Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **39** (2013), 331–354

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE OPTIMAL CONTROL OF SOME PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS ARISING IN ECONOMICS*

R. Boucekkine, C. Camacho, G. Fabbri

Communicated by M. I. Krastanov

This paper was written to pay a tribute to Vladimir Veliov on the occasion of his 60th birthday. His friendship and brilliant scientific guidance have been a precious support to the three co-authors during the last decade.

Abstract. We review an emerging application field to parabolic partial differential equations (PDEs), that's economic growth theory. After a short presentation of concrete applications, we highlight the peculiarities of optimal control problems of parabolic PDEs with infinite time horizons. In particular, the heuristic application of the maximum principle to the latter leads to single out a serious ill-posedness problem, which is, in our view, a barrier to the use of parabolic PDEs in economic growth studies as the latter are interested in long-run asymptotic solutions, thus requiring the solution to infinite time horizon optimal control problems. Adapted dynamic programming methods are used to dig deeper into the identified ill-posedness issue.

2010 *Mathematics Subject Classification:* 91B62, 91B72, 49K20, 49L20.

Key words: Parabolic partial differential equations, optimal control, infinite dimensional problems, infinite time horizons, ill-posedness, dynamic programming.

*Giorgio Fabbri was partially supported by the *Post-Doc Research Grant* of *Unicredit & Universities*. His research has been developed in the framework of the center of excellence LABEX MME-DII (ANR-11-LABX-0023-01).The usual disclaimer applies.

1. Introduction. Many economic problems involve diffusion mechanisms: production factor mobility or technological dissemination are among these phenomena. Other phenomena studied in economics like pollution spreading or migrations can be also modeled as diffusion processes. In all cases, the use of parabolic partial differential equations (PDE) might be adequate. Nonetheless, to this date, only a very limited studies using the latter tool have been published in the economic literature. An early one is due to Issard and Liossatos [22], who consider different economic problems of diffusion across space. However, the first serious attempt to integrate such a modeling into full-fledged optimization settings only came out 25 years later thanks to Brito [9]. Since then, several authors have tried this avenue in different economic contexts, mostly in spatio-temporal frameworks: Boucekkine et al. [8], Boucekkine et al. [7] and Camacho et al. [13] elaborate on Brito's work on economic growth theory with spatial diffusion through capital mobility; Brock and Xepapadeas [10] develop an alternative version with technological diffusion without capital mobility; Camacho and Perez Barahona [12] have used similar tools to deal with pollution diffusion across space and land use dynamics, and finally Camacho [11], following Alvarez and Mossay [2], explores the economic, distributional and geographic consequences of migrations where population dynamics are driven by a parabolic PDE.

This paper reflects on the use of parabolic PDEs in economics, with a special focus on economic growth theory. Because growth theorists are principally interested in the long-term economic performances, the typical analysis performed is asymptotic, therefore requiring the study of infinite time horizon optimization problems (see for example, Barro and Sala-i-Martin, [5, chapter 2]). In short, optimal economic growth problems with parabolic PDEs to model diffusion are infinitely dimensioned (because of the PDEs) and have infinite time horizons. We will show in a simple and intuitive way that this double characteristic may cause serious methodological problems, including ill-posedness issues in a sense that will be clarified later. Indeed, an overwhelming part of the mathematical literature devoted to the control of parabolic PDEs concern finite time horizons problems: this is true for textbook expositions like in Barbu and Precupanu [4] or for specialized articles like in the recent sequence of papers by Raymond and Zidani [32, 33, 34]. To the best of our knowledge, there is no result on necessary and sufficient optimality conditions for the general class of problems we are interested in, which are typically nonlinear, non-quadratic, infinite dimensional and with an infinite time support.

The heuristic application of the maximum principle to this class of problems gives rise to a serious difficulty that neither Brito [9] nor Boucekkine et al. [7]

could solve satisfactorily. The resulting adjoint equation is a "reverse heat equation" of the backward type, and since the time horizon is infinite, there is no way to run a standard reverse timing technique (see [29] for example) to recover the more manageable framework of initial value parabolic PDEs. Integral representations of the solutions to these adjoint equations show that the use of standard (and necessary) transversality conditions is not generally enough to ensure existence or uniqueness of the solutions. We refer to this finding as ill-posedness. Resorting to dynamic programming methods well adapted to the infinite-dimensional setting (see Benssoussan et al. [6]) looks then interesting because it allows to circumvent the adjoint equations. The use of this complementary technique is indeed shown to visualize much better the contours of the ill- posedness problem identified (Boucekkine et al.[8]).

The paper is organized as follows. Section 2 presents a key example of economic problems with parabolic PDE modeling. Section 3 presents a benchmark optimal control economic problem of a parabolic PDE with finite time horizon and the typical treatment in the related economic literature. Section 4 is devoted to the case of infinite time horizon problems, and is particularly devoted to the identification of the ill-posedness issue defined above. Section 5 digs deeper in the latter issue using an adapted dynamic programming technique. Section 6 concludes.

2. A representative example: capital mobility across space, growth and inequalities. Capital mobility is one of the most crucial assumptions in economics: if capital can flow from rich to poor regions, then the latter can eventually catch up and regional inequalities will end up drastically reduced. Modelling capital mobility across space is therefore a key issue. A pioneering work on this question leading to parabolic PDEs is due to Issard and Liossatos [22]. About 30 years later, Brito [9] and Boucekkine et al. [7] have resumed research along the line of Issard and Liossatos within the framework of economic growth theory. Here comes a short description of how the parabolic PDE comes out.

Denote by $k(x,t)$ the capital stock held by a household located at x at date t. Without capital mobility, the unique way for the household to increase $k(x,t)$ is by consuming less, thus saving more and using this saving to invest more. Because we allow for capital to flow across space, $k(x,t)$ is also affected by the net flows of capital to a given location or space interval. Suppose that the technology at work in location x is simply $y(x,t) = A(x,t)f[k(x,t)]$, where $A(x,t)$ stands for total factor productivity at location x and date t, and $f(\cdot)$, only depending on capital available at (x,t), satisfies the following assumptions:

(A1) $f(\cdot)$ is non-negative, increasing and concave;

(A2) $f(\cdot)$ verifies the Inada conditions, that is,

$$f(0) = 0, \quad \lim_{k\to 0} f'(k) = +\infty, \quad \lim_{k\to +\infty} f'(k) = 0.$$

The budget constraint of household $x \in \mathbb{R}$ is

$$\frac{\partial k(x,t)}{\partial t} = A(x,t)f[k(x,t)] - \delta k(x,t) - c(x,t) - \tau(x,t), \tag{1}$$

where δ is the depreciation rate of capital, $c(x,t)$ is consumption of the household at x and t, and $\tau(x,t)$ is the household's net trade balance of household x at time t. A key step is to derive the **aggregate** resource constraint of all the households distributed across space. At this stage, a formal presentation of this aggregation step requires an explicit description of the spacial sets considered. Brito [9], Camacho et al. [13] and Boucekkine et al. [7] considered the whole real line; Issard and Liossatos [22] and Brock and Xepapadeas [10] work on bounded segments of the real line, and Boucekkine et al. [8] work on a circle. In all cases, the main argument is the same. From (1), it is easy to see for any $[a,b] \subset \mathbb{R}$, we get

$$\int_a^b \left\{ \frac{\partial k(x,t)}{\partial t} - A(x,t)f[k(x,t)] + \delta k(x,t) + c(x,t) + \tau(x,t) \right\} dx = 0. \tag{2}$$

The total net trade balanced in region $X = [a,b]$, for any a and b real numbers, is by definition $\int_a^b \tau(x,t)\, dx$. It is the symmetric of total capital account balance, which equals to capital flows received from locations lying to the left of a minus that flowing away to the right of b, that is $\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x}$. The latter applies for $a = x$ and $b = x + dx$:

$$\int_x^{x+dx} \tau(u,t)dt = -\left(\frac{\partial k(x+dx,t)}{\partial x} - \frac{\partial k(x,t)}{\partial x} \right).$$

Applying the mean value theorem for the integral term on the left side of the equality just above, one finds that it exists a real number ξ such that $x < \xi < x + dx$ and

$$\tau(\xi,t)\, dx = -\left(\frac{\partial k(x+dx,t)}{\partial x} - \frac{\partial k(x,t)}{\partial x} \right),$$

or

$$\tau(\xi, t) = -\left(\frac{\frac{\partial k(x+dx,t)}{\partial x} - \frac{\partial k(x,t)}{\partial x}}{dx} \right).$$

Now by making dx tending to 0, since then ξ tends to x, one gets

$$\tau(x, t) = -\lim_{dx \to 0} \frac{\frac{\partial k(x+dx,t)}{\partial x} - \frac{\partial k(x,t)}{\partial x}}{dx} = -\frac{\partial^2 k}{\partial x^2}.$$

Substituting the equation just above into equation (2), we have $\forall X \subset \mathbb{R}$, $\forall t$

$$\int_X \left\{ \frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} - A(x,t) f[k(x,t)] - c(x,t) - \delta k(x,t) \right\} dx = 0.$$

The budget constraint can be written as

$$\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} = A(x,t) f[k(x,t)] - \delta k(x,t) - c(x,t), \ \forall (x,t), \tag{3}$$

which is the key state equation in the parabolic PDEs economic literature. As it transpires from derivations above, the term $\dfrac{\partial^2 k(x,t)}{\partial x^2}$ in (3) comes entirely from capital mobility across space. It's therefore absent in the standard economic theory, notably economic growth theory, which ignores space, and only focuses on time, therefore relying on ordinary differential equations even when discussing issues with a strong geographic flavor like economic convergence across nations and regions (see Barro and Sala-i-Martin, [5, for standard growth theory, chapter 2]). It's not difficult to understand this omission: the inclusion of the spacial term $\dfrac{\partial^2 k(x,t)}{\partial x^2}$ renders the motion of capital dynamics infinite-dimensional, and optimization of this kind of motion is much less trivial than the standard growth theory counterpart. This is true for finite time horizon problems and even more intricate in the standard infinite time horizon optimal economic growth problems. The term $\dfrac{\partial^2 k(x,t)}{\partial x^2 (x,t)}$ is frequently identified with physical capital diffusion, since it makes capital to flow from locations with a high level of physical capital.

Remark 1. The derivation above assumes no institution barriers to capital flows, that's adjustment speed is ignored. The important aspect is considered in the related economic and geographic literatures (see for example Ten Raa [37] and Puu, [31]). Introducing these barriers may not change substantially the mathematical setting. For example, if one assumes that the barriers are independent of capital k, the parabolic PDE formalism will still apply after some affine transformations (see Issard and Liossatos [22], for this kind of treatment). However, if the barriers are functions of k, we face *nonlinear problems* not covered by the class of PDEs considered in this paper. If instead transportation costs were to be included, again the outcome depends on the way they are modeled. If transportation costs are proportional to output, then one gets the parabolic PDE above. In a more general case with space velocity, we would have to deal with a non-local problem which is out of the scope of this paper.

Remark 2. Needless to say, the PDE (3) is completed by adequate boundary conditions, depending on the time and space supports of the problem. Suppose that the time horizon is finite at the minute and focus on the spacial support. In case space is unbounded, the real line for example. In such a case, beside the initial distribution of capital, $k_0(x)$, one might need to fix how capital flows should behave at the locations which are far away from the origin. One might assume that there is no capital flow at infinitely distant locations,

$$\lim_{x\to\pm\infty} \frac{\partial k(x,t)}{\partial x} = 0,$$

meaning that there is no trade at these too distant locations. In such a configuration, one gets a Neumann problem. Alternatively, one can impose the Dirichlet condition, that is, $\lim_{x\to\pm\infty} k(x,t) = g(t)$, with $g(t)$ a given continuous function in t, which also implies that the stock of capital does not depend on trade for infinitely distant locations. In case space is bounded, two possibilities emerge. Either the space has no boundary set (case of the circle) or it does (case of an interval $[a\ b]$ of the real line). A substantial part of the economic geography literature is based on the former starting with an early influential framework developed by Salop [35]. In such a case, no space-induced boundary conditions are needed. In the case of the interval $[a\ b]$, boundary conditions on $k(a,t)$ and $k(b,t)$ (or alternatively on capital flows at the frontiers a and b) are instead needed.

3. A benchmark economic optimal control problem of a parabolic PDE. The benchmark problem proposed in this paper is linked to the economic issue exposed in Section 2. Capital is mobile across space, not individuals, and production uses capital (with time and space-independent productivity to unburden the presentation, $A(x,t) \equiv A > 0$). Capital spatio-temporal dynamics are therefore described by the parabolic PDE (3): capital stock at (x,t) depends on the saving and investment capacity of individuals established at x and on trade as well. A typical economic problem is to identify optimal saving of individuals, which amount to determine optimal consumption $c(x,t)$. A standard policymaking problem would therefore consist in searching for a control $c(x,t)$ in order to maximize the welfare or utility of all the individuals present in the space considered over a certain period of time. A benchmark problem suggested by Camacho et al. [13] is:

$$\max_{c} \int_0^T \int_{\mathbb{R}} \psi(x)\, u(c(x,t))e^{-\rho t} dt dx + \int_{\mathbb{R}} \phi\left(k(x,T),x\right) e^{-\rho T} dx, \tag{4}$$

subject to:

(5)

$$\begin{cases} \dfrac{\partial k(x,t)}{\partial t} - \dfrac{\partial^2 k(x,t)}{\partial x^2} = Af\left(k(x,t)\right) - \delta k(x,t) - c(x,t), \quad (x,t) \in \mathbb{R} \times [0,T], \\ k(x,0) = k_0(x) > 0, \quad x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} \dfrac{\partial k(x,t)}{\partial x} = 0, \quad \forall t \in [0,T]. \end{cases}$$

where $c(x,t)$ is the consumption level of an individual located at x at time t, $x \in \mathbb{R}$ and $t \in [0,T]$, $u(c(x,t))$ is a standard utility function and $\rho > 0$ stands for the time discounting rate. The second integral term in the objective function is the scrap value. While time discounting could be dropped in the problem above because the time support is finite ($T < \infty$), spatial discounting through the choice of a "rapidly decreasing" $\psi(x)$ is needed (see Boucekkine et al. [7], for examples): the convergence of the first integral term of the objective function requires such a spatial discounting given that the spacial support is here unbounded.[1] Similarly, function $\phi(\cdot)$ in the scrap value of the problem should be "rapidly decreasing" with respect to its second argument to assure the convergence of the second integral term. The initial distribution of capital, $k(x,0) \in C(\mathbb{R})$, is assumed to be a known positive bounded function, that is, $0 < k(x,0) \leq K_0 < \infty$. Moreover, we assume that, if the location is far away from the origin, there is no capital flow, yielding the typical Neumann boundary condition

$$\lim_{x \to \pm\infty} \frac{\partial k(x,t)}{\partial x} = 0. \tag{6}$$

[1] As cleverly pointed out by Brito [9], spatial discouting has the unpleasant outcome to introduce a preference relation over locations in space and tends to force rejection of an homogeneous spatial distribution as an optimal distribution. He therefore proposes an alternative objective function in terms of spatial averages of utilities of the forme $\lim\limits_{x \to +\infty} \dfrac{1}{2x} \int_{-x}^{x} \int_0^T u(c(y,t))e^{-\rho t} dt dy$. We keep here the spatial discounting formalization for simplicity; it goes without saying that the principal methodological challenges discussed later are independent of this question.

In this benchmark, space is taken to be the whole real line, and the time horizon is finite, equal to a given $T > 0$. While the nature of the space support is manageable from the optimal control point of view provided $T < \infty$, it will be shown in Section 4 that the case $T = \infty$ is problematic in a precise sense to be made explicite. A typical treatment in the economic literature is to adapt the numerous results in the related mathematical literature on the maximum principle for finite time horizon control problems of parabolic PDEs quoted in the introduction. Most of the time, the first-order conditions are derived using simple adapted methods of calculus of variations (see proof of Proposition 1 in [9], for example). In the case of benchmark problem considered here, Camacho et al. [13] have used the same elementary method to extract the Pontryagin conditions under some mild conditions (see [13, Theorem 1]): if $c \in C^{2,1}(\mathbb{R} \times [0,T])$ is an optimal control and $k \in C^{2,1}(\mathbb{R} \times [0,T])$ is its corresponding state, then there exists a function $q(x,t) \in C^{2,1}(\mathbb{R} \times [0,T])$, the adjoint variable associated to the parabolic PDE in the state $k(x,t)$ (3), such that:

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial^2 q(x,t)}{\partial x^2} + q(x,t)\left(Af'\left(k(x,t)\right) - \delta - \rho\right) = 0, \tag{7}$$

with the transversality condition

$$q(x,T) = \phi_1'\left(k(x,T), x\right), \forall x \in \mathbb{R}, \tag{8}$$

and the associated conditions dual to the Neumann conditions on capital flows at infinitely distant locations:

$$\lim_{x\to\infty} \frac{\partial q(x,t)}{\partial x} = \lim_{x\to-\infty} \frac{\partial q(x,t)}{\partial x} = 0, \quad \forall t \in [0,T].$$

The value of the co-state variable at $t = T$ depends on the scrap value function. Understanding q as the shadow price of capital, $q(x,T) = \phi_1(k(x,T),x)$ implies that the price of capital at location x equals the increase in felicity it generates. The adjoint equation (7) is also a PDE, and it's quite similar to the state equation (3), a noticeable difference is the opposite signs premultiplying the second-order spacial derivatives. The latter is referred to as the “heat equation", the former as "the reverse heat equation". Note also that the condition (8) is the usual transversality condition for finite time horizon problems with free terminal states and with a scrap value. Finally, it's worth pointing out that optimal economic growth models deliver a one-to-one relationship between $c(x,t)$ and $q(x,t)$ thanks to the first-order condition with respect to the control $c(x,t)$: $c(x,t) = (u')^{-1}\left(\frac{q(x,t)}{\psi(x)}\right)$. Therefore, solving for the co-state is solving for the control. In general, computing the optimal solutions paths require the solution of the following system of PDEs with the corresponding boundary and transversality conditions:

$$
(9) \qquad \begin{cases}
\dfrac{\partial k(x,t)}{\partial t} - \dfrac{\partial^2 k(x,t)}{\partial x^2} = A(x,t) f\left(k(x,t)\right) - \delta k(x,t) - c(x,t), \\
\hfill (x,t) \in \mathbb{R} \times [0,T], \\
\dfrac{\partial q(x,t)}{\partial t} + \dfrac{\partial^2 q(x,t)}{\partial x^2} = -q(x,t)\left(A f'\left(k(x,t)\right) - \delta - \rho\right), \\
\hfill (x,t) \in \mathbb{R} \times [0,T], \\
k(x,0) = k_0(x) > 0, \ \ \forall x \in \mathbb{R}, \\
q(x,T) = \phi_1'\left(k(x,T), x\right), \forall x \in \mathbb{R}, \\
\lim\limits_{x \to \pm\infty} \dfrac{\partial k(x,t)}{\partial x} = 0, \forall t \in [0,T], \\
\lim\limits_{x \to \pm\infty} \dfrac{\partial q(x,t)}{\partial x} = 0, \forall t \in [0,T].
\end{cases}
$$

Notice that when function $f(\cdot)$ is linear, the adjoint equation becomes independent of the state variable. This peculiarity will be exploited to explore the ill-posedness issue described below. Beside the existence of optimal solutions and the issue of sufficiency of the Pontryagin conditions, globally settled by the mathematical literature for finite time horizons problems (see for example, Barbu and Precupanu [4], for a textbook overview), the issue of computability of the solutions to systems like system (9) is worth mentioning. Since the time horizon T is finite, one can invert time in (7) to obtain a system of parabolic partial differential equations with spatial boundary conditions where the initial level of capital and the time-inverted co-state variable are known. Indeed, calling $h(x,t) = q(x, T-t)$, one obtains:

$$
(10) \qquad \begin{cases}
\dfrac{\partial h(x,t)}{\partial t} - \dfrac{\partial^2 h(x,t)}{\partial x^2} = h(x,t)\left(A f'\left(k(x,T-t)\right) - \delta - \rho\right), \\
\hfill (x,t) \in \mathbb{R} \times [0,T], \\
h(x,0) = \phi_1'\left(k(x,T), x\right), \forall x \in \mathbb{R}, \\
\lim\limits_{x \to \pm\infty} \dfrac{\partial h(x,t)}{\partial x} = 0, \forall t \in [0,T].
\end{cases}
$$

Unfortunately, inverting time makes nonsense when the time horizon is infinite. In fact, this failure is also a signal of the ill-posedness problem detailed just below.

4. The ill-posedness problem when time horizons are infinite. The ill-posedness problem is described in [7, 8] provide with a characterization of the latter problem using dynamic programming as explained in the next section. As mentioned in the introduction, and to the best of our knowledge, there is no general result in the mathematical literature on necessary and sufficient optimality conditions for infinite time horizons problems. Brito [9] and Boucekkine et al. [7] propose heuristic derivations of these conditions using simple calculus of variations techniques. A key point is the specification of the “limit" transversality condition. Intuitively, as T increases, the scrap value should decrease to zero, yielding the standard transversality condition in economics:[2]

$$\lim_{T\to\infty} q(x,T) = 0, \forall x \in \mathbb{R} \tag{11}$$

This is the key piece in this heuristic approach, the rest of the first-order conditions are straightforwardly derived from the finite time horizon case, yielding the following system when $T=\infty$

$$\begin{cases} \dfrac{\partial k(x,t)}{\partial t} - \dfrac{\partial^2 k(x,t)}{\partial x^2} = A(x,t) f\left(k(x,t)\right) - \delta k(x,t) - c(x,t), \\ \hfill (x,t) \in \mathbb{R}\times[0,\infty[, \\ \dfrac{\partial q(x,t)}{\partial t} + \dfrac{\partial^2 q(x,t)}{\partial x^2} = -q(x,t)\left(Af'\left(k(x,t)\right) - \delta - \rho\right), \\ \hfill (x,t) \in \mathbb{R}\times[0,\infty[, \\ k(x,0) = k_0(x) > 0, \quad \forall x\in\mathbb{R}, \\ \lim\limits_{T\to\infty} q(x,T) = 0, \forall x\in\mathbb{R}, \\ \lim\limits_{x\to\pm\infty} \dfrac{\partial k(x,t)}{\partial x} = 0, \forall t\in[0,\infty[, \\ \lim\limits_{x\to\pm\infty} \dfrac{\partial q(x,t)}{\partial x} = 0, \forall t\in[0,\infty[. \end{cases} \tag{12}$$

Our ill-posedness problem arises when trying to solve the system just above. Once the observation that the time-inversion method does not apply to the infinite time horizon counterpart of the benchmark problem is made, one can hope to visualize better the situation by looking at integral forms of the PDEs. Explicit integral representations of the solutions to parabolic PDEs are quite known (see

[2]It goes without saying that this transversality condition, though highly intuitive, is not always necessary, even in the non-spatial case. Regularity conditions are needed to this end, see Michel [28].

for example, [29] for a textbook presentation, and [35, 36] for some refinements). For a general parabolic PDE in variable $u(x,t)$:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = G[u(x,t), z(x,t)], \tag{13}$$

where $G(\cdot)$ is any given continuous function, and $z(x,t)$ a forcing variable, with initial continuous function $u(x,0) = u_0(x)$ given, and under some conditions, (13) has a unique solution $u \in C^{2,1}[\mathbb{R} \times (0,T)]$, given by

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \Gamma(x-y,t) u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-\tau) \left\{ G[u(y,\tau), z(y,\tau)] \right\} dy d\tau, \end{aligned} \tag{14}$$

where

$$\Gamma(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}, \ t > 0, \\ 0, \ t < 0. \end{cases}$$

For a backward parabolic equation with terminal condition like the adjoint equation in $q(x,t)$ (7), one can deduce easily the corresponding integral representation by using the time inversion method seen above. Suppose we have to deal with the PDE:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} + \dfrac{\partial^2 w(x,t)}{\partial x^2} = H[w(x,t), z(x,t)], \quad x \in \mathbb{R}, \quad t \in [0,T), \\ w(x,T) = w_1(x), \text{given}, \quad x \in \mathbb{R}, \end{cases}$$

then using the variable change $v(x,t) = w(x, T-t)$ to recover the forward PDE with initial value, one gets after ultimately reverting the variable change:

$$\begin{aligned} w(x,t) &= \int_{\mathbb{R}} \Gamma(x-y, T-t) \phi(y) dy \\ &- \int_t^T \int_{\mathbb{R}} \Gamma(x-y, T-\tau) H[w(y, T+t-\tau), z(y, T+t-\tau)] dy d\tau. \end{aligned}$$

Applied to our adjoint equation (7), one gets:

$$\begin{aligned} q(x,t) = &\int_{\mathbb{R}} \Gamma(x-y, T-t) q_0(y) dy \\ &- \int_t^T \int_{\mathbb{R}} \Gamma(x-y, T-\tau) q(y, T+t-\tau) \left\{ A f'[k(y, T+t-\tau)] - \delta \right\} dy d\tau, \end{aligned} \tag{15}$$

with $q_0(y) = \phi_1'(k(y,T), y)$. It's easy to understand intuitively why the integral representation of a the backward PDE may be problematic when T becomes infinite. Now focus on the integral representation of the backward PDE (15). For fixed T, it is the sum of two integrals, one is related to the transversality condition via function $q_0(\cdot)$ and the second simply states that the adjoint variable $q(x,t)$ is a weighted average (across space and time) of the realization of the same variable from t to T for given capital spatio-temporal profiles. If function $f(\cdot)$ is linear, then (15) only involves the adjoint variable.

When T goes to infinity, the first term should vanish (because the scrap value should in principle vanish) but the second term remains essentially the same, that is the weighted average of all the realizations of the adjoint variable from t onwards, here from t to infinity. In other terms, the transversality condition (11) does not make this second term more explicit, it rather makes the backwardness of the PDE in $q(x,t)$ more apparent. In the standard non-spatial optimal growth model in economics (see [5, chapter 2]), transversality conditions do allow to directly identify the (unique) optimal solutions. With space, such a nice picture does not show up, and there is no reason to believe that when T goes to infinity, the integral equations involved are free of existence or multiplicity problems. Boucekkine et al. [7] call these potential problems "ill-posedness". It is important to have in mind two things in this respect. First, ill-posedness regards the existence or uniqueness problems inherent in the system (12), the set of heuristically extracted first-order conditions, and does not necessarily regard the original optimal control problem. Second, it is essential to realize that the potential ill-posedness problem comes from the adjoint equation, that is from the backward PDE which cannot be "time-inverted" when the time horizon is infinite, not from the state equation in $k(x,t)$ which integral representation (14) is not problematic at all.

5. Investigating ill-posedness using a dynamic programming approach. Boucekkine et al. [8] have provided with the first analysis of the nature and consequences of this potential ill-posedness problem within the class of models considered here. Precisely, the authors use a dynamic programming method well adapted to the infinite-dimensional characteristic of the problem under scrutiny. A fundamental property of the latter methods is to avoid the direct use of adjoint equations, which are precisely the source of the problem. Before getting to the analysis of the ill-posedness problem, we describe the basic ingredients of dynamic programming in Hilbert spaces. We shall do this in a benchmark textbook case with a finite time horizon and on abounded spatial set

$\Omega \subset \mathbb{R}^n$ with non-empty boundary. The adaptation of this dynamic programming to infinite time horizons and to other spatial settings (unbounded or bounded without boundary in $\mathbb{R}^n$) is quite manageable, the first step being systematically rewriting the problem in $L^2(\Omega)$. In the infinite time horizon case, the dynamic programming approach allows to avoid the two disturbing ingredients mentioned above, the adjoint backward PDE and the transversality condition. So the steps taken in the infinite time horizon case can be comfortably replicated for infinite time horizons problems (see Section 5.1.2 for explicite arguments). Indeed, the method is conclusive in the infinite time horizon case studied by Boucekkine et al. [8] as detailed in Section 5.2.

5.1. Dynamic programming in Hilbert spaces.

5.1.1. Rewriting the problem in $L^2(\Omega)$. First of all we need to recall some introductory facts and notions. The first notion is the defintion of "Neumann maps". Given a bounded open set $\Omega \subset \mathbb{R}^n$ with the same properties used in the previous section, and called Γ its boundary, we look at the following problem:

$$\begin{cases} \Delta_x y(x) = \lambda y(x), & x \in \Omega \\ \dfrac{\partial}{\partial n} y(x) = g(x), & x \in \partial\Omega \end{cases} \tag{16}$$

(where $\dfrac{\partial}{\partial n}$ is the normal derivative). It is a standard result (see e.g. [26]) that, for $\lambda \geq 0$ big enough, given a certain $s \geq 0$ and some boundary condition $g \in H^s(\partial\Omega)$ (where $H^s(\partial\Omega)$ denotes the Sobolev space of index s, see e.g. [1]) there exists a unique solution $N(g) \in H^{s+3/2}(\Omega)$ of (16). The operator N is continuous and is called "Neumann map".

Observe now what happens in the homogeneous case (i.e. when the (Neumann) boundary condition is always null). Consider some $y_0 \colon \Omega \to \mathbb{R}$ belonging to $L^2(\Omega)$ and look at the following heat equation on Ω:

$$\begin{cases} \dfrac{\partial}{\partial s} y(s,x) = \Delta_x y(s,x), & (s,x) \in (0,T) \times \Omega \\ \dfrac{\partial}{\partial n} y(s,x) = 0, & (s,x) \in (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & x \in \Omega \end{cases} \tag{17}$$

For $s \geq 0$ denote by $S(s)x$ the solution of (17) at time s. It can be proved (see e.g. [30]) that $S(\cdot)$ is a (analytic) C_0-semigroup on $L^2(\Omega)$ (for the notion of C_0-semigroup and for the related concepts the reader is referred e.g. to Bensoussan et al., 2007, or again to Pazy, 1983) whose generator (in the sense of C_0-semigroups) is defined as follows

$$\begin{cases} D(A) := \left\{ u \in H^2(\Omega) \ : \ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \right\} \\ A(\phi) := \Delta\phi. \end{cases} \tag{18}$$

If we endow, as usual the domain $D(A)$ with the graph topology (see [7]) and we define the fractional powers of A, and the related norms, as in [30], it can be shown that the Neumann map N is continuous when defined on $L^2(\partial\Omega)$ with values in $((-A+\lambda)^{3/4-\epsilon})$.

We denote with A^* (with domain $D(A^*)$) the adjoint of A.

Let us consider now the following heat equation

$$\begin{cases} \dfrac{\partial}{\partial s} y(s,x) = \Delta_x y(s,x) + b(s,x), & (s,x) \in (0,T) \times \Omega \\ \dfrac{\partial}{\partial n} y(s,x) = g(s,x), & (s,x) \in (0,T) \times \partial\Omega \\ y(0,x) = y_0(x). \end{cases} \tag{19}$$

It can be shown (see e.g. [6]) that the solution of this equation can be rewritten, if f and g are regular enough, as follows:

$$Y(s) = S(s)y_0 - (A-\lambda) \int_0^s S(s-r)N(g(r))dr + \int_0^s S(s-r)b(r)dr \tag{20}$$

that (see again [6], it is a "variation of parameters" argument) can be seen as the integral (or "mild") solution of the following evolution equation in $L^2(\Omega)$:

$$\begin{cases} \dfrac{d}{ds} Y(s) = AY(s) + (\lambda - A)Ng(s) + b(s) \\ Y(0) = y_0. \end{cases} \tag{21}$$

5.1.2. The HJB equation. Similarly, whenever the original parabolic (state) equation is driven by a control $u(t)$ belonging to some set of admissible controls $\mathcal{U}$ with values in some Polish space U and the function b depends also on the state y, we have the following state equation:

$$\begin{cases} \dfrac{d}{ds} Y(s) = AY(s) + (\lambda - A)Ng(s, Y(s)) + b(s, Y(s), u(s)) \\ Y(t) = y_0. \end{cases} \tag{22}$$

where we have considered a generic initial time t. Typically the set of optimal controls is given by $\mathcal{U} := L^1(t, T; U)$. Moreover one need to assume conditions on b and g in order to be able to prove existence and uniqueness of the solution of such a state equation (see e.g. [6]).

So far we have recalled how a (state) parabolic equation can be rewritten as an evolution equation in the Hilbert space $L^2(\Omega)$. Now we can see how to use the general theory of optimal control problem for infinite dimensional systems in this specific case.

First of all we introduce some functional to be maximized. Consider for example the following

$$J(y_0; u) = \int_t^T l(s, u(s)), Y(s))ds + \phi(Y(T)). \tag{23}$$

At this point, using (22) and (23) we can obtain, formally, the Hamilton-Jacobi-Bellman (HJB) equation of the system (see e.g. [25]):

$$\begin{cases} v_t + \langle AY + (\lambda - A)Ng(s, Y), Dv\rangle + \\ \qquad\qquad + \sup_{u \in U} \{\langle b(s, Y, u), Dv\rangle + l(s, Y, u)\} = 0, \\ \\ v(T, Y) = \phi(Y), \end{cases} \tag{24}$$

where we denoted with $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega)$.

We can introduce the notion of "current value Hamiltonian" of the system. It is defined as follows.

$$H_{CV}(s, Y, p, u) = \langle b(s, Y, u), p\rangle + l(s, Y, u) \tag{25}$$

We can use it to define the "maximal value Hamiltonian" (or simply Hamiltonian):

$$H(s, Y, , p) = \inf_{u \in U} (\langle b(s, Y, u), p\rangle + l(s, T, u)) \tag{26}$$

so we can rewrite the HJB as follows

$$\begin{cases} v_t + \langle AY + (\lambda - A)Ng(s, Y), Dv\rangle + H(s, Y, Dv) = 0, \\ \\ v(T, Y) = \phi(Y), \end{cases} \tag{27}$$

Similarly we can consider the case in which the functional to be maximized has infinite horizon as the following case:

$$J(y_0; u) = \int_0^{+\infty} e^{-\rho s} l(u(s)), Y(s))ds. \tag{28}$$

In this case we consider as set of admissible controls the following space.

$$\mathcal{U} = \{u\colon [0,+\infty) \to U \ : \ u \quad \text{locally integrable}\} \tag{29}$$

In this case, if the state equation (22) is homogeneous (i.e. g and b do not depend directly on s) the HJB becomes the following:

$$\rho v = \langle AY + (\lambda - A)Ng(Y), Dv\rangle + \sup_{u\in U} \{\langle b(Y,u), Dv\rangle + l(Y,u)\}\,, \tag{30}$$

Similarly to what we have done before we can introduce the notion of "current value Hamiltonian".

$$H_{CV}(Y,p,u) = \langle b(Y,u), p\rangle + l(Y,u) \tag{31}$$

and that of Hamiltonian:

$$H(Y,p) = \inf_{u\in U} \left(\langle b(Y,u), p\rangle + l(T,u)\right) \tag{32}$$

and rewrite the HJB as follows

$$\rho v = \langle AY + (\lambda - A)Ng(Y), Dv\rangle + H(s,Y,Dv). \tag{33}$$

5.1.3. Solution of the HJB equation and solution of the optimal control problem in the regular. As in the finite-dimensional case (see e.g. [20]), studying the solution of the HJB equation provides information on the associated optimal control problem. We describe the exact situation in the most regular case.

Definition 1. *We will say that* $w\colon [0,T] \times L^2(\Omega) \to \mathbb{R}$ *is a strict solution of the HJB equation* (24) *if it is in* $C^1([t,T] \times L^2(\Omega))$, $Dw \in C([0,T] \times L^2(\Omega); D(A^*))$ *and*

$$\begin{cases} \partial_t w(s,Y) + \langle Y, A^* Dw(Y)\rangle + \langle Ng(s,Y), (\lambda - A)^* Dw(Y)\rangle + \\ \qquad\qquad\qquad\qquad + H(s,Y,Dw(s,Y)) = 0 \qquad \text{in } [0,T] \times L^2(\Omega) \\ w(T,Y) = \phi(Y) \end{cases}$$

We define the value function of the problem (22)–(23) as follows.

$$V(t,y_0) = \inf_{u(\cdot)\in\mathcal{U}} \left(J(t,y_0,u(\cdot))\right). \tag{34}$$

The two following results are proved e.g. in [25].

Proposition 1. *Assume that the value function* $V: [t,T] \times L^2(\Omega) \to \mathbb{R}$ *is in* $C^1([t,T] \times L^2(\Omega))$ *and that* $V \in C([0,T] \times L^2(\Omega); D(A^*))$. *Then it is a strict solution of the HJB equation.*

Proposition 2. *If* $v \in C^1([0,T] \times L^2(\Omega))$ *is a strict solution of the HJB equation then* $v(t,Y) \geq V(t,Y)$ *for every* $(t,Y) \in [0,T] \times L^2(\Omega)$. *Moreover if we have an admissible pair* $(\bar{Y}(\cdot), \bar{u}(\cdot))$ *such that*

$$\bar{u}(s) \in \operatorname*{argmax}_{u \in U} H_{CV}(s, \bar{Y}(s), Dv(s, \bar{Y}(s)), u) \qquad a.e.\ in\ [t,T] \tag{35}$$

Then the couple $(\bar{Y}(\cdot), \bar{u}(\cdot))$ *is optimal at* (t,Y).

This second proposition shows that, when we can find an explicit solution of the HJB equation, it can be used to solve the optimal control problem in feedback form.

Whenever we cannot find a regular (strict) solution of the HJB equation we need to introduce weaker notion of solution. We can find in the literature several different possible generalization of solution. In any of these possibilities there are two features that are still there:

– The valued function of the optimal control problem is a solution of the HJB equation. And, under mild assumptions, it is the unique solution

– The solution of the HJB equation can be use to give an optimal feedback solution to the optimal control problem.

The following are examples of generalizations of solution for HJB related with optimal control problems driven by parabolic PDE:

(i) The strong solution approach. In this case the solution is defined as the limit of families of more regular (approximating) problems. It has been introduced by Barbu and Da Prato (see e.g. [3]) and developed in several ways, see e.g. Cannarsa and Di Blasio [15],Gozzi [21] and, for the linear quadratic case (even for the boundary control case) to Lasiecka and Triggiani [24] and to Bensoussan et al. [6]. All these works apply to the case of an HJB related to an optimal control problem driven by a parabolic PDE.

(ii) The viscosity solution approach method. Here the solution is defined using test functions that "touch" the solution from above and from below. Classes of optimal control problems driven by parabolic equations can be treated using the material contained for example in [17, 18, 23, 36].In the boundary control case (parabolic systems) we can quote [14, 16].

5.2. Characterizing ill-posedness with the dynamic programming approach. We now use the previous approach to analyse the ill-posedness problem described in Section 4. Boucekkine et al. [8] consider an infinite time horizon problem similar to our benchmark. In order to make the argument transparent, they assume a linear production function $f(\cdot)$ and they replace the real line by the circle. Both choices have been made to have an explicit solution to the resulting HJB equation.[3] The complete adapted dynamic programming strategy is detailed in [8, Appendix]. Let's sketch briefly the version of the capital mobility problem considered. Individuals are distributed homogeneously along the unit circle in the plane, denoted by $\mathbb{T}$. Using polar coordinates $\mathbb{T}$ can be described as the set of spatial parameters θ in $[0, 2\pi]$ with $\theta = 0$ and $\theta = 2\pi$ being identified. Capital is mobile along the circle $\mathbb{T}$, and the spatio-temporal capital dynamics are shown by the authors to follow the same type of parabolic PDE as in the benchmark case detailed in Section 2:

$$
(36) \qquad \begin{cases} \dfrac{\partial k}{\partial t}(t,\theta) = \dfrac{\partial^2 k(t,\theta)}{\partial \theta^2} + Ak(t,\theta) - c(t,\theta), \quad \forall t \geq 0, \quad \forall \theta \in \mathbb{T} \\ k(t,0) = k(t,2\pi), \quad \forall t \geq 0 \\ k(0,\theta) = k_0(\theta), \quad \forall \theta \in [0,2\pi]. \end{cases}
$$

Provided an initial distribution of physical capital $k_0(\cdot)$ on $\mathbb{T}$, the policy maker has to choose a control $c(\cdot,\cdot)$ to maximize the following functional

$$
(37) \qquad J(k_0, c(\cdot,\cdot)) := \int_0^{+\infty} e^{-\rho t} \int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} d\theta dt
$$

The value function of our problem starting from k_0 is defined as

$$
(38) \qquad V(k_0) := \sup_{c(\cdot,\cdot)} J(k_0, c(\cdot,\cdot)).
$$

where the supremum is taken over the controls that ensure the capital to remain non-negative at every time and at every point of the space.The method employed involves regular enough functions $k(\cdot,\cdot)$, $c(\cdot,\cdot)$, so that for any time $t \in [0, +\infty)$ the functions $k(t,\cdot)$, $c(t,\cdot)$ of the space variable can be considered as elements of the Hilbert space $L^2(\mathbb{T})$. $L^2(\mathbb{T})$ is the set of the functions $f \colon \mathbb{T} \to \mathbb{R}$ s.t.

[3] As explained by the authors, see Remark B.2, working with alternative popular manifolds like the real line or segments of the real line would make the computation of explicit solutions to the HJB equations much less comfortable though not impossible. Considering a manifold like the circle which has no boundary sets is therefore made for simplicity.

$\int_0^{2\pi} |f(\theta)|^2 d\theta < +\infty$. This simplifying feature allows to apply dynamic programming techniques in $L^2(\mathbb{T})$ exactly along the lines of Section 5.1 (again see the detailed Appendix in [8]). Thanks to the linear production function and the choice of the circle, it is possible to solve explicitly the HJB equation.

Theorem 1. *Suppose that*

$$A(1-\sigma) < \rho \tag{39}$$

and consider $k_0 \in L^2(\mathbb{T})$, a positive initial distribution of physical capital. Define

$$\eta := \frac{\rho - A(1-\sigma)}{2\pi\sigma}. \tag{40}$$

Provided that the trajectory $k^*(t,\theta)$, driven by the feedback control (constant in θ)

$$c^*(t,\theta) = \eta \int_0^{2\pi} k^*(t,\varphi) d\varphi \tag{41}$$

remains positive, $c^*(t,\theta)$ is the unique optimal control of the problem. Moreover the value function of the problem is finite and can be written explicitly as

$$V(k_0) = \alpha \left(\int_0^{2\pi} k_0(\theta) d\theta \right)^{1-\sigma} \tag{42}$$

where

$$\alpha = \frac{1}{1-\sigma} \left(\frac{\rho - A(1-\sigma)}{2\pi\sigma} \right)^{-\sigma}. \tag{43}$$

This statement above is Theorem 3.1 in [8], the proof is given in this paper. As one can see, the problem is well-behaved from the point of view of dynamic programming as one can identify a simple unique solution to HJB, and a simple and unique optimal control in feedback form. Of course, using dynamic programming allows to circumvent the problematic adjoint equations as mentioned above. Still one could be surprised that a potentially ill-posed problem taking the avenue of the maximum principle (albeit heuristically applied) could be so tractable through dynamic programming. Indeed, using the maximum principle as in [7], the resulting set of first-order necessary conditions are (with $q(t,\theta)$ the adjoint variable): (i) the state equation (36), (ii) the maximum condition $q(t,\theta) = e^{-\rho t} c(t,\theta)^{-\sigma}$, (iii) the adjoint equation $\frac{\partial q(t,\theta)}{\partial t} = -\frac{\partial^2 q(t,\theta)}{\partial \theta^2} - Aq(t,\ theta)$ and (iv) the transversality condition $\lim_{t\to\infty} q(t,\theta) = 0$ for all $\theta \in [0, 2\pi]$. Still we are in the potential ill-posedness case described in Section 4 because of the backward adjoint equation. We can go a step further in the analysis of this potential ill-posedness. The adjoint variable $q(t,\theta)$ is connected to the value function, V, as follows: $q(t,\theta) = e^{-\rho t}\nabla V(k(t))(\theta)$, where ∇V is the Gâteaux derivative of V. One can study the dynamics of q since the value function is fully identified. We have

$$q(t,\theta) = e^{-\rho t}\nabla V(k(t))(\theta) = e^{-\rho t}\alpha(1-\sigma)\langle k(t), \mathbb{1}\rangle^{-\sigma}\mathbb{1}(\theta) = \\ = \left(\frac{\rho - A(1-\sigma)}{2\pi\sigma}K(0)\right)^{-\sigma}\mathbb{1}(\theta)e^{-At}.$$

One can directly see that such a q satisfy the adjoint equation (iii): indeed $\frac{\partial^2 q(t,\theta)}{\partial \theta^2} = 0$ since q is constant in θ and

$$\frac{\partial q(t,\theta)}{\partial t} = -A\left(\frac{\rho - A(1-\sigma)}{2\pi\sigma}K(0)\right)^{-\sigma}\mathbb{1}(\theta)e^{-At} = -Aq(t,\theta).$$

Clearly this q also satisfies the transversality condition (iv). In other words, the first-order conditions derived above, including the transversality conditions, are necessary, which is good. However, one can use the explicit solutions to notice that the the adjoint variable (iii), together with the transversality condition (iv), admits more than one solution, for example all the functions of the form $c\mathbb{1}(\theta)e^{-At}$ for some real constant c satisfy both. So the first-order conditions found above are only necessary and not sufficient to determine the optimum. This finding of Boucekkine et al. [8]clarifies, among other things, the sources of the ill-posedness described in Section 4. It turns out that contrary to the finite horizon case where under standard regularity conditions the first-order conditions are necessary and sufficient, they are only necessary in the infinite time horizon case. Of course, this is a partial finding based on a specific problem but we believe that it brings out a significant insight into the tricky ill-posedness issue identified.

6. Concluding remarks. We have shed light on an emerging application field to parabolic PDEs, that's economic growth theory. After a brief survey of concrete application, we have outlined a serious ill-posedness problem, which is, in our view, a barrier to the use of parabolic PDEs in economic growth studies. The latter are interested in long-run asymptotic solutions, which requires the solution to infinite time horizon optimal control problems. To the best of our knowledge, there is no general result on necessary and sufficient conditions for the maximum principle in the infinite time horizon case for infinite dimensional optimization problems. In particular, none of the few existing partial results apply to general economic growth theory frames, they focus instead on quadratic, or similar, functionals, see [19] for example. We believe that this is an important open question that should be of interest to the mathematicians working in the area.

REFERENCES

[1] R. A. Adams, J. F. F. Fournier. Sobolev spaces. Pure and Applied Mathematics Series, vol. **140**, Amsterdam, Elsevier/Academic Press, 2003.

[2] F. J. Alvarez, P. Mossay. Estimation of a continuous spatio-temporal population model. *Journal of Geographical Systems.* **8** (2006), 307–316.

[3] V. Barbu, G. Da Prato. Hamilton-Jacobi equations in Hilbert spaces. Research Notes in Mathematics, vol. **86**, Boston, MA, Pitman (Advanced Publishing Program), 1983.

[4] V. Barbu, Th. Precupanu. Convexity and Optimization in Banach Spaces. Springer Monographs in Mathematics, 2013.

[5] R. J. Barro, X. Sala-i-Martin. Economic Growth. MIT Press, 1995.

[6] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter. Representation and control of infinite-dimensional systems. second ed., Systems and Control: Foundations and Applications, Boston, MA,, Birkhauser Boston Inc. 2007.

[7] R. Boucekkine, C. Camacho, B. Zou. Bridging the gap between growth theory and the new economic geography: The spatial Ramsey model. *Macroeconomic Dynamics.* **13** (2009), 20–45.

[8] R. Boucekkine, C. Camacho, G. Fabbri. Spatial dynamics and convergence: The spatial AK model. *Journal of Economic Theory* **148** (2013), 2719–2736.

[9] P. Brito. The dynamics of growth and distribution in a spatially heterogenous world. WP 13/2004, ISEG Working Papers, 2004.

[10] W. Brock, A. Xepapadeas. Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. *Journal of Economic Dynamics and Control.* **32** (2008), 2745–2787.

[11] C. Camacho. Spatial Migration. Working paper 2013.17, C. E. Sorbonne, 2013.

[12] C. Camacho, A. Perez-Barahona. Land use dynamics and the environment. Working paper 2012.12, C. E. Sorbonne, 2012.

[13] C. Camacho, B. Zou, M. Briani. On the dynamics of capital accumulation across space. *European Journal of Operational Research.* **186** (2008), 451–465.

[14] P. Cannarsa, F. Gozzi, H. M. Soner. A dynamic programming approach to nonlinear boundary control problems of parabolic type. *J. Funct. Anal.* **117** (1993), 25–61.

[15] P. Cannarsa, G. Di Blasio. A direct approach to infinite-dimensional Hamilton-Jacobi equations and applications to convex control with state constraints. *Differential Integral Equations* **8** (1995), 225–246.

[16] P. Cannarsa, M. E. Tessitore. Infinite-dimensional Hamilton-Jacobi equations and Dirichlet boundary control problems of parabolic type. *SIAM J. Control Optim.* **34** (1996), 1831–1847.

[17] M. G. Crandall, P. L. Lions. Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. IV. Hamiltonians with unbounded linear terms. *J. Funct. Anal.* **90** (1990), 237–283.

[18] M. G. Crandall, P. L. Lions. Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. V. Unbounded linear terms and B-continuous solutions. *J. Funct. Anal.* **97** (1991), 417–465.

[19] S. Faggian. Maximum principle for boundary control problems arising in optimal investment with vintage capital. Working paper **181**, Department of Applied Mathematics, University of Venezia, 2008.

[20] W. H. Fleming, R. W. Rishel. Deterministic and stochastic optimal control. Applications of Mathematics Vol. **1**, Berlin-Heidelberg-New York, Springer-Verlag, 1975.

[21] F. Gozzi. Some results for an optimal control problem with semilinear state equation. *SIAM J. Control Optim.* **29** (1991), 751–768.

[22] W. Isard, P. Liossatos. Spatial dynamics and optimal space-time development. 1979, North Holland.

[23] H. Ishii. Viscosity solutions for a class of Hamilton-Jacobi equations in Hilbert spaces. *J. Funct. Anal.* **105** (1992), 301–341.

[24] I. Lasiecka, R. Triggiani. Control theory for partial differen- tial equations: continuous and approximation theories. Encyclopedia of Mathematics and its Applications, vol. **74**, Cambridge, Cambridge University Press, 2000.

[25] X. J. Li, J.M. Yong. Optimal control theory for infinite-dimensional systems, Systems and Control: Foundations and Applications. Boston, Birkhauser, 1995.

[26] J. L. Lions, E. Magenes. Non-homogeneous boundary value problems and applications. vol. **1**, Berlin, Springer 1972.

[27] J. L. Lions, E. Magenes. Non-homogeneous boundary value problems and applications. vol. **2**, Berlin, Springer 1973.

[28] P. Michel. On the transversality condition in infinite horizon optimal problems. *Econometrica.* **50** (1982), 975–985.

[29] C. Pao. Nonlinear Parabolic and Elliptic Equations. New York, Plenum Press, 1992.

[30] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, vol. **44**, New York, Springer-Verlag, 1983

[31] T. Puu. On growth and dispersal of populations. *The Annals of Regional Science* **23** (1989), 171–186.

[32] J. Raymond, H. Zidani. Pontryagin's principle for state-constrained control prob- lems governed by parabolic equations with unbounded controls. *SIAM Journal on Control and Optimization.* **36** (1998), 1853–1879.

[33] J. Raymond, H. Zidani. Hamiltonian Pontryagin's Principles for Control Problems Governed by Semilinear Parabolic Equations. *Applied Mathematics and Optimization* **39** (1999), 143–177.

[34] J. Raymond, H. Zidani. Time optimal problems with Boundary controls. *Differential and Integral Equations* **13** (2000), 1039–1072.

[35] S. Salop. Monopolistic competition with outside goods. *Bell Journal of Economics* **10** (1979), 141–156.

[36] D. Tataru. Viscosity solutions for Hamilton-Jacobi equations with unbounded nonlinear term: a simplified approach. *J. Differential Equations* **111** (1994), 123–146.

[37] T. Ten Raa. Dynamic Input-Output Analysis with Distributed Activities. *The Review of Economics and Statistics.* **68** (1986), 300–310.

[38] G. C. Wen, B. Zou. Initial-irregular oblique derivative problems for nonlinear parabolic complex equations of second order with measurable coefficients. *Nonlinear Analysis TMA* **39** (2000), 937–953.

[39] G. C. Wen, B. Zou. Initial-boundary value problems for nonlinear parabolic equations in higher dimensional domains, Beijing/New York, Science Press, 2002.

R. Boucekkine
Aix-Marseille University
(Aix-Marseille School of Economics)
EHESS and CNRS, France
e-mail: raouf.boucekkine@univ-amu.fr

C. Camacho
CNRS
Université Paris I, France
e-mail: maria.camacho-perez@univ-paris1.fr

G. Fabbri
EPEE, Université d'Evry-Val-d'Essonne
(TEPP, FR-CNRS 3126)
Département d'Economie, France
e-mail: giorgio.fabbri@univ-evry.fr

Received May 22, 2013