also (10)) which satisfies:

$$
\begin{aligned}
\|\tilde{x} - x_k\| \leq \|T^{-1}(x_{k-1}, x_k)\| \Big(&\| - f(x_k) + f(x_{k-1}) + f'(x_{k-1})(x_k - x_{k-1})\| \\
&+ \|g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1}; g](x_k - x_{k-1})\|\Big).
\end{aligned}
$$

By definition, one has

$$
\begin{aligned}
&\|g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1}; g](x_k - x_{k-1})\| \\
&\qquad = \|[x_{k-2}, x_{k-1}, x_k; g](x_k - x_{k-2})(x_k - x_{k-1})\|.
\end{aligned}
$$

Using the theorem's assumptions and setting $M = \dfrac{B}{1 - BM_1}$, we have

$$
\|\tilde{x} - x_k\| \leq M \left(\frac{L}{2} \|x_k - x_{k-1}\|^2 + K \|x_k - x_{k-2}\| \|x_k - x_{k-1}\| \right).
$$

Thanks to (9), we obtain

$$
\|\tilde{x} - x_k\| \leq M \left(\frac{L}{2} (t_k - t_{k-1})^2 + K (t_k - t_{k-1})(t_k - t_{k-2}) \right).
$$

The right-hand side of the last inequality invites us to consider the sequence defined by

$$
t_{k+1} - t_k = M \left(\frac{L}{2} (t_k - t_{k-1})^2 + K (t_k - t_{k-1})(t_k - t_{k-2}) \right),
$$

with $t_0 = 0$, $t_1 = \alpha$ and $t_2 = \beta$.

We can apply the Proposition 4.1 with the theorem's assumptions, we conclude that the sequence (t_k) is strictly increasing and converges to a number t^*.

With the help of this sequence we obtain

$$\|x_{k+1} - x_k\| \leq \|\tilde{x} - x_k\| \leq t_{k+1} - t_k.$$

In other words the sequence (x_k) is majorized by the sequence (t_k) and this completes the proof. $\square$

5. Numerical results. In this part, we consider one example in finite dimension where X is taken to be $I\!R^2$, $Y = I\!R^3$ and $K = I\!R^2_- \times \{0\}$. The numerical experiments were conducted in MAPLE Sofware with a processor "Intel® core i7".

Example. Let us consider the system:

$$\left\{ \begin{array}{l} x_1^2 + x_2^2 - |x_1 - 0.5| - 1 \leq 0 \\ x_1^2 + (x_2 - 1)^2 - |x_1 - 0.5| - 1 \leq 0, \\ (x_1 - 1)^2 + (x_2 - 1)^2 - 1 = 0 \end{array} \right. \tag{12}$$

We can remark that the point $x^* = \left(\frac{1}{2}, 1 - \frac{1}{2}\sqrt{3}\right)$ is one of the solution of the system.

For a better understanding, we specify that the method introduced in [17] consists in replacing (5)

$$\text{minimize } \{\|x - x_k\| / f(x_k) + g(x_k) + f'(x_k)(x - x_k) \in K\}. \tag{13}$$

The system (12) has been treated by the method introduced in [17], taking:

$$f(x) := \big(x_1^2 + x_2^2 - 1, x_1^2 + (x_2 - 1)^2 - 1, (x_1 - 1)^2 + (x_2 - 1)^2 - 1\big),$$

$$g(x) := \big(-|x_1 - 0.5|, -|x_1 - 0.5|, 0\big),$$

and $x_0 = (0.55, 0.1)$ for the guess point.

The results are in Table 1 where M_k is the value of the minimization problem at the step k.

Table 1. Solutions of (12) starting from
$x_0 = (0.55, 0.1)$ and using [17]

step k	$\|x_k - x^*\|_\infty$	M_k
0	5×10^{-2}	
1	2.5×10^{-2}	10^{-3}
4	3.12×10^{-3}	1.3×10^{-5}
7	1.95×10^{-4}	5.08×10^{-8}
10	4.88×10^{-5}	3.18×10^{-9}
13	6.1×10^{-6}	4.97×10^{-11}
16	15.3×10^{-6}	3.1×10^{-12}
17	7.63×10^{-7}	7.76×10^{-13}
21	4.77×10^{-8}	3.03×10^{-15}

If now we use the algorithm introduced in the present paper, considering for the first order $[x, y; g]$, the matrix whose components are

$$[x, y; g]_{i,1} = \frac{g_i(y_1, y_2) - g_i(x_1, y_2)}{y_1 - x_1}, \quad \forall i = 1, 2, 3$$

$$[x, y; g]_{i,2} = \frac{g_i(x_1, y_2) - g_i(x_1, x_2)}{y_2 - x_2}, \quad \forall i = 1, 2, 3.$$

By starting with $x_0 = (0.55, 0.1)$ and $x_1 = (0.5, 0.12)$, we find

Table 2. Solutions of (12) starting from
$x_0 = (0.55, 0.1)$, $x_1 = (0.5, 0.12)$ and using (5)

step k	$\|x_k - x^*\|_\infty$	M_k
0	5×10^{-2}	
1	2×10^{-3}	3.5×10^{-3}
4	7.10×10^{-9}	1.23×10^{-8}
7	6.11×10^{-24}	1.05×10^{-23}
10	2.12×10^{-34}	1.00×10^{-50}

By comparing both tables, we indeed notice that the method proposed in this paper is faster than the one introduced in [17].

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Alain Pietrus, Célia Jean-Alexis
Laboratoire LAMIA, EA 4540
Université des Antilles et de la Guyane
Département de Mathématiques et Informatique
Campus de Fouillole
F–97159 Pointe–à–Pitre, France
e-mail: apietrus@univ-ag.fr
cjeanale@univ-ag.fr

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