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GENERALIZED STRONG CESÀRO SPACES OF LACUNARY STATISTICAL CONVERGENT SEQUENCES*

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Abstract. In this paper, we introduce and study the concepts of f-lacunary statistical convergence of order α and strong Cesàro summability of order α with respect to modulus function f and lacunary sequence $\theta = (k_r)$. Further, using these concepts we define some sequence spaces $S^{\alpha}(f, \theta, \Delta_v^m, u)$ of all f-lacunary statistical convergence of order α and $w^{\alpha}(f, \theta, \Delta_v^m, u)$ of all strong Cesàro summability of order α. We also investigate some inclusion relations between these spaces.

1. Introduction and preliminaries. In 1935 the notion of statistical convergence was given by Zygmund [28] in the first edition of his monograph published in Warsaw. The concept of statistical convergence was introduced by

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Steinhaus [26] and Fast [6] and further reintroduced by Schoenberg [25] independently. Throughout the years and under various names statistical convergence was discussed in the hypothesis of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further elaborated from the sequence spaces perspective and connected with summability theory by Fridy [7], Connor [4], Mursaleen [13], Fridy and Orhan [8], Šalát [24] and many others. In recent years, wide ranging of statistical convergence appeared in the investigation of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations were also connected with subsets of the Stone-Čech compactification of the natural numbers. Futhermore, the statistical convergence is closely related to the concept of convergence in probability. Throughout the paper we denote the space of all, bounded, convergent and null sequences of complex numbers by s, l_∞, c and c_0, respectively.

Let $\mathbb{N}$ denote the set of natural numbers. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, where the natural density of a subset A of $\mathbb{N}$ (see [17]) is defined by

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|,$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of $A \subseteq \mathbb{N}$ not exceeding n. It is clear that any finite subset of $\mathbb{N}$ have zero natural density and $d(A^c) = 1 - d(A)$. In this case we write $S - \lim x_k = L$ and we denote the set of all statistically convergent sequences by S.

A sequence $x = (x_k)$ is said to be strongly Cesàro summable to a number L if $\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0$. The set of all strongly Cesàro summable sequences is denoted by $[C, 1]$ and defined as

$$[C, 1] = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \ \text{ for some } L\Big\}.$$

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x)=\dfrac{x}{x+1}$, then $f(x)$ is bounded. If $f(x)=x^p, 0<p<1$ then the modulus function $f(x)$ is unbounded. For more details about sequence spaces (see [11], [15], [16], [21], [19], [18], [23], [27]) and references therein.

Definition 1.1 ([1]). *Let f be an unbounded modulus function. The f-density of a set $A\subset\mathbb{N}$ is defined by*

$$d^f(A)=\lim_{n\to\infty}\frac{f(|\{k\le n: k\in A\}|)}{f(n)}$$

in this case the limit exists.

Definition 1.2 ([1]). *Let f be an unbounded modulus function. A sequence $x=(x_k)$ is said to be f-statistically convergent to L or S^f-convergent to L, if for each $\epsilon>0$,*

$$d^f(\{k\in\mathbb{N}: |x_k-L|\ge\epsilon\})=0,$$

$$i.e.\ \lim_{n\to\infty}\frac{1}{f(n)}f(|\{k\le n: |x_k-L|\ge\epsilon\}|)=0$$

and we write it as $S^f-\lim x_k=L$. The set of all f-statistically convergent sequences is denoted by S^f.

Definition 1.3 ([2]). *Let α be any real number such that $0<\alpha\le 1$. The α-density of a set $A\subset\mathbb{N}$ is defined by*

$$d_\alpha(A)=\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\le n: k\in A\}|$$

in this case limit exists.

Definition 1.4 ([2]). *Let $0<\alpha\le 1$. A sequence $x=(x_k)$ is said to be statistically convergent of order α to L or S_α-convergent to L, if for each $\epsilon>0$,*

$$d^\alpha(\{k\in\mathbb{N}: |x_k-L|\ge\epsilon\})=0,$$

$$i.e.\ \lim_{n\to\infty}\frac{1}{n^\alpha}|(\{k\le n: |x_k-L|\ge\epsilon\})|=0$$

and we write it as $S_\alpha-\lim x_k=L$. The set of all statistically convergent sequences of order α is denoted by S_α. In case $\alpha=1$, the statistical convergence of order α reduces to the statistical convergence.

The main purpose of this paper is to introduce the spaces $S^{\alpha}(f,\theta,\Delta_v^m,u)$ and $w^{\alpha}(f,\theta,\Delta_v^m,u)$ of all f-lacunary statistical convergent sequences of order α and all strong Cesàro summable sequences of order α respectively. We also make an effort to study some topological properties and inclusion relations between these sequence spaces.

2. Main results.

Definition 2.1. *Let E be any set of sequences, the space of multipliers of E, denoted by $M(E)$ and is given by*

$$M(E) = \{a \in s : ax \in E \text{ for all } x \in E\}.$$

Definition 2.2. *A sequence space E is called normal if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.*

Definition 2.3. *A sequence space E is called monotone if it contains the canonical preimages of all its step spaces.*

The notion of difference sequence spaces was conceptualized as $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ initially by Kizmaz [9]. Further, the notion was generalized by Et and Çolak [5] as they introduced the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let m, v be non-negative integers, then for $Z = c, c_0$ and l_∞, we have

$$Z(\Delta_v^m) = \{x = (x_k) \in s : (\Delta_v^m x_k) \in Z\},$$

where $\Delta_v^m x = (\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+v})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+vi}.$$

If $v = 1$, we get the spaces $l_\infty(\Delta^m), c(\Delta^m)$ and $c_0(\Delta^m)$ as studied by Et and Çolak [5].

If $v = m = 1$, we get the spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ as introduced and studied by Kizmaz [9].

By a lacunary sequence $\theta = (k_r), r = 0, 1, 2, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\dfrac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Recently, lacunary sequence spaces were studied in ([14], [20]).

Now we introduce a new concept of f-lacunary statistical convergence of order α as follows:

Definition 2.4. *Let $\theta = (k_r)$ be a lacunary sequence, f be an unbounded modulus function, $u = (u_k)$ be a sequence of strictly positive real numbers and $0 < \alpha \leq 1$. A sequence $x = (x_k)$ is said to be **f-lacunary statistically convergent of order** α to L or $S^\alpha(f, \theta, \Delta_v^m, u)$-convergent to L if for each $\epsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |u_k \Delta_v^m x_k - L| \geq \epsilon\}|) = 0,$$

where $I_r = (k_{r-1}, k_r]$ and $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \ldots, h_r^\alpha, \ldots)$. In this case, we write $S^\alpha(f, \theta, \Delta_v^m, u) - \lim x_k = L$. The set of all sequences which are f-lacunary statistically convergent of order α is denoted by $S^\alpha(f, \theta, \Delta_v^m, u)$ and the set of all f-lacunary statistically null sequences of order α is denoted by $S_0^\alpha(f, \theta, \Delta_v^m, u)$. It is clear that $S_0^\alpha(f, \theta, \Delta_v^m, u) \subset S^\alpha(f, \theta, \Delta_v^m, u)$ for any unbounded modulus function f. For $\theta = (2^r)$, $m = 0$, $(v_k) = (1, 1, 1, \ldots)$ and $(u_k) = 1$ for all k we shall write S_α^f instead of $S^\alpha(f, \theta, \Delta_v^m, u)$ and in special case $\alpha = 1, \theta = (2^r)$, $f(x) = x$, $m = 0$, $(v_k) = (1, 1, 1, \ldots)$ and $(u_k) = 1$ for all k we write S instead of $S^\alpha(f, \theta, \Delta_v^m, u)$.

The f-lacunary statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general.

Example. Let $x = (x_k)$ be a sequence defined as follows:

$$(x_k) = \begin{cases} 1, & k = 2r, \\ & \qquad r = 1, 2, 3, \ldots \\ 0, & k \neq 2r. \end{cases}$$

For $m = 0$, $(v_k) = (1, 1, 1, \ldots)$ and $(u_k) = 1$ for all k, we have

$$\frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |x_k - 1| \geq \epsilon\}|) \leq \frac{f(k_r - k_{r-1})}{f(2h_r^\alpha)} = \frac{f(h_r)}{f(2h_r^\alpha)}$$

and

$$\frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |x_k - 0| \geq \epsilon\}|) \leq \frac{f(k_r - k_{r-1})}{f(2h_r^\alpha)} = \frac{f(h_r)}{f(2h_r^\alpha)}.$$

Since $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$, we have

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |x_k - 1| \geq \epsilon\}|) = 0$$

and

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |x_k - 0| \geq \epsilon\}|) = 0,$$

for $\alpha > 1$ and for each $\epsilon > 0$, the sequence $x = (x_k) \in S^\alpha(f, \theta, \Delta_v^m, u)$ which converges to both 1 and 0, i.e. $S^\alpha(f, \theta, \Delta_v^m, u) - \lim x_k = 1$ and $S^\alpha(f, \theta, \Delta_v^m, u) - \lim x_k = 0$. But this is not possible.

Theorem 2.5. *Let f be an unbounded modulus function and $0 < \alpha \leq 1$. Let $x = (x_k)$, $y = (y_k)$ be sequences of complex numbers. Then the following statements hold.*

(1) *If $S^\alpha(f, \theta, \Delta_v^m, u) - \lim x_k = L$ and $c \in \mathbb{C}$, then $S^\alpha(f, \theta, \Delta_v^m, u) - \lim cx_k = cL$.*

(2) *If $S^\alpha(f, \theta, \Delta_v^m, u) - \lim x_k = L_1$ and $S^\alpha(f, \theta, \Delta_v^m, u) - \lim y_k = L_2$, then $S^\alpha(f, \theta, \Delta_v^m, u) - \lim(x_k + y_k) = L_1 + L_2$.*

Proof. It is clear for the case $c = 0$. Consider $c \neq 0$, then the proof of (1) follows from

$$\frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |cu_k \Delta_v^m x_k - cL| \geq \epsilon\}|) = \frac{1}{f(h_r^\alpha)} f\left(\left|\left\{k \in I_r : |u_k \Delta_v^m x_k - L| \geq \frac{\epsilon}{|c|}\right\}\right|\right)$$

and that of (2) follows from

$$\frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |u_k \Delta_v^m (x_k + y_k) - (L_1 + L_2)| \geq \epsilon\}|)$$
$$\leq \frac{1}{f(h_r^\alpha)} f\left(\left|\left\{k \in I_r : |u_k \Delta_v^m x_k - L_1| \geq \frac{\epsilon}{2}\right\}\right|\right) + \frac{1}{f(h_r^\alpha)} f\left(\left|\left\{k \in I_r : |u_k \Delta_v^m y_k - L_2| \geq \frac{\epsilon}{2}\right\}\right|\right). \qquad \square$$

3. Strong Cesàro summability of order α. In this section, we have extended the notion of strong Cesàro summability of order α to that of strong Cesàro summability of order α ($\alpha > 0$) with respect to modulus function and lacunary sequence.

Let $\theta = (k_r)$ be a lacunary sequence, f be a modulus function, α be a real number and $u = (u_k)$ be a sequence of strictly positive real numbers. We define the following sequence spaces:

$$w_0^{\alpha}(f, \theta, \Delta_v^m, u) = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|u_k \Delta_v^m x_k|) = 0 \right\},$$

$$w^{\alpha}(f, \theta, \Delta_v^m, u) \\ = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|u_k \Delta_v^m x_k - L|) = 0, \text{ for some number } L \right\}$$

and

$$w_\infty^{\alpha}(f, \theta, \Delta_v^m, u) = \left\{ x = (x_k) : \sup_r \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|u_k \Delta_v^m x_k|) < \infty \right\}.$$

If we take $\alpha = 1$, then the above spaces reduces to $w_0(f, \theta, \Delta_v^m, u)$,$w(f, \theta, \Delta_v^m, u)$ and $w_\infty(f, \theta, \Delta_v^m, u)$.

If we take $f(x) = x$, then we have the above spaces as $w_0^{\alpha}(\theta, \Delta_v^m, u)$, $w^{\alpha}(\theta, \Delta_v^m, u)$ and $w_\infty^{\alpha}(\theta, \Delta_v^m, u)$.

Remark 3.1. In the spaces w_p^{α} and $w_p^{0\alpha}$ of Çolak [2], α is a positive real number less than or equal to 1, whereas in our spaces $w_0^{\alpha}(f, \theta, \Delta_v^m, u)$ and $w^{\alpha}(f, \theta, \Delta_v^m, u)$, α is any positive real, i.e., there is no restriction on α.

Theorem 3.2. *Let $\theta = (k_r)$ be a lacunary sequence, f be any modulus function, $u = (u_k)$ be a sequence strictly positive real numbers and α is any positive real. Then the sequence spaces $w_0^{\alpha}(f, \theta, \Delta_v^m, u)$, $w^{\alpha}(f, \theta, \Delta_v^m, u)$ and $w_\infty^{\alpha}(f, \theta, \Delta_v^m, u)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We prove the theorem only for the space $w^{\alpha}(f, \theta, \Delta_v^m, u)$ and for the other spaces it will follow on applying similar argument. Let $x, y \in w^{\alpha}(f, \theta, \Delta_v^m, u)$ and $\lambda, \mu \in \mathbb{C}$. Then there exist positive integers K_λ and M_μ such that $|\lambda| \le K_\lambda$ and $|\mu| \le M_\mu$, we have

$$\frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|u_k \Delta^m (\lambda x_k + \mu y_k) - (\lambda L_1 + \mu L_2)|)$$

$$\le \quad K_\lambda \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|u_k \Delta^m x_k - L_1|) + M_\mu \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|u_k \Delta^m y_k - L_2|)$$

$\rightarrow \quad 0$ as $r \rightarrow \infty$.

Therefore, $(\lambda x_k + \mu y_k) \in w^\alpha(f, \theta, \Delta_v^m, u)$. Hence the space $w^\alpha(f, \theta, \Delta_v^m, u)$ is linear. □

Theorem 3.3. *Let $\theta = (k_r)$ be a lacunary sequence and f be any modulus function. Then*

(1) *for a positive real number α, $w_0^\alpha(f, \theta, \Delta_v^m, u) \subset w_\infty^\alpha(f, \theta, \Delta_v^m, u)$.*

(2) *for $\alpha \geq 1$, $w^\alpha(f, \theta, \Delta_v^m, u) \subset w_\infty^\alpha(f, \theta, \Delta_v^m, u)$.*

Proof. We prove only for the second inclusion and for the first it will follow on applying similar argument. Let $x \in w^\alpha(f, \theta, \Delta_v^m, u)$. Then, we have

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|u_k \Delta_v^m x_k|) \quad \leq \quad \frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|u_k \Delta_v^m x_k - L|) + f(|L|) \frac{1}{h_r^\alpha} \sum_{k \in I_r} 1.$$

Since $\alpha \geq 1$ and $x \in w^\alpha(f, \theta, \Delta_v^m, u)$, we have $x \in w_\infty^\alpha(f, \theta, \Delta_v^m, u)$. This completes the proof. □

Theorem 3.4. *Consider $\theta = (k_r)$ be a lacunary sequence, f be any modulus function and $\alpha \geq 1$. Then we have $w^\alpha(\theta, \Delta_v^m, u) \subset w^\alpha(f, \theta, \Delta_v^m, u)$, $w_0^\alpha(\theta, \Delta_v^m, u) \subset w_0^\alpha(f, \theta, \Delta_v^m, u)$ and $w_\infty^\alpha(\theta, \Delta_v^m, u) \subset w_\infty^\alpha(f, \theta, \Delta_v^m, u)$.*

Proof. We prove only the last inclusion and the first two inclusions are easily proved. Let $x \in w_\infty^\alpha(\theta, \Delta_v^m, u)$. Hence, we have

$$\sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} |u_k \Delta_v^m x_k| < \infty.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t \leq \delta$. Suppose $\frac{1}{h_r^\alpha} \sum_{k \in I_r} f(|u_k \Delta_v^m x_k|) = \sum_1 + \sum_2$, where the first summation is over $|u_k \Delta_v^m x_k| \leq \delta$ and the second summation is over $|u_k \Delta_v^m x_k| > \delta$. Then $\sum_1 \leq \epsilon \frac{1}{h_r^{\alpha - 1}}$ and for $|u_k \Delta_v^m x_k| > \delta$ we use the fact that

$$|u_k \Delta_v^m x_k| < |u_k \Delta_v^m x_k| / \delta < 1 + [|u_k \Delta_v^m x_k| / \delta],$$

where $[t]$ denotes the integral part of t. Thus, by definition of modulus function we have for $|u_k \Delta_v^m x_k| > \delta$,

$$f(|u_k \Delta_v^m x_k|) \leq (1 + [|u_k \Delta_v^m x_k| / \delta]) f(1) \leq 2f(1) |u_k \Delta_v^m x_k| / \delta.$$

Hence, $\sum_2 \leq 2f(1)\delta^{-1}\dfrac{1}{h_r^\alpha}\sum_{k\in I_r}|u_k\Delta_v^m x_k|$ with $\sum_1 \leq \epsilon\dfrac{1}{h_r^{\alpha-1}}$ gives

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k|) \quad\leq\quad \epsilon\frac{1}{h_r^{\alpha-1}} + 2f(1)\delta^{-1}\frac{1}{h_r^\alpha}\sum_{k\in I_r}|u_k\Delta_v^m x_k|.$$

Since $\alpha \geq 1$ and $x \in w_\infty^\alpha(\theta, \Delta_v^m, u)$, we have $x \in w_\infty^\alpha(f, \theta, \Delta_v^m, u)$. This completes the proof. □

Theorem 3.5. *Let f be a modulus function and α be a positive real number. If $\lim_{t\to\infty}\dfrac{f(t)}{t} > 0$, then $w^\alpha(f, \theta, \Delta_v^m, u) \subset w^\alpha(\theta, \Delta_v^m, u)$.*

Proof. Since $\beta = \lim_{t\to\infty}\dfrac{f(t)}{t} = \inf\{f(t)/t; t > 0\}$. Thus, by definition of β we have $f(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq \beta^{-1}f(t)$ for all $t \geq 0$. Hence,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}|u_k\Delta_v^m x_k - L| \leq \beta^{-1}\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k - L|).$$

It follows that $x \in w^\alpha(\theta, \Delta_v^m, u)$ whenever $x \in w^\alpha(f, \theta, \Delta_v^m, u)$. □

Theorem 3.6. *Let f be a modulus function and α be a positive real number. If $\lim_{t\to\infty}\dfrac{f(t)}{t} > 0$ and $\alpha \geq 1$, then $w^\alpha(f, \theta, \Delta_v^m, u) = w^\alpha(\theta, \Delta_v^m, u)$.*

Proof. On combining Theorem (3.4) and (3.5), we get the proof of the theorem. □

Theorem 3.7. *Let $\theta = (k_r)$ be lacunary sequence, f be any modulus function and α be a positive real number. Then*

(1) $l_\infty \subset M(w_\infty^\alpha(f, \theta, \Delta_v^m, u))$,

(2) $M(w_\infty^\alpha(f, \theta, \Delta_v^m, u)) \subset w_\infty^\alpha(f, \theta, \Delta_v^m, u)$ *if* $\alpha \geq 1$,

(3) *If f is bounded and $\alpha \geq 1$, then* $M(w_\infty^\alpha(f, \theta, \Delta_v^m, u)) = w_\infty^\alpha(f, \theta, \Delta_v^m, u) = s$.

Proof. (1) Let $b = (b_k) \in l_\infty$ which implies that $|b_k| < 1 + [H]$ for some $H > 0$ and for all k. Hence,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|b_k u_k\Delta_v^m x_k|)$$

$$\leq (1+[H])\frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k|), \text{ for all } x \in w_\infty^\alpha(f,\theta,\Delta_v^m,u),$$

which gives the first inclusion.

(2) The second inclusion is given from the fact that $e = (1,1,1,\dots) \in w_\infty^\alpha(f,\theta,\Delta_v^m,u)$ for $\alpha \geq 1$.

(3) Suppose f is bounded and $\alpha \geq 1$, then for any $x = (x_k) \in s$,

$$\frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k|) \leq \sup\{f(y) : y \geq 0\}\frac{1}{h_r^{\alpha-1}}.$$

Hence, $w_\infty^\alpha(f,\theta,\Delta_v^m,u) = s$. Similarly, by (2) and the above argument, we have $M(w_\infty^\alpha(f,\theta,\Delta_v^m,u)) = w_\infty^\alpha(f,\theta,\Delta_v^m,u) = s$. □

Theorem 3.8 ([22])**.** *If E is a sequence space, the following are equivalent:*

(1) *E is normal,*

(2) *$l_\infty \subset M(E)$,*

(3) *$M(E)$ is normal.*

Lemma 3.9 ([10])**.** *Every normal sequence space is monotone.*

By using Theorems 3.7, 3.8 and Lemma 3.9, we have the following result:

Theorem 3.10. *For any modulus function f, lacunary sequence $\theta = (k_r)$ and positive real number α, the spaces $w_\infty^\alpha(f,\theta,\Delta_v^m,u)$ and $M(w_\infty^\alpha(f,\theta,\Delta_v^m,u))$ are normal as well as monotone.*

P r o o f. We shall prove the result for $w_\infty^\alpha(f,\theta,\Delta_v^m,u)$. For $M(w_\infty^\alpha(f,\theta,\Delta_v^m,u))$, the result can be proved similarly. Let $x = x_k \in w_\infty^\alpha(f,\theta,\Delta_v^m,u)$, then we have

$$\sup_r \frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k|) < \infty.$$

Let (α_k) be a sequence scalar with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we get

$$\begin{aligned}\sup_r \frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|\alpha_k u_k\Delta_v^m x_k|) &\leq |\alpha_k| \sup_r \frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k|)\\ &\leq \sup_r \frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k|)\end{aligned}$$

$$< \quad \infty.$$

The spaces $w_\infty^\alpha(f,\theta,\Delta_v^m,u)$ and $M(w_\infty^\alpha(f,\theta,\Delta_v^m,u))$ are monotone follows from Lemma 3.9. □

4. Relation between f-lacunary statistical convergence of order α and strong cesàro summability of order α. In this section, we have studied the relationship between the spaces $S^\alpha(f,\theta,\Delta_v^m,u)$ and $w^\alpha(f,\theta,\Delta_v^m,u)$.

In [12] Maddox showed the existence of an unbounded modulus function f for which there exists a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$.

Theorem 4.1. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq \beta \leq 1$. Let f be an unbounded modulus function there is a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Cesàro summable of order α with respect to f and θ to L, then it is f-lacunary statistically convergent of order β to L.*

P r o o f. For any sequence $x = (x_k)$ and $\epsilon > 0$ and by definition of modulus function, we have

$$\begin{aligned}
\sum_{k\in I_r} f(|u_k\Delta_v^m x_k - L|) &\geq f\left(\sum_{k\in I_r} |u_k\Delta_v^m x_k - L|\right) \\
&\geq f(|\{k \in I_r : |u_k\Delta_v^m x_k - L| \geq \epsilon\}|\epsilon) \\
&\geq cf(|\{k \in I_r : |u_k\Delta_v^m x_k - L| \geq \epsilon\}|)f(\epsilon)
\end{aligned}$$

and since $\alpha \leq \beta$,

$$\begin{aligned}
\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|u_k\Delta_v^m x_k - L|) &\geq \frac{cf(|\{k \in I_r : |u_k\Delta_v^m x_k - L| \geq \epsilon\}|)f(\epsilon)}{h_r^\alpha} \\
&\geq \frac{cf(|\{k \in I_r : |u_k\Delta_v^m x_k - L| \geq \epsilon\}|)f(\epsilon)}{h_r^\beta} \\
&= \frac{cf(|\{k \in I_r : |u_k\Delta_v^m x_k - L| \geq \epsilon\}|)f(\epsilon)f(h_r^\beta)}{h_r^\beta f(h_r^\beta)}.
\end{aligned}$$

Using the fact that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $x \in w^\alpha(f,\theta,\Delta_v^m,u)$, it follows that $x \in S^\beta(f,\theta,\Delta_v^m,u)$. This completes the proof. □

If we take $\beta = \alpha$ in Theorem 4.1, we have the following result:

Corollary 4.2. *Let $\theta = (k_r)$ be lacunary sequence and $0 < \alpha \leq 1$. Let for an unbounded modulus function f, there is a positive constant c such that $f(xy) \geq cf(x)f(y)$, for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Cesàro summable of order α with respect to f and θ to L, then it is f-lacunary statistically convergent of order α to L.*

If we take $\alpha = 1$ in Corollary 4.2. Then, we have

Corollary 4.3. *Let $\theta = (k_r)$ be a lacunary sequence and for an unbounded modulus function f, there is a positive constant c such that $f(xy) \geq cf(x)f(y)$, for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Cesàro summable with respect to f and θ to L, then it is f-lacunary statistically convergent to L.*

Remark 4.4. If we take $\theta = (2^r)$, $f(x) = x$, $m = 0, (v_k) = (1, 1, 1, \dots)$ and $(u_k) = 1$, for all k in Theorem 4.1, we have Theorem 3.8 of Çolak [2] for the case $p = 1$.

Theorem 4.5. *Let $\theta = (k_r)$ be a lacunary sequence and f be a modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\alpha \in (0, 1]$. If a sequence is strongly Cesàro summable of order α with respect to f and θ to L, then it is lacunary statistically convergent of order α to L.*

If we take $\theta = (2^r)$, $\alpha = 1$, $m = 0, (v_k) = (1, 1, 1, \dots)$ and $(u_k) = 1$, for all k in Theorem 4.5, we have the following result which is a particular case of part (a) of Theorem 8 of Connor [3].

Corollary 4.6. *Let f be a modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Cesàro summable with respect to f to L, then it is statistically convergent to L.*

By taking $\theta = (2^r)$, $\alpha = 1$, $f(x) = x$, $m = 0, (v_k) = (1, 1, 1, \dots)$ and $(u_k) = 1$, for all k in Theorem 4.5, we obtain the following result, which is in Theorem 2.1 of Connor [4], for the case $q = 1$.

Corollary 4.7. *If a sequence is strongly Cesàro summable to L, then it is statistically convergent to L.*

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