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ON 2-HOMOGENEOUS C^*-ALGEBRAS OVER TWO-DIMENSIONAL ORIENTED MANIFOLDS GENERATED BY THREE IDEMPOTENTS

Mikhail Shchukin

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Abstract. We consider algebraic bundles over a two-dimensional compact oriented connected manifold. In 1961 J. Fell, J. Tomiyama, M. Takesaki showed that every n-homogeneous C^*-algebra is isomorphic to the algebra of all continuous sections for the appropriate algebraic bundle. By using this realization we prove in the work that every 2-homogeneous C^*-algebra over two-dimensional compact oriented connected manifold can be generated by three idempotents. Such algebra can not be generated by two idempotents.

1. Introduction. Banach algebras generated by idempotents are naturally appear in the theory of singular integral operators. Remind that an element a from the algebra A is called idempotent if $a^2 = a$. The theory of Banach algebras generated by two idempotents has applications to the symbol calculus for the algebra of singular integral operators over a simple contour [2]. Such algebras can

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have irreducible representations of order 1 or 2. The structure of Banach algebras generated by N idempotents with some concrete relations between generators was described in the work [2]. Such relations between generators are naturally appear in the theory of singular integral operators. In the work [10] it was proved that every n-homogeneous C^*-algebra over sphere S^2 can be generated by three idempotents. Moreover, such algebra can not be generated by two idempotents. The set of n-homogeneous C^*-algebras over an oriented two-dimensional compact connected manifold was described in the work [8]. In the work we find the minimal number of idempotent generators for such algebras.

Denote by A a n-homogeneous C^*-algebra. Suppose $Prim(A)$ is the space of primitive ideals for the algebra A in the hull-kernel topology. In this paper we consider such algebra A that the space $Prim(A)$ is homeomorphic to a two-dimensional compact oriented manifold.

Proposition 1.1 ([5]). *Every compact connected oriented manifold is homeomorphic to the sphere P_k with k handles.*

Suppose A is a n-homogeneous C^*-algebra over the set P_k. It means that the set of primitive ideals for the algebra A is homeomorphic to the set P_k in the topology. Let the space P_k is the sphere S^2 with k handles such that all handles are attached to the upper half of S^2. We will impose conditions on the handles below. Let D be a lower half of the sphere P_k. In this case, the set D is homeomorphic to the open unit disk. The next statements have place for such selection of P_k and D. Denote by $P_k \backslash D$ the set P_k without D. For every n-homogeneous C^*-algebra there exists an algebraic bundle $\zeta_A = (E, B, p)$ such that the algebra A is isomorphic to the algebra $\Gamma(E)$ of all continuous sections for the bundle [3].

Proposition 1.2 ([8]). *The restriction of the bundle ζ_A to the set $P_k \backslash D$ is trivial.*

Consider the cartesian coordinate system $Oxyh$ in R^3. We denote by h the point applicate. Further, let P_k be the sphere S^2 with k-handles attached. Suppose all k handles are attached to the upper half of S^2. Let the handles be so small that the projection of P_k to the (x,y) plane is the unit disc and the projection of the set P_k to the axe Oh is the [-1,1] interval. As above, let D be the lower half of S^2. Let $z = x + iy$ be the complex point on the plane Oxy. For all points from P_k the projection to the plane Oxy has the next property: $|z| \leq 1$. Denote by B_V the algebra of continuous matrix-functions from $P_k \backslash D$ to $\mathbb{C}^{n\times n}$

with additional condition on the boundary: $a(z) = V^{-1}(z) \cdot a(1) \cdot V(z), a(z) \in B_V$,

$$V(z) = \begin{pmatrix} z^m & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Proposition 1.3 ([8]). *The algebra A is isomorphic to the one of the algebras B_V for some matrix-function $V(z)$.*

2. Some results on the structure of C^*-algebras B_V. Let $C(P_k)$ be the algebra of all continuous functions on P_k. The set of all functions $a(z) \in C(P_k \backslash D)$ such that $a(z) = z^m \cdot a(1)$ for all $z \in S^1 = \delta D$ has the structure of a module. Denote by B_m the module. The integer m belongs to the next range: $-n+1 \leq m \leq n-1$. Suppose E_{ij} is the matrix $n \times n$ that has 1 on ij place. All other elements for the matrix are equal to zero.

Lemma 2.1. *The algebra B_V can be considered as the module over its center. The center of B_V is isomorphic to the algebra $C(P_k)$. Since the algebra B_V is the module over $C(P_k)$, we obtain*

$$B_V = E_{11}C(P_k) \bigoplus E_{12}B_m \bigoplus \dots \bigoplus E_{1n}B_m \bigoplus_{2 \leq s \leq n} E_{s1}B_{-m} \bigoplus_{2 \leq s,t \leq n} E_{st}C(P_k).$$

P r o o f. Suppose g is an element of the algebra B_V. Note that $g \in C(P_k \backslash D, \mathbb{C}^{n \times n})$. The element $g(x,y,h)$ is the matrix-function. For $h = 0$ we obtain $|z| = 1$ and

$$g(z) = \begin{pmatrix} z^m & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} g_{11}(1) & g_{12}(1) & \dots & g_{1n}(1) \\ g_{21}(1) & g_{22}(1) & \dots & g_{2n}(1) \\ \dots & \dots & \dots & \dots \\ g_{n1}(1) & g_{n2}(1) & \dots & g_{nn}(1) \end{pmatrix} \cdot \begin{pmatrix} \overline{z}^m & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Thus we have $g(z) = \begin{pmatrix} g_{11}(1) \cdot |z|^{2m} & g_{12}(1) \cdot z^m & \dots & g_{1n}(1) \cdot z^m \\ g_{21}(1) \cdot \overline{z}^m & g_{22}(1) & \dots & g_{2n}(1) \\ \dots & \dots & \dots & \dots \\ g_{n1}(1) \cdot \overline{z}^m & g_{n2}(1) & \dots & g_{nn}(1) \end{pmatrix}.$

Since $z \in S^1$ it follows that $|z| = 1$. Thus we have $g_{11}(z) = g_{11}(1)$. Therefore the function g_{11} can be considered as the function on the set P_k. Similarly, the functions $g_{ij}(2 \leq i,j \leq n)$ can be considered as the functions from $C(P_k)$. The functions $g_{1j}(x,y,h)(2 \leq j \leq n)$ generate the module B_m. In the same way, the functions $g_{i1}(x, y, h)(2 \le i \le n)$ generate the module B_{-m}. This concludes the proof. □

Lemma 2.2. *Let $f_1, \dots, f_t$ be functions from $B_m(-(n-1) \le m \le n-1)$. Suppose for any point $x_0 \in P_k \backslash D$ there is a integer i such that $1 \le i \le t$ and $f_i(x_0) \ne 0$. In this case, $B_m = f_1 \cdot C(P_k) + \dots + f_t \cdot C(P_k)$.*

P r o o f. Select a point x_0 from $P_k \backslash D$. Let U_{x_0} be an open ball with center x_0 and radius $r(x_0)$. Denote by $2U_{x_0}$ the open ball with center x_0 and radius $2r(x_0)$. Suppose $r(x_0)$ is so small that $f_i(x) \ne 0, x \in 2U_{x_0} \bigcap (P_k \backslash D)$ for some function f_i. Suppose x_0 is a point from $\delta D = S^1$. In this case, $f_i(z) = z^m \cdot f_i(1) \ne 0$. For any point $x_0 \in S^1$ let U_{x_0} be an open set containing the set S^1. In this case, $U_{x_0} = \bigcup\limits_{z \in S^1} U_z$. Since $P_k \backslash D$ is compact, we have a finite subcover $U_1, \dots, U_s$ for the set $P_k \backslash D$. Let $h_1, \dots, h_s$ be a partition of unity for the cover $U_1, \dots, U_s$. In this case, any function $f \in B_m$ has the next form: $f = fh_1 + \dots + fh_s$. Further, for any positive integer $i \in \overline{1, t}$, there exists a integer $n(i)$ such that $f_{n(i)}(x) \ne 0, x \in (2U_i \bigcap (P_k \backslash D))$, by construction of the set U_i. Since the set $\overline{U_i}$ is the compact, then the function $\dfrac{1}{f_{n(i)}}$ is bounded. In addition, there is a continuous function $\dfrac{1}{f^*_{n(i)}}$ on the compact P_k, by the Titze-Brower-Uryson lemma. Here the function $\dfrac{1}{f^*_{n(i)}}$ is the continuous extension for the function $\dfrac{1}{f_{n(i)}}$ to the compact P_k. This implies that

$$f = f \cdot h_1 \cdot f_{n(1)} \cdot \frac{1}{f^*_{n(1)}} + \dots + f \cdot h_s \cdot f_{n(s)} \cdot \frac{1}{f^*_{n(s)}}.$$

The functions $f \cdot h_i \cdot \dfrac{1}{f^*_{n(i)}}$ belong to the class $C(P_k)$. This completes the first part of the proof.

On the other hand, the module $f_i \cdot C(P_k)$ is a subset of B_m, by definition of the module B_m. Thus $f_1 \cdot C(P_k) + \dots + f_t \cdot C(P_k) \subset B_m$. This completes the proof. □

Lemma 2.3. *Suppose f is a function from $C(P_k \backslash D)$ such that $f(z) = f(1)$ for any $z \in \delta D$. In this case, there are $f_i \in B_m, g_i \in B_{-m}$ such that $f = f_1 g_1 + f_2 g_2$.*

P r o o f. The algebra of continuous functions $C(P_k \backslash D)$ such that $f(z) = f(1)$ for any $z \in \delta D$ is isomorphic to the algebra $C(P_k)$. Suppose x_1 is a point of intersection $P_k \backslash D$ and the axe Oh. Let U_1, U_2 be an open covering for the set P_k such that $x_1 \in U_1$ and $\delta D \bigcap U_1 = \emptyset$. Suppose $\overline{U_2}$ does not contain the point x_1. Let h_1, h_2 be the partition of unity for the open covering by U_1 and U_2. Denote by f_2 and g_2 the functions $f_2 = \sqrt{fh_2} \cdot \frac{z}{|z|}, g_2 = \sqrt{fh_2} \cdot \frac{\overline{z}}{|z|}$. In this case, $\sqrt{fh_2}$ denotes the same complex number for two-valued function $\sqrt{w}$ in both cases. Since the function $\frac{1}{|z|} > 0$ on the closed set $\overline{U_2}$, then the functions f_2 and g_2 are well defined. The function $\frac{1}{|z|}$ is bounded on the top and bottom on the set $\overline{U_2}$. Denote by f_1 the function $\sqrt{fh_1}$, let $g_1 = \sqrt{fh_1}$. Here $\sqrt{fh_1}$ denotes the same value for the two-valued function. Since $\sqrt{fh_1}(z) = 0$ for all $z \in S^1 = \delta D$, we have $f_1 \in B_m$ and $g_1 \in B_{-m}$.

Thus $f = fh_1 + fh_2 = \sqrt{fh_1} \cdot \sqrt{fh_1} + \sqrt{fh_2} \cdot \frac{z}{|z|} \cdot \sqrt{fh_2} \cdot \frac{\overline{z}}{|z|} = f_1 g_1 + f_2 g_2$. This concludes the proof. □

3. 2-homogeneous C^*-algebras over two-dimensional manifolds. Our main result is the following

Theorem 3.1. *Denote by A a 2-homogeneous C^*-algebra over the compact two-dimensional oriented connected manifold P_k. In this case, the algebra A can be generated by three idempotents.*

P r o o f. Let A_1 be the 2-homogeneous C^*-algebra B_V, $V(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. Suppose Q_1 is a matrix-function $\begin{pmatrix} 1 & h_1 \\ 0 & 0 \end{pmatrix}$. Denote by h_1 the next function: $h_1(x, y, h) = h$. In this case, the matrix-function Q_1 is idempotent. Indeed, $Q_1^2 = Q_1$, by direct calculation. Similarly, denote by Q_2 the matrix-function $\begin{pmatrix} 1 & 0 \\ h_1 & 0 \end{pmatrix}$ and let Q_3 be $\frac{1}{1+|f|^2} \cdot \begin{pmatrix} 1 & f \\ \overline{f} & |f|^2 \end{pmatrix}$. Suppose $f(x, y, h) = x + iy$ and $\overline{f}(x, y, h) = x - iy$. It is not hard to prove that $Q_2^2 = Q_2$ and $Q_3^2 = Q_3$, by direct calculation. This implies that Q_2 and Q_3 are idempotents. Note that the idempotents Q_1, Q_2, Q_3 belong to the algebra A_1, by definition of the algebra A_1. Suppose B is the smallest Banach algebra containing the idempotents Q_1, Q_2, Q_3.

Multiplying Q_1 by Q_2, we obtain $Q_1 \cdot Q_2 = \begin{pmatrix} 1 + h_1^2 & 0 \\ 0 & 0 \end{pmatrix} \in A_1$. Since the function $1+h_1^2$ separates the points of the set $[0,1]$ and $1+h_1^2 \neq 0$ on the set, it follows that the function generates the algebra of all continuous functions $C(h)$. In other words, for for all $h_2 \in C(h)$ there is a sequence of polynomials $M_n(1+h_1^2)$ such that $\lim\limits_{n\to\infty} M_n(1+h_1^2) = h_2$. This means that $\lim\limits_{n\to\infty} M_n(Q_1 \cdot Q_2) = \begin{pmatrix} h_2 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore the algebra of matrix-functions $\begin{pmatrix} C(h) & 0 \\ 0 & 0 \end{pmatrix}$ is a subset of the algebra B. This implies that the matrix-function $\begin{pmatrix} 1+|f|^2 & 0 \\ 0 & 0 \end{pmatrix}$ belongs to the algebra B. In addition, $\begin{pmatrix} 1+|f|^2 & 0 \\ 0 & 0 \end{pmatrix} \cdot Q_3 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \in B$. Also, $Q_3 \cdot \begin{pmatrix} 1+|f|^2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{f} & 0 \end{pmatrix} \in B$.

On the other hand, $Q_2 - E_{11} = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} \in B$ and $Q_1 - E_{11} = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \in B$ Further, $\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \overline{f} & 0 \end{pmatrix} = \begin{pmatrix} |f|^2 & 0 \\ 0 & 0 \end{pmatrix} \in B$.

Furthermore, $\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \overline{f} & 0 \end{pmatrix} = \begin{pmatrix} h\overline{f} & 0 \\ 0 & 0 \end{pmatrix} \in B$ and $\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} = \begin{pmatrix} hf & 0 \\ 0 & 0 \end{pmatrix} \in B$. Let D_{h_0} be the complex plane with applicate h_0 over the plane $h = 0$. Notice that the functions $hf, h\overline{f}$ separate the points of $P_k \bigcap D_{h_0}$. By construction, the function h separates the points of P_k with different height over the plane Oxy. Gluing together the circle $S^1 = D$, we obtain the set P_k^*. Since the set P_k^* is homeomorphic to P_k, we obtain that $C(P_k^*) \cong C(P_k)$. Finally, the functions $hf, h\overline{f}, h$ and 1 generate the algebra $C(P_k^*)$, by Stone-Weierstrass theorem. Since the functions f and h are not equal zero together on $P_k \backslash D$, we obtain that the matrix-functions $\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}$ generate the module $\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}$ over the algebra $\begin{pmatrix} C(P_k) & 0 \\ 0 & 0 \end{pmatrix}$, by lemma 2.2.

On the other hand, the matrix-functions $\begin{pmatrix} 0 & 0 \\ \overline{f} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}$ generate the module $\begin{pmatrix} 0 & 0 \\ B_{-1} & 0 \end{pmatrix}$ over the algebra $\begin{pmatrix} C(P_k) & 0 \\ 0 & 0 \end{pmatrix}$.

Furthermore, the elements from the modules $\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ B_{-1} & 0 \end{pmatrix}$ generate the algebra $\begin{pmatrix} 0 & 0 \\ 0 & C(P_k) \end{pmatrix} \subset B$. This means that the sets $\begin{pmatrix} C(P_k) & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ B_{-1} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & C(P_k) \end{pmatrix}$ are the subsets of the algebra B. Therefore, we have $B = A_1$, by lemma 2.1. This completes the proof of the Theorem 3.1. □

Theorem 3.2. *Suppose A is a 2-homogeneous C^*-algebra over the compact two-dimensional oriented connected manifold P_k. In this case, the algebra A can not be generated by two idempotents.*

P r o o f. Assume the converse, then the algebra A is generated by two idempotents $a_1, a_2 \in A$. Let $M(A)$ be the set of maximal ideals of the algebra A. In this case, the set $M(A)$ is homeomorphic to a subset of the plane C ([1]). But it is well known that the two-dimensional manifold P_k is not homeomorphic to a subset of the plane C. This contradiction proves the theorem. □

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Department of Higher Mathematics
Information Technologies and Robotics Faculty
Belarusian National Technical University
9, B. Khmelnitsky Str.
220013, Minsk, Belarus
e-mail: `mvs777777@gmail.com`

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