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ON CLUSTER POINTS OF ALTERNATING PROJECTIONS

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*Dedicated to Asen Dontchev on the occasion of his 65th birthday
and to Vladimir Veliov on the occasion of his 60th birthday*

ABSTRACT. Suppose that A and B are closed subsets of a Euclidean space such that $A \cap B \neq \emptyset$, and we aim to find a point in this intersection with the help of the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ generated by the *method of alternating projections*. It is well known that if A and B are convex, then $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ converge to some point in $A \cap B$. The situation in the nonconvex case is much more delicate. In 1990, Combettes and Trussell presented a dichotomy result that guarantees either convergence to a point in the intersection or a nondegenerate compact continuum as the set of cluster points.

In this note, we construct two sets in the Euclidean plane illustrating the continuum case. The sets A and B can be chosen as countably infinite unions of closed convex sets. In contrast, we also show that such behaviour is impossible for finite unions.

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1. Motivation. Let X be a real Euclidean space, and let A and B be closed subsets of X . Our aim is to find a point in $A \cap B$ which we assume to be nonempty. One classical algorithm is the *method of alternating projections*: Given a starting point $b_{-1} \in X$, generate sequences

$$(1) \quad (\forall n \in \mathbb{N}) \quad a_n \in P_A(b_{n-1}) \quad \text{and} \quad b_n \in P_B(a_n)$$

where $P_C x := \{c \in C \mid \|x - c\| = d_C(x) := \inf_{y \in C} \|x - y\|\}$ denotes the *projection* of x onto C . When A and B are convex, then the projectors P_A and P_B are single-valued and the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ converge to some point in $A \cap B$. This classical result goes back to Bregman [6], and it has found a huge number of extensions (see, e.g., [3], [8], [10], [11]). In the general case, when A and B are not necessarily convex, the situation is much more delicate. In their 1990 paper [9], Combettes and Trussell gave quite general sufficient conditions for the following dichotomy: either $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ converge to a point in $A \cap B$ or the set of cluster points is a nondegenerate continuum. (For recent results in the nonconvex case, see [4] and [5] and the references therein.)

The goal of this note is to explicitly construct two sets A and B illustrating the continuum case.

The main ingredient of our construction is a spiral in the Euclidean plane from which we pick points in an alternating fashion¹.

The sets A and B may be chosen to be countably infinite unions of closed convex sets. In contrast, we also prove that the continuum case cannot occur when A and B are finite unions of closed convex sets.

The remainder of the paper is organized as follows. In Section 2, we lay the ground work by studying a certain curve in the Euclidean plane. In Section 3, we use this curve to construct a sequence of points in the plane that is crucial in obtaining the sets A and B . Some remarks and the announced positive result conclude the paper.

2. A useful spiral. We will mostly work in the Euclidean plane \mathbb{R}^2 . As usual, angles will be measured in radians, but sometimes we shall use degrees as in writing $\pi/2 = 90^\circ$.

Let us recall that the distance d between $(r \cos(\alpha), r \sin(\alpha))$ and $(s \cos(\beta), s \sin(\beta))$, where $r \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$, satisfies

$$(2a) \quad d^2 = \|(r \cos(\alpha), r \sin(\alpha)) - (s \cos(\beta), s \sin(\beta))\|^2 = r^2 + s^2 - 2rs \cos(\alpha - \beta)$$

$$(2b) \quad \geq r^2 + s^2 - 2rs = (r - s)^2;$$

¹The reader is invited to take a peek at the figure below for an illustration of the location of these points.

hence,

$$(3) \quad r - d \leq s \leq r + d.$$

Define the function ρ by

$$(4) \quad \rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+: t \mapsto 1 + \exp(-t).$$

This function will represent the distance of a point on the curve at time t to the origin. Clearly, ρ is strictly decreasing with $\rho(0) = 2$ and $\lim_{t \rightarrow +\infty} \rho(t) = 1$. Also define

$$(5) \quad \varepsilon: \mathbb{R}_+ \rightarrow \mathbb{R}_{++}: t \mapsto \frac{\rho(t) - \rho(t + 2\pi)}{2}.$$

Then $\varepsilon' = -\varepsilon$ and hence ε is strictly decreasing to $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$. Note that

$$(6) \quad \mathbb{R}_+ \rightarrow \mathbb{R}_{++}: \alpha \mapsto \frac{\varepsilon(\alpha)}{\rho(\alpha)} = \frac{1}{2} \frac{1 - e^{-2\pi}}{1 + e^\alpha} \text{ is strictly decreasing.}$$

We now define the curve

$$(7) \quad x: \mathbb{R}_+ \rightarrow \mathbb{R}^2: \alpha \mapsto \rho(\alpha) \cdot (\cos(\alpha), \sin(\alpha)).$$

Note that x describes a spiral traversing counter-clockwise; x is *injective* because ρ is strictly decreasing. Now let α and β be in \mathbb{R}_+ , and assume that $\|x(\alpha) - x(\beta)\| \leq \varepsilon(\alpha)$. By (3), $\rho(\alpha) - \varepsilon(\alpha) \leq \rho(\beta) \leq \rho(\alpha) + \varepsilon(\alpha)$. Using the definitions, we solve these inequalities for β and obtain

$$(8) \quad \alpha - 0.40 \approx \alpha + \ln(2) - \ln(3 - e^{-2\pi}) \leq \beta \leq \alpha + \ln(2) - \ln(1 + e^{-2\pi}) \approx \alpha + 0.69;$$

in degrees, this implies $\alpha - 24^\circ \leq \beta \leq \alpha + 40^\circ$. To summarize,

$$(9) \quad \|x(\alpha) - x(\beta)\| \leq \varepsilon(\alpha) \quad \Rightarrow \quad \alpha - 24^\circ \leq \beta \leq \alpha + 40^\circ.$$

We will now discuss the monotonicity of the function

$$(10) \quad f: t \mapsto \|x(\alpha + t) - x(\alpha)\|^2.$$

Recall that

$$(11) \quad t \in]0, \pi/2[\quad \Rightarrow \quad \sin(t) + \cos(t) > 1.$$

One checks that

$$(12) \quad f'(t) \frac{\exp(2(\alpha + t))}{2} = g_1(t) + g_2(t) + g_3(t),$$

where

$$(13a) \quad g_1(t) = \sin(t) \exp(2t + \alpha)(1 + \exp(\alpha)),$$

$$(13b) \quad g_2(t) = \exp(\alpha + t)(\sin(t) + \cos(t) - 1),$$

$$(13c) \quad g_3(t) = \exp(t)(\sin(t) + \cos(t) - \exp(-t)).$$

Since each g_i is strictly positive on $]0, \pi/2[$, it follows from the mean value theorem that

$$(14) \quad f \text{ is strictly increasing on } [0, \pi/2].$$

Combining with (9), we deduce²

$$(15) \quad (\forall \alpha \in \mathbb{R}_+) (\exists ! \beta > \alpha) \quad \|x(\beta) - x(\alpha)\| = \varepsilon(\alpha).$$

Furthermore, denoting the unit sphere by S , we have

$$(16) \quad (\forall \alpha \in \mathbb{R}_+) \quad d_S(x(\alpha)) = \rho(\alpha) - 1 = \exp(-\alpha) > \varepsilon(\alpha).$$

3. A useful sequence. We now construct a sequence $(x_n)_{n \in \mathbb{N}}$ in the Euclidean plane with remarkable properties. Let us initialize

$$(17) \quad \alpha_0 := 0, \quad x_0 := x(\alpha_0), \quad \rho_0 := \rho(\alpha_0), \quad \varepsilon_0 := \varepsilon(\alpha_0).$$

In Cartesian coordinates, $x_0 = (2, 0)$, and $\varepsilon_0 \approx 0.5$. Now suppose $n \in \mathbb{N}$ and α_n , x_n , ρ_n , and ε_n are given. In view of (15), there exists a unique $\beta > \alpha_n$ such that

$$(18) \quad \|x(\beta) - x(\alpha_n)\| = \varepsilon_n.$$

We then update

$$(19) \quad \alpha_{n+1} := \beta, \quad x_{n+1} := x(\alpha_{n+1}), \quad \rho_{n+1} := \rho(\alpha_{n+1}), \quad \text{and} \quad \varepsilon_{n+1} := \varepsilon(\alpha_{n+1}).$$

(The picture illustrates the beginning of the spiral and x_0, \dots, x_{15} along with the radii used to construct the next iterate.) We also set

$$(20) \quad \delta_n := \alpha_{n+1} - \alpha_n.$$

By construction,

$$(21) \quad (\forall n \in \mathbb{N}) \quad \|x_n - x_{n+1}\| = \varepsilon_n \quad \text{and} \quad \sum_{k=0}^n \delta_k = \alpha_{n+1} - \alpha_0.$$

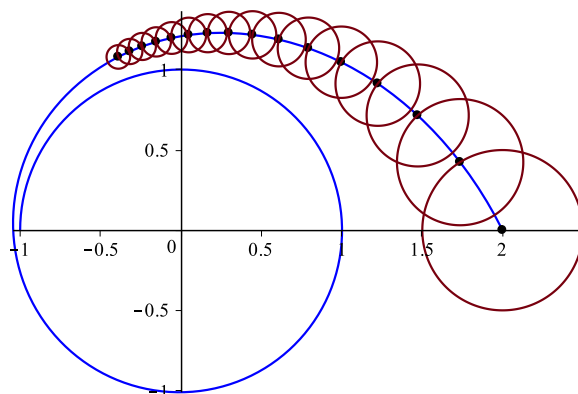
Note that

$$(22) \quad (\alpha_n)_{n \in \mathbb{N}} \text{ is strictly increasing, and } (\varepsilon_n)_{n \in \mathbb{N}} \text{ is strictly decreasing}$$

because the function ε is strictly decreasing. Set

$$(23) \quad \alpha_\infty := \lim_{n \in \mathbb{N}} \alpha_n \in]0, +\infty].$$

²“ $\exists !$ ” stands for “there exists a *unique*”



Since ρ is strictly decreasing we also note that

$$(24) \quad (\rho_n)_{n \in \mathbb{N}} \text{ is strictly decreasing, with } \lim_{n \in \mathbb{N}} \rho_n =: \rho_\infty \in [1, 2[.$$

Hence the corresponding sequence of quotients satisfies

$$(25) \quad 1 > q_n := \frac{\rho_{n+1}}{\rho_n} \rightarrow 1.$$

Using (2a) and the half-angle identity for sine, we have

$$(26a) \quad (\forall n \in \mathbb{N}) \quad \varepsilon_n^2 = \|x_n - x_{n+1}\|^2$$

$$(26b) \quad = \rho_n^2 + \rho_{n+1}^2 - 2\rho_n\rho_{n+1}\cos(\delta_n)$$

$$(26c) \quad = (\rho_n - \rho_{n+1})^2 + 2\rho_n\rho_{n+1}(1 - \cos(\delta_n))$$

$$(26d) \quad = (\rho_n - \rho_{n+1})^2 + 4\rho_n\rho_{n+1}\frac{1 - \cos(\delta_n)}{2}$$

$$(26e) \quad = (\rho_n - \rho_{n+1})^2 + 4\rho_n\rho_{n+1}\sin^2(\delta_n/2).$$

Dividing by ρ_n^2 and recalling (6), we obtain

$$(27) \quad (\forall n \in \mathbb{N}) \quad \left(\frac{1}{2} \frac{1 - e^{-2\pi}}{1 + e^{\alpha_n}} \right)^2 = \frac{\varepsilon_n^2}{\rho_n^2} = (1 - q_n)^2 + 4q_n \sin^2(\delta_n/2).$$

Taking limits, we learn that

$$(28) \quad \left(\frac{1}{2} \frac{1 - e^{-2\pi}}{1 + e^{\alpha_\infty}} \right)^2 = 4 \lim_n \sin^2(\delta_n/2).$$

Since δ_n , in degrees, belongs to $]0^\circ, 40^\circ]$ by (9), we deduce that $(\delta_n)_{n \in \mathbb{N}}$ is convergent as well. If $\alpha_\infty = +\infty$, then $\delta_n \rightarrow 0$ by (28); however, if $\alpha_\infty < +\infty$, then $\delta_n = \alpha_{n+1} - \alpha_n \rightarrow \alpha_\infty - \alpha_\infty = 0$. Either way,

$$(29) \quad \delta_n \rightarrow 0.$$

Again by (28), we have

$$(30) \quad \alpha_n \rightarrow \alpha_\infty = +\infty,$$

which by (21) implies

$$(31) \quad \sum_{n \in \mathbb{N}} \delta_n = +\infty,$$

$$(32) \quad \varepsilon_n \rightarrow 0,$$

and

$$(33) \quad \rho_n \rightarrow \rho_\infty = 1.$$

Note also that in view of (26), we have

$$(34) \quad \varepsilon_n^2 > 4 \sin^2(\delta_n/2) \geq \frac{\delta_n^2}{4} \quad \text{eventually,}$$

where we used (29) and the Taylor estimate

$$(35) \quad \sin(t/2) \geq \frac{1}{2}t - \frac{1}{48}t^3 = \frac{t}{2} \left(1 - \frac{1}{24}t^2\right) \geq \frac{t}{4} \quad \text{for } t \text{ sufficiently close to } 0.$$

Combining with (31), we record that

$$(36) \quad (\forall n \in \mathbb{N}) \quad \|x_n - x_{n+1}\| > \|x_{n+1} - x_{n+2}\| \rightarrow 0, \\ \text{and} \quad \sum_{n \in \mathbb{N}} \|x_n - x_{n+1}\| = +\infty.$$

Furthermore, (30) and (33) imply that

$$(37) \quad \text{the set of cluster points of } (x_n)_{n \in \mathbb{N}} \text{ is the unit sphere } S.$$

Define

$$(38) \quad (\forall n \in \mathbb{N}) \quad C_n := \{x_0, x_1, \dots\} \setminus \{x_n\}$$

We claim that

$$(39) \quad (\forall n \in \mathbb{N}) \quad P_{C_n} x_n = \{x_{n+1}\}.$$

Let $n \in \mathbb{N}$. Since $D_n := \{x_{n+1}, x_{n+2}, \dots\} \subset x(]\alpha_n, +\infty[)$, it follows from (9), (14), and (15) that $P_{D_n} x_n = \{x_{n+1}\}$. We show that there is no $k \in \mathbb{N}$ such

that $k < n$ and $\|x_k - x_n\| < \|x_n - x_{n+1}\|$. Suppose the contrary. Then, by (9), $\alpha_n - 24^\circ \leq \alpha_k < \alpha_n$. Hence $\alpha_k < \alpha_n \leq \alpha_k + 24^\circ$. By (14), $\|x_k - x_{k+1}\| = \|x(\alpha_k) - x(\alpha_{k+1})\| \leq \|x(\alpha_k) - x(\alpha_n)\| = \|x_k - x_n\| < \|x_n - x_{n+1}\| < \|x_k - x_{k+1}\|$, which is absurd. This verifies (39). Furthermore, by (16),

$$(40) \quad (\forall n \in \mathbb{N}) \quad d_S(x_n) > \|x_n - x_{n+1}\|.$$

Let us summarize our findings.

Theorem 3.1. *The sequence $(x_n)_{n \in \mathbb{N}}$ and the set $Y := \{x_n \mid n \in \mathbb{N}\}$ satisfy the following:*

- (i) $(\|x_n - x_{n+1}\|)_{n \in \mathbb{N}}$ is strictly decreasing.
- (ii) $x_n - x_{n+1} \rightarrow 0$.
- (iii) $\sum_{n \in \mathbb{N}} \|x_n - x_{n+1}\| = +\infty$.
- (iv) $(\forall n \in \mathbb{N}) \quad P_{(S \cup Y) \setminus \{x_n\}} x_n = \{x_{n+1}\}$.
- (v) *The set of cluster points of $(x_n)_{n \in \mathbb{N}}$ is the compact continuum S .*
- (vi) $S \cup D$ is closed, where D is an arbitrary subset of Y .

We now obtain the announced example concerning an instance of the method of alternating projections whose set of cluster points is a nondegenerate compact continuum.

Corollary 3.2. *Set*

$$(41) \quad A := \{x_{2n} \mid n \in \mathbb{N}\} \cup S, \quad B := \{x_{2n+1} \mid n \in \mathbb{N}\} \cup S, \quad \text{and} \quad b_{-1} := x_0.$$

Then A and B are nonempty compact subsets of \mathbb{R}^2 . The corresponding sequences of alternating projections satisfy

$$(42) \quad (\forall n \in \mathbb{N}) \quad a_n = P_A b_{n-1} = x_{2n} \quad \text{and} \quad b_n = P_B a_n = x_{2n+1}.$$

Moreover, $a_n - b_{n-1} \rightarrow 0$, $b_n - a_n \rightarrow 0$, and S is the set of cluster points of $(a_n)_{n \in \mathbb{N}}$ and of $(b_n)_{n \in \mathbb{N}}$.

Remark 3.3. Some comments on Corollary 3.2 are in order.

- (i) We note that Corollary 3.2 is the first example constructed where the set of limit points of alternating projections is a nondegenerate compact continuum. This complements the analysis of Combettes and Trussell [9] who conceived this case.
- (ii) If the starting point b_{-1} is an arbitrary point, then either $a_0 \in S$ or $a_0 \in A \setminus S$. In the first case, we have $(\forall n \in \mathbb{N}) \quad a_n = b_n = a_0$; in the second case, the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are tails of $(x_{2n})_{n \in \mathbb{N}}$ and $(x_{2n+1})_{n \in \mathbb{N}}$

respectively. A more involved analysis shows that if b_{-1} is outside the closed unit disk, then $P_A b_{-1} \in A \setminus S$ and we are in the second case. Hence one obtains a nondegenerate compact continuum of cluster points exactly when b_{-1} lies outside the closed unit disk.

- (iii) Suppose that, in (41), we replace S by the closed unit disk and we consider all possible orbits, i.e., the starting point b_{-1} ranges over the entire space X instead of being pinned at x_0 . Then the corresponding union of the sets of cluster points of $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ is the closed unit disk. Note that in this case, both A and B are *countably infinite* unions of convex sets. In the following result, we show that a degenerate continuum cannot occur as the set of cluster points when A and B are *finite* unions of nonempty closed convex sets.

Theorem 3.4 (finite unions of convex sets). *Suppose that I and J are nonempty finite index sets, let $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ be families of nonempty closed convex subsets of a Euclidean space X , and set $A := \bigcup_{i \in I} A_i$ and $B := \bigcup_{j \in J} B_j$. Consider a sequence of alternating projections $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ generated by A and B : $b_{-1} \in X$, and $(\forall n \in \mathbb{N})$ $a_n \in P_A b_{n-1}$ and $b_n \in P_B a_n$. Suppose that $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are bounded, and that $b_n - a_n \rightarrow 0$ and $a_{n+1} - b_n \rightarrow 0$. Then there exists a point $c \in A \cap B$ such that $a_n \rightarrow c$ and $b_n \rightarrow c$.*

Proof. After relabeling and considering the tails of the sequences if necessary, we assume that

$$(43a) \quad (\forall i \in I) \ A_i \text{ is projected upon infinitely often}$$

and that

$$(43b) \quad (\forall j \in J) \ B_j \text{ is projected upon infinitely often.}$$

The pigeonhole principle gives $(i_+, j_+) \in I \times J$ and subsequences $(a_{k_n})_{n \in \mathbb{N}}$ and $(b_{k_n})_{n \in \mathbb{N}}$ lying in A_{i_+} and B_{j_+} respectively. After passing to further subsequences if necessary, we also assume that there is $c \in A_{i_+} \cap B_{j_+}$ such that $a_{k_n} \rightarrow c$ and $b_{k_n} \rightarrow c$. Set $I_- := \{i \in I \mid c \notin A_i\}$, $I_+ := I \setminus I_-$, $J_- := \{j \in J \mid c \notin B_j\}$, $J_+ := J \setminus J_-$, $\delta := \min\{\min_{i \in I_-} d_{A_i}(c), \min_{j \in J_-} d_{B_j}(c), 1\}$, $A_- := \bigcup_{i \in I_-} A_i$, and $B_- := \bigcup_{j \in J_-} B_j$. Now assume that there exists $m \in \mathbb{N}$ such that $\|a_m - c\| < \delta/2$. Then $d_{B_-}(a_m) \geq d_{B_-}(c) - \|a_m - c\| > \delta - \delta/2 = \delta/2 > \|a_m - c\| \geq d_{B \setminus B_-}(a_m)$. Hence $(\forall j \in J_-)$ $b_m \notin P_{B_j}(a_m)$, and thus $b_m \in \{P_{B_j}(a_m) \mid j \in J_+\}$. Since projectors are nonexpansive, it follows that $\|b_m - c\| \leq \|a_m - c\| < \delta/2$. Therefore,

$$(44a) \quad \|a_m - c\| < \delta/2 \quad \Rightarrow \quad \|b_m - c\| < \delta/2 \text{ and } (\forall j \in J_-) \ b_m \notin P_{B_j}(a_m),$$

and a similar argument yields

$$(44b) \quad \|b_m - c\| < \delta/2 \quad \Rightarrow \quad \|a_{m+1} - c\| < \delta/2 \text{ and } (\forall i \in I_-) \ a_{m+1} \notin P_{A_i}(b_m).$$

Since $a_{k_n} \rightarrow c$, there does exist $M \in \mathbb{N}$ such that $\|a_M - c\| < \delta/2$. But then (44) has two consequences. First, starting with iteration index M , $(\forall i \in I_-)$ A_i is not projected upon and $(\forall j \in J_-)$ B_j is not projected upon. In view of (43), we conclude that $I_- = J_- = \emptyset$, i.e., $c \in \bigcap_{i \in I} A_i \cap \bigcap_{j \in J} B_j$. The second consequence of (44) is $(\forall m \geq M)$ $\|a_m - c\| \geq \|b_m - c\| \geq \|a_{m+1} - c\|$. Finally, since c is a cluster point of $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, it thus follows that $\|a_n - c\| \rightarrow 0$ and $\|b_n - c\| \rightarrow 0$. \square

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REFERENCES

- [1] P.-A. ABSIL, R. MAHONY, B. ANDREWS. Convergence of the iterates of descent methods for analytic cost functions. *SIAM J. Optim.* **16** (2005), 531–547.
- [2] H. ATTOUCH, X. GOUDOU, P. REDONT. The heavy ball with friction method I. The continuous dynamical system: global exploration of the local minima of a real-valued function by asymptotic analysis of a dissipative dynamical system. *Commun. Contemp. Math.* **2** (2000), 1–34.
- [3] H. H. BAUSCHKE, P. L. COMBETTES. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics. Berlin, Springer, 2011.
- [4] H. H. BAUSCHKE, D. R. LUKE, H. M. PHAN, X. WANG. Restricted normal cones and the method of alternating projections: theory. *Set-Valued Var. Anal.* **21** (2013), 431–473.

- [5] H. H. BAUSCHKE, D. R. LUKE, H. M. PHAN, X. WANG. Restricted normal cones and the method of alternating projections: applications. *Set-Valued Var. Anal.* **21** (2013), 475–501.
- [6] L. M. BREGMAN. The method of successive project for finding a common point of convex sets. *Dokl. Akad. Nauk SSSR* **162** (1965), 487–490; English translation in: *Sov. Math. Dokl.* **6** (1965), 688–692.
- [7] A. CEGIELSKI. Iterative Methods for Fixed Point Problems in Hilbert Spaces. Lecture Notes in Mathematics, vol. **2057**, Berlin, Springer, 2012.
- [8] Y. CENSOR, S. A. ZENIOS. Parallel Optimization. Theory, Algorithms, and Applications. Numerical Mathematics and Scientific Computation. Oxford, Oxford University Press, 1997.
- [9] P. L. COMBETTES, H. J. TRUSSELL. Methods of successive projections for finding a common point of sets in metric spaces. *J. Optim. Theory Appl.* **67**, 3 (1990), 487–507.
- [10] K. GOEBEL, W. A. KIRK. Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics, vol. **28**, Cambridge, Cambridge University Press, 1990.
- [11] K. GOEBEL, S. REICH. Uniform Convexity, Hyperbolic Geometry, and Non-expansive Mappings. New York, Marcel Dekker, 1984.
- [12] J. PALIS, JR., W. DE MELO. Geometric Theory of Dynamical Systems. New York–Heidelberg–Berlin, Springer, 1982.

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