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ON CAUCHY'S RESIDUE THEOREM IN $\mathbb{R}^{N*}$

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Abstract. A method of spatial (integral) differentiation of multivector fields in a k-dimensional hyper-rectangle $[a, b]$ immersed in $\mathbb{R}^N$ has been introduced. For a class of discontinuous multivector fields a new concept of a residual field as well as the concept of total $\mathcal{KH}$-integrability have been defined. Finally, this has led naturally to an extension of Cauchy's residue theorem in $\mathbb{R}^N$.

1. Introduction. The Kurzweil–Henstock integral [2, 4], defined using Riemann sums, and with certain modification of the fineness of a partition of the interval $[a, b] \subset \mathbb{R}$, was the first generalized Riemann integral. This integral is equivalent to the integrals of Denjoy and Peron, [4]. The McShane integral [3], which is equivalent to the Lebesque integral [4], was the second generalized Riemann integral. In contrast to the one-dimensional case, the Kurzweil–Henstock integral in $\mathbb{R}^N$ does not integrate all derivatives (see Pfeffer [11]). To remove this

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flow Mawhin [10] added a condition restricting the class of admissible partitions of an N-dimensional interval. This led to another Riemann type integral named regular partition integral, which would integrate all derivatives in $\mathbb{R}^N$. Macdonald [9] used the regular partition integral to overcome the deficiency in Hestenes' proof of Stokes' theorem, [5]. Sarić [12] defined a new integral named the total Kurzweil–Henstock integral. This integral solves the problem in formulating the fundamental theorem of calculus in $\mathbb{R}$ whenever a primitive F is defined at the endpoints of $[a,b] \subset \mathbb{R}$. Accordingly, in what follows, we shall try to extend Cauchy's integral formula to a k-dimensional hyper-rectangle $[a,b]$ immersed in $\mathbb{R}^N$, for a large scale class of multivector fields F, and in the spirit of Hestenes' appealing proof. To do this, we must first define a so-called spatial (integral) derivative of F in $[a,b]$. After that, it remains to define an integral that would integrate this derivative.

2. Preliminaries. The ambient space of this note is the N-dimensional Euclidean space $\mathbb{R}^N$. The measure of a set E in $\mathbb{R}^N$, denoted by $|E|$, is the Lebesgue outer measure. Let $(\boldsymbol{e}_i)_{i=1}^N = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N)$ be the standard orthonormal basis for $\mathbb{R}^N$. With the Cartesian coordinate system every point x in $\mathbb{R}^N$ has an ordered set $\left(x^i\right)_{i=1}^N$ associated with it. We work in $\mathbb{R}^N$ with the usual inner (dot) product $x \cdot y = \sum_{i=1}^N x^i y^i$ and the associated Euclidean norm $\|\cdot\|$. Given a positive integer $\hat{N} \leq N$, by a brick $[a,b]$ in $\mathbb{R}^{\hat{N}}$ we mean an $\hat{N}$-dimensional hyper-rectangle (also called an orthotope) formally defined as the Cartesian product of $\hat{N}$ non-degenerate compact intervals $\left[a^{\hat{n}}, b^{\hat{n}}\right] \in \mathbb{R}$ $(\hat{n} = 1, 2, \ldots, \hat{N})$. In symbols,

$$[a,b] := \prod_{\hat{n}=1}^{\hat{N}} \left[a^{\hat{n}}, b^{\hat{n}}\right] = \left[a^1, b^1\right] \times \left[a^2, b^2\right] \times \cdots \times [a^{\hat{N}}, b^{\hat{N}}].$$

The collection $\mathcal{I}([a,b])$ is a family of all non-degenerate compact hyper-rectangles $I = \prod_{\hat{n}=1}^{\hat{N}} \left[u^{\hat{n}}, v^{\hat{n}}\right]$ such that $\left[u^{\hat{n}}, v^{\hat{n}}\right] \subseteq \left[a^{\hat{n}}, b^{\hat{n}}\right]$, [8].

By $\mathfrak{L}(\hat{N})$ we denote the set of all multi-indices $\boldsymbol{\hat{\imath}} = (i_{\hat{n}})_{\hat{n}=1}^{\hat{N}} = (i_1, i_2, \ldots, i_{\hat{N}})$ with $i_{\hat{n}} = 1, 2, \ldots, \nu_{\hat{n}}$ for each $\hat{n} = 1, 2, \ldots, \hat{N}$. The product partition of $[a,b] \in \mathbb{R}^{\hat{N}}$, denoted by $P[a,b]$, is a finite collection of all orthotope-point pairs

$([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}})$ such that

$$[a_{\hat{\imath}}, b_{\hat{\imath}}] = \prod_{\hat{n}=1}^{\hat{N}} [a^{\hat{n}}_{i_{\hat{n}}}, b^{\hat{n}}_{i_{\hat{n}}}] = [a^1_{i_1}, b^1_{i_1}] \times [a^2_{i_2}, b^2_{i_2}] \times \cdots \times [a^{\hat{N}}_{i_{\hat{n}}}, b^{\hat{N}}_{i_{\hat{n}}}]$$

and $x_{\hat{\imath}} = (x^{\hat{n}}_{i_{\hat{n}}})^{\hat{N}}_{\hat{n}=1} = (x^1_{i_1}, x^2_{i_2}, \ldots, x^{\hat{N}}_{i_{\hat{n}}})$, for each $\hat{\imath} \in \mathfrak{L}(\hat{N})$. In addition, for each $\hat{n} = 1, 2, \ldots, \hat{N}$, the intervals $[a^{\hat{n}}_{i_{\hat{n}}}, b^{\hat{n}}_{i_{\hat{n}}}]$ are non-overlapping (they have pairwise disjoint interiors), $\bigcup_{i_{\hat{n}}=1}^{\nu_{\hat{n}}} [a^{\hat{n}}_{i_{\hat{n}}}, b^{\hat{n}}_{i_{\hat{n}}}] = \left[a^{\hat{n}}, b^{\hat{n}}\right]$ and $x^{\hat{n}}_{i_{\hat{n}}} \in [a^{\hat{n}}_{i_{\hat{n}}}, b^{\hat{n}}_{i_{\hat{n}}}]$. It is evident that a given $P([a,b])$ can be tagged in infinitely many ways by choosing different points as tags. If E is a set of points belonging to $[a,b]$, then the restriction of $P([a,b])$ to E is a finite collection of $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a,b])$ such that each pair of $[a_{\hat{\imath}}, b_{\hat{\imath}}]$ and E intersects in at least one point and all $x_{\hat{\imath}}$ are tagged in E. In symbols,

$$P([a,b])\,|_E = \{([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a,b]) \mid x_{\hat{\imath}} \in [a_{\hat{\imath}}, b_{\hat{\imath}}] \cap E \neq \emptyset \;\text{ and }\; \hat{\imath} \in \mathfrak{L}(\hat{N})\}.$$

The distance of the $[a_{\hat{\imath}}, b_{\hat{\imath}}]$, denoted by $\operatorname{diam}([a_{\hat{\imath}}, b_{\hat{\imath}}])$, is defined as follows:

$$\operatorname{diam}([a_{\hat{\imath}}, b_{\hat{\imath}}]) = \sup\{\|x-y\| : x, y \in [a_{\hat{\imath}}, b_{\hat{\imath}}]\},$$

where $\|x-y\|$ is computed using the Euclidean norm of a vector in $\mathbb{R}^{\hat{N}}$. Given $\delta : [a,b] \to (0,1)$, named a gauge, a product partition

$$P([a,b]) = \{([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \mid \hat{\imath} \in \mathfrak{L}(\hat{N})\}$$

is called *δ-fine* if $\operatorname{diam}([a_{\hat{\imath}}, b_{\hat{\imath}}]) \leq \delta(x_{\hat{\imath}})$ for every $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a,b])$. Let $\mathcal{P}([a,b])$ be the family of all product partitions $P([a,b])$ of $[a,b]$. For $E \subset [a,b]$ we denote by $\mathcal{P}_\delta([a,b])\,|_E$ the family of all *δ-fine* product partitions $P([a,b]) \in \mathcal{P}([a,b])$ of $[a,b]$, such that $P([a,b])\,|_E \subset P([a,b])$.

In addition to the above mentioned inner product we work in $\mathbb{R}^{\hat{N}}$ with the geometric product too, [7]. The geometric product of vectors in $\mathbb{R}^{\hat{N}}$, which can be decomposed into the symmetric inner and anti-symmetric outer (wedge) product, has the following properties: associativity, distributivity, and $\boldsymbol{v}\boldsymbol{v} = \boldsymbol{v}^2 = \|\boldsymbol{v}\|^2$. Although the vector space $\mathbb{R}^{\hat{N}}$ is closed under vector addition, it is not closed under multiplication. Instead, by multiplication and addition the vectors of $\mathbb{R}^{\hat{N}}$ generate a larger linear space $\mathcal{G}(\mathbb{R}^{\hat{N}})$ called the geometric algebra of $\mathbb{R}^{\hat{N}}$. Given an integer $k \leq \hat{N}$, we denote by $\mathfrak{I}(\hat{N}, k)$ the set of all multi-indices $\boldsymbol{i}_k = (i_j)^k_{j=1}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq \hat{N}$, and for every $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N}, k)$ the geometric product $\boldsymbol{e}_{\boldsymbol{i}_k} = \boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_k}$ of the orthonormal basis vectors $\{\boldsymbol{e}_{i_j}\}^k_{j=1}$ is reduced to the outer product

$$\Lambda^k_{j=1}\boldsymbol{e}_{i_j} = \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2} \wedge \cdots \wedge \boldsymbol{e}_{i_k},$$

since the inner product $\Theta_{j=1}^{k}\boldsymbol{e}_{i_j} = \boldsymbol{e}_{i_1}\cdot\boldsymbol{e}_{i_2}\cdot\;\cdots\;\cdot\boldsymbol{e}_{i_k}$ vanishes (there are no repeated factors in the product). The outer product is completely determined by the following properties: associativity, linearity in both arguments, $\boldsymbol{e}_i\wedge\boldsymbol{e}_j = -\boldsymbol{e}_j\wedge\boldsymbol{e}_i$ for every $i \neq j$, and $\boldsymbol{e}_i\wedge\boldsymbol{e}_i = 0$ for every i, [1]. A k-vector in $\mathbb{R}^{\hat{N}}$ is any formal linear combination $\sum\limits_{\boldsymbol{i}_k\in\Im(\hat{N},k)} \alpha_{\boldsymbol{i}_k}\boldsymbol{e}_{\boldsymbol{i}_k}$ with $\alpha_{\boldsymbol{i}_k} \in \mathbb{R}$ for every $\boldsymbol{i}_k \in \Im(\hat{N},k)$. The space of k-vectors is denoted by $\mathcal{G}_k(\mathbb{R}^{\hat{N}})$. In particular, $\mathcal{G}_1(\mathbb{R}^{\hat{N}}) = \mathbb{R}^{\hat{N}}$. For reasons of formal convenience, we set $\mathcal{G}_0(\mathbb{R}^{\hat{N}}) := \mathbb{R}$ and $\mathcal{G}_k(\mathbb{R}^{\hat{N}}) := \{0\}$ for $k > \hat{N}$. As $\mathcal{G}(\mathbb{R}^{\hat{N}}) = \sum\limits_{k=0}^{\hat{N}} \mathcal{G}_k(\mathbb{R}^{\hat{N}})$, the elements $\mathcal{F}$ of $\mathcal{G}(\mathbb{R}^{\hat{N}})$ are called multivectors which can be expressed uniquely as a sum of its k-vector parts. The norm of $\mathcal{F} \in \mathcal{G}(\mathbb{R}^{\hat{N}})$ is defined by

$$\|\mathcal{F}\| = \left(\sum_{k=0}^{\hat{N}} \mathcal{F}_k^{\dagger}\mathcal{F}_k\right)^{1/2} \geq 0,$$

where $\mathcal{F}_k^{\dagger}$ is reverse (or adjoint) of $\mathcal{F}$, [5]. A simple k-vector is the outer product of k linearly independent 1-vectors, that is, $v = \boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\cdots\wedge\boldsymbol{v}_k$, [1]. Since k linearly independent vectors also span a k-dimensional subspace V of $\mathbb{R}^{\hat{N}}$, it is apparent that to every simple k-vector it corresponds a unique k-dimensional subspace of $\mathbb{R}^{\hat{N}}$. In fact, every simple k-vector can be interpreted geometrically as an oriented volume of some k-dimensional subspace of $\mathbb{R}^{\hat{N}}$, [5]. Hence, if $[a,b]_{\boldsymbol{i}_k}$ is the projection of $[a,b]$ onto the Euclidean vector space spanned by $\{\boldsymbol{e}_{i_j}\}_{j=1}^{k}$, then for any $\boldsymbol{i}_k \in \Im(\hat{N},k)$ the simple k-vector $\mathbb{I}_{\boldsymbol{i}_k} = \Lambda_{j=1}^{k}(-1)^{i_j - j}\,\boldsymbol{e}_{i_j}$ is an orientation of $[a,b]_{\boldsymbol{i}_k}$ determined by the convention

$$\mathbb{I}_{\boldsymbol{i}_{\hat{N}}} = \boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\cdots\wedge\boldsymbol{e}_{\hat{N}} = \mathbb{I}_{\boldsymbol{i}_k}\wedge\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\cdots\wedge\underbrace{\overset{\wedge}{\boldsymbol{e}}_{i_j}}_{i_j\in\boldsymbol{i}_k}\wedge\cdots\wedge\boldsymbol{e}_{\hat{N}},$$

where the hat indicates that $\overset{\wedge}{\boldsymbol{e}}_{i_j}$ is omitted for each $i_j \in \boldsymbol{i}_k$, and

$$\mathbb{I}_{\boldsymbol{i}_k}\triangle x_{\boldsymbol{i}_k} = \Lambda_{j=1}^{k}(-1)^{i_j - j}\,\boldsymbol{e}_{i_j}\triangle x^{i_j}$$

is the oriented hyper-rectangle's volume of $I_{\boldsymbol{i}_k} \in \mathcal{I}(\,[a,b]_{\boldsymbol{i}_k})$, where

$$(\triangle x^{i_j})_{j=1}^{k} = \left(\triangle x^{i_1}, \triangle x^{i_2}, \ldots, \triangle x^{i_k}\right)$$

is an ordered set of the corresponding coordinate intervals and $\triangle x_{\boldsymbol{i}_k}$ is the Lebesgue outer measure $|I_{\boldsymbol{i}_k}|$ of $I_{\boldsymbol{i}_k}$.

A multivector field $f : [a,b] \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ associates with each point $x \in [a,b]$ a multivector $f(x) \in \mathcal{G}(\mathbb{R}^{\hat{N}})$. By an orthotope field on $\mathcal{I}([a,b])$ we mean $\mathcal{F} : \mathcal{I}([a,b]) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ which associates with each orthotope $I \in \mathcal{I}([a,b])$ a multivector $\mathcal{F}(I) \in \mathcal{G}(\mathbb{R}^{\hat{N}})$. Such an orthotope field is called additive on $\mathcal{I}([a,b])$ if for any $I \in \mathcal{I}([a,b])$ and any collection of non-overlapping orthotopes $(I_i)_{i=1}^n$, whose union is I, the equality $\mathcal{F}(I) = \sum_{i=1}^{n} \mathcal{F}(I_i)$ holds. There are many different ways to define the limit of an orthotope field. The definition given below is one of them.

Definition 1. *Let* $\mathcal{F} : \mathcal{I}([a,b]) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ *be an orthotope field and* $E \subset [a,b]$. *A multivector field* f *is the limit of* $\mathcal{F}$ *on* $[a,b] \setminus E$ *if for every* $\varepsilon > 0$ *there exists a gauge* δ *on* $[a,b]$ *such that*

$$\|\mathcal{F}([a_{\hat{\imath}}, b_{\hat{\imath}}]) - f(x_{\hat{\imath}})\| < \varepsilon,$$

whenever $([a_{\hat{\imath}}, b_{\hat{\imath}}], x_{\hat{\imath}}) \in P([a,b]) \setminus P([a,b])|_E$ *and* $P([a,b]) \in \mathcal{P}_\delta([a,b])|_E$. *In symbols,* $f(x) = \lim_{I \to x} \mathcal{F}(I)$.

If $\mathcal{F}$ converges to its limit f almost everywhere on $[a,b]$, which means for every $x \in [a,b]$ except for a set $E \subset [a,b]$ of Lebesgue outer measure zero, then the domain of f may not be all of $[a,b]$. If the set E is countable, then $\mathcal{F}$ is said to converge to f nearly everywhere on $[a,b]$. Now, we are in a position to define a differential k-form $d\mathcal{F}_{\boldsymbol{i}_k} : [a,b]_{\boldsymbol{i}_k} \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ on $[a,b]_{\boldsymbol{i}_k}$, as follows.

Definition 2. *For any* $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N}, k)$ *let* $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ *and let a multivector field* f *be the limit of an orthotope field* $\mathcal{F} : \mathcal{I}([a,b]) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ *on* $[a,b]_{\boldsymbol{i}_k} \setminus E_{\boldsymbol{i}_k}$. *Then, a differential* k*-form* $d\mathcal{F}_{\boldsymbol{i}_k} = f dx_{\boldsymbol{i}_k}$ *on* $[a,b]_{\boldsymbol{i}_k}$ *is the limit of* $\triangle\mathcal{F}_{\boldsymbol{i}_k} = \mathcal{F}\triangle x_{\boldsymbol{i}_k}$ *on* $[a,b]_{\boldsymbol{i}_k}$. *In symbols,* $d\mathcal{F}_{\boldsymbol{i}_k}(x_{\boldsymbol{i}_k}) = \lim_{I_{\boldsymbol{i}_k} \to x_{\boldsymbol{i}_k}} \triangle\mathcal{F}_{\boldsymbol{i}_k}(I_{\boldsymbol{i}_k})$.

In spite of the fact that the limit of $\triangle x_{\boldsymbol{i}_k}$ vanishes identically on $[a,b]_{\boldsymbol{i}_k}$, a differential k-form $d\mathcal{F}_{\boldsymbol{i}_k} = f dx_{\boldsymbol{i}_k}$ could be a null multivector field on $[a,b]_{\boldsymbol{i}_k}$. (A multivector field $F : [a,b] \to \mathcal{G}(R^{\hat{N}})$ is said to be a null multivector field on $[a,b]_{\boldsymbol{i}_k}$ if the set $\{x_{\boldsymbol{i}_k} \in [a,b]_{\boldsymbol{i}_k} \mid F(x_{\boldsymbol{i}_k}) \neq 0\}$ is a set of Lebesgue outer measure zero, see [2, 2.4 Definition]). In what follows, we shall use the notations

$$\sum_{\hat{\imath} \in \mathfrak{L}(k)} \mathbb{I}_{\boldsymbol{i}_k}(x_{\hat{\imath}}) \triangle\mathcal{F}_{\boldsymbol{i}_k}([a_{\hat{\imath}}, b_{\hat{\imath}}]) = \Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k}))$$

and

$$\sum_{\hat{\imath}\in\mathfrak{L}(k)} \mathbb{I}_{\boldsymbol{i}_k}(x_{\hat{\imath}})f(x_{\hat{\imath}})\,|[a_{\hat{\imath}},b_{\hat{\imath}}]| = \Xi_{\mathbb{I}f\triangle x}(P([a,b]_{\boldsymbol{i}_k})),$$

whenever $([a_{\hat{\imath}},b_{\hat{\imath}}]\,,x_{\hat{\imath}}) \in P([a,b]_{\boldsymbol{i}_k})$.

The following definition of the Kurzweil–Henstock directed integral of an oriented differential k-form $\mathbb{I}_{\boldsymbol{i}_k}d\mathcal{F}_{\boldsymbol{i}_k} = \mathbb{I}_{\boldsymbol{i}_k}f dx_{\boldsymbol{i}_k}$ comes from [12] and [5].

Definition 3. *For a multivector field $f : [a,b] \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ a multivector $\mathcal{L}_{\boldsymbol{i}_k} \in \mathcal{G}(\mathbb{R}^{\hat{N}})$ is the Kurzweil–Henstock directed integral of an oriented differential k-form $\mathbb{I}_{\boldsymbol{i}_k}f dx_{\boldsymbol{i}_k}$ over $[a,b]_{\boldsymbol{i}_k}$, if for every $\varepsilon>0$ there exists a gauge δ on $[a,b]_{\boldsymbol{i}_k}$ such that*

$$\left\|\Xi_{\mathbb{I}f\triangle x}(P([a,b]_{\boldsymbol{i}_k})) - \mathcal{L}_{\boldsymbol{i}_k}\right\| < \varepsilon,$$

whenever $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}_\delta([a,b]_{\boldsymbol{i}_k})$. In symbols, $\mathcal{L}_{\boldsymbol{i}_k} = \mathcal{KH}\!-\!\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k}f dx_{\boldsymbol{i}_k}$.

When working with multivector fields, which have a finite number of discontinuities on $[a,b]$, it does not matter how these fields are defined on the set of discontinuities $E \subset [a,b]$. The validity of this statement will be clarified as the theory unfolds. As this situation will arise frequently, we adopt the convention that, unless mentioned otherwise, such multivector fields are equal to 0 at all points at which they can take the infinite values or not be defined at all. Accordingly, we may define a multivector field $D_{ex}f : [a,b] \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ by extending f from $[a,b]\setminus E$ to E by $D_{ex}f\,(x) = 0$ for $x \in E$, so that

$$D_{ex}f\,(x) = \begin{cases} f\,(x)\,, & \text{if } x \in [a,b]\setminus E \\ 0, & \text{if } x \in E \end{cases}.$$

3. Main results. Let for $\mathcal{F} : \mathcal{I}(\,[a,b]) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ the limit f of $\mathcal{F}$ is defined on $[a,b]_{\boldsymbol{i}_k}$. Then, it follows from Definitions 1 and 2 that for every $\varepsilon>0$ there exists a gauge δ on $[a,b]_{\boldsymbol{i}_k}$ such that

$$\left\|\Xi_{\mathbb{I}f\triangle x}(P([a,b]_{\boldsymbol{i}_k})) - \Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k}))\right\| < \varepsilon\,|[a_{\hat{\imath}},b_{\hat{\imath}}]|\,, \tag{3.1}$$

whenever $([a_{\hat{\imath}},b_{\hat{\imath}}]\,,x_{\hat{\imath}}) \in P([a,b]_{\boldsymbol{i}_k})$ and $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}_\delta([a,b]_{\boldsymbol{i}_k})$. Consequently,

$$\mathcal{KH}\!-\!\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k}d\mathcal{F}_{\boldsymbol{i}_k} = \mathcal{KH}\!-\!\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k}f dx_{\boldsymbol{i}_k}.$$

In the opposite case, when the multivector field f, as the limit of $\mathcal{F}$ on $[a,b]_{\boldsymbol{i}_k}$, at some points of $[a,b]_{\boldsymbol{i}_k}$, can take infinite values or not be defined at all, the Kurzweil–Henstock directed integrals of $\mathbb{I}_{\boldsymbol{i}_k} f dx_{\boldsymbol{i}_k}$ and $\mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k}$ over $[a,b]_{\boldsymbol{i}_k}$ can be distinguished from each other. It would be reasonable to make use of $\Xi_{\mathbb{I}\triangle\mathcal{F}}(P_n([a,b]_{\boldsymbol{i}_k}))$ instead of $\Xi_{\mathbb{I}f\triangle x}(P_n([a,b]_{\boldsymbol{i}_k})$ to define an integral of both $\mathbb{I}_{\boldsymbol{i}_k} f dx_{\boldsymbol{i}_k}$ and $\mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k}$ over $[a,b]_{\boldsymbol{i}_k}$. This is obviously our way of attempting to totalize the Kurzweil–Henstock directed integral afore defined. The definition of the total Kurzweil–Henstock directed integral which follows is more general one since it includes one more manifold field.

Definition 4. *Let for any $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N},k)$ the sets $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ and $G_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ be the disjoint sets of points at each of which, respectively, the limits f and g of orthotope fields $\mathcal{F} : \mathcal{I}([a,b]) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ and $\wp : \mathcal{I}([a,b]) \to \mathcal{G}(\mathbb{R}^{\hat{N}})$, can take infinite values or not be defined at all. A multivector $\mathcal{L}_{\boldsymbol{i}_k} \in \mathcal{G}(\mathbb{R}^{\hat{N}})$ is the total Kurzweil–Henstock directed integral of $g\mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k} = g\mathbb{I}_{\boldsymbol{i}_k} f dx_{\boldsymbol{i}_k}$ over $[a,b]_{\boldsymbol{i}_k}$ if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]_{\boldsymbol{i}_k}$ such that*

$$\left\| \Xi_{\wp\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k})) - \mathcal{L}_{\boldsymbol{i}_k} \right\| < \varepsilon,$$

whenever $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}_\delta([a,b]_{\boldsymbol{i}_k})$. In symbols, $\mathcal{L}_{\boldsymbol{i}_k} = \mathcal{KH}{-}vt \int_{[a,b]_{\boldsymbol{i}_k}} g\mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k}$.

The crucial advantage of the integration process established by Definition 4, in comparison with any other integration process defined until now, including all the generalized Riemann approach to integration, lies in the fact that it is not necessary that g and $d\mathcal{F}_{\boldsymbol{i}_k}$, as the limits of $\wp$ and $\triangle\mathcal{F}_{\boldsymbol{i}_k}$, respectively, to be defined at all points of $[a,b]_{\boldsymbol{i}_k}$. This fact, upon which our theory is based, in what follows, gives us the possibility to include the calculus of residues in the process of integration of multivector fields in $[a,b]_{\boldsymbol{i}_k}$. Before that, we are prepared to prove the extended version of the fundamental theorem of calculus in $[a,b]_{\boldsymbol{i}_k}$. As we shall see, the proof becomes trivial if the definition of the total Kurzweil–Henstock directed integral is applied. In fact, in this way, we shall attempt to put into a rigorous form Hestenes' proof based on the integral definition of the derivative and the Riemann integral, [5]. A major motivation for the formulation of integration in this manuscript has been to achieve as simple and general a statement of the fundamental theorem as possible, just as was Hestenes' motivation too.

For any $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N},k)$ a hyper-rectangle $[a,b]_{\boldsymbol{i}_k}$ has k pairs of the boundary faces of $(k-1)$-dimension with corresponding set of opposite orientations $\{\pm\mathbb{I}^j_{\boldsymbol{i}_k-1}\}_{j=1}^k$ determined by the convention $\mathbb{I}_{\boldsymbol{i}_k} = \mathbb{I}^j_{\boldsymbol{i}_k-1} \wedge \boldsymbol{e}_{i_j}$. This implies that

$$\mathbb{I}^j_{\boldsymbol{i}_k-1} = (-1)^{i_1-1}\,\boldsymbol{e}_{i_1} \wedge \cdots \wedge (-1)^{i_j-j}\,\overset{\wedge}{\boldsymbol{e}_{i_j}} \wedge \cdots \wedge (-1)^{i_k-j}\,\boldsymbol{e}_{i_k}.$$

Let $\partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$ be the family of the boundaries $\boldsymbol{\partial} I_{\boldsymbol{i}_k}$ of $I_{\boldsymbol{i}_k} \in \mathcal{I}([a,b]_{\boldsymbol{i}_k})$ and for an orthotope fields $\Phi_{i_j} : [a,b] \to \mathcal{G}(\mathbb{R}^{\hat{N}})$ let F_{i_j} be the limit of Φ_{i_j} on $[a,b]_{\boldsymbol{i}_k} \setminus (E_{\boldsymbol{i}_k})_{i_j}$, where $(E_{\boldsymbol{i}_k})_{i_j} \subset [a,b]_{\boldsymbol{i}_k}$. Given $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N},k)$, any formal linear combination

$$\mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k} = \sum_{j=1}^{k} \mathbb{I}^j_{\boldsymbol{i}_k-1} F_{i_j} dx^j_{\boldsymbol{i}_k-1},$$

as the limit of $\sum_{j=1}^{k} \mathbb{I}^j_{\boldsymbol{i}_k-1} \Phi_{i_j} \triangle x^j_{\boldsymbol{i}_k-1}$ on $[a,b]_{\boldsymbol{i}_k}$, where $\triangle x^j_{\boldsymbol{i}_k-1} = \triangle x^{i_1} \cdots \widehat{\triangle x^{i_j}} \cdots \triangle x^{i_k}$, is an oriented differential $(k-1)$-form on $[a,b]_{\boldsymbol{i}_k}$. An oriented differential $(k-1)$-form $\mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k}$ on $[a,b]_{\boldsymbol{i}_k}$ is said to be totally $\mathcal{KH}$-directly integrable with respect to $\partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$ if it is totally $\mathcal{KH}$-directly integrable on every $\boldsymbol{\partial} I_{\boldsymbol{i}_k} \in \partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$.

Definition 5. *Let for any $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N},k)$ an oriented differential $(k-1)$-form $\mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k}$ be totally $\mathcal{KH}$-directly integrable with respect to $\partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$. Then F is said to be spatially (integrally) differentiable to f on $[a,b]_{\boldsymbol{i}_k}$, if f is the limit on $[a,b]_{\boldsymbol{i}_k}$ of $\mathcal{F}$ defined by*

$$\mathcal{F}(I_{\boldsymbol{i}_k}) = \frac{\mathbb{I}^\dagger_{\boldsymbol{i}_k}}{\triangle x_{\boldsymbol{i}_k}} \mathcal{KH}{-}vt \int^{\circlearrowleft}_{\boldsymbol{\partial} I_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k}. \tag{3.2}$$

Theorem 1. *For any $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N},k)$ let $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ be a set at whose points the multivector field f, as the limit on $[a,b]_{\boldsymbol{i}_k} \setminus E_{\boldsymbol{i}_k}$ of $\mathcal{F}$ defined by (3.2), can take the infinite values or not be defined at all. Then, the oriented differential k-form $\mathbb{I}_{\boldsymbol{i}_k} f dx_{\boldsymbol{i}_k} = \mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k}$ on $[a,b]_{\boldsymbol{i}_k}$ is totally $\mathcal{KH}$-directly integrable on $[a,b]_{\boldsymbol{i}_k}$ and*

$$\mathcal{KH}{-}vt \int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k} f dx_{\boldsymbol{i}_k} = \mathcal{KH}{-}vt \int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k} \tag{3.3}$$

$$= \mathcal{KH}{-}vt \int^{\circlearrowleft}_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k}.$$

P r o o f. As the orthotope field $\triangle\mathcal{F}_{\boldsymbol{i}_k} = \mathcal{F} \triangle x_{\boldsymbol{i}_k}$ defined by (3.2) is additive and hence

$$\Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k})) = \mathcal{KH}{-}vt \int^{\circlearrowleft}_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k},$$

for each $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}([a,b]_{\boldsymbol{i}_k})$, it follows from Definition 4 that

$$\mathcal{KH}\text{−}vt \int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k} f dx_{\boldsymbol{i}_k} = \mathcal{KH}\text{−}vt \int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k} d\mathcal{F}_{\boldsymbol{i}_k} = \mathcal{KH}\text{−}vt \int^{\circlearrowleft}_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k - 1} F d\sigma_{\boldsymbol{i}_k}.$$

□

In spite of the fact that the differential k-form $d\mathcal{F}_{\boldsymbol{i}_k}$, as the limit of $\mathcal{F}\triangle x_{\boldsymbol{i}_k}$, can take the infinite values at some points of $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ or be a null multivector field on $[a,b]_{\boldsymbol{i}_k}$, in both cases, (3.3) is valid also. All this refers us to the following definitions.

Definition 6. *For any $\boldsymbol{i}_k \in \Im(\hat{N},k)$ let $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$. For an arbitrary multivector field F such that the oriented differential $(k-1)$-form $\mathbb{I}_{\boldsymbol{i}_k - 1} F d\sigma_{\boldsymbol{i}_k}$ on $[a,b]_{\boldsymbol{i}_k}$ is totally $\mathcal{KH}$-directly integrable with respect to $\partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$, the differential k-form $d\mathcal{F}_{\boldsymbol{i}_k}$, as the limit of $\mathbb{I}^{\dagger}_{\boldsymbol{i}_k} \mathcal{KH}\text{−}vt \int^{\circlearrowleft}_{\boldsymbol{\partial} I_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k - 1} F d\sigma_{\boldsymbol{i}_k}$ on $[a,b]_{\boldsymbol{i}_k}$, is basically summable (BS_δ) in $E_{\boldsymbol{i}_k}$ to the sum $\Re$, if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]_{\boldsymbol{i}_k}$ such that*

$$\left\| \Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k}) \Big|_{E_{\boldsymbol{i}_k}}) - \Re \right\| < \varepsilon,$$

whenever $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}_\delta([a,b]_{\boldsymbol{i}_k}) \Big|_{E_{\boldsymbol{i}_k}}$. If in addition $E_{\boldsymbol{i}_k}$ can be written as a countable union of sets on each of which $d\mathcal{F}_{\boldsymbol{i}_k}$ is BS_δ, then $d\mathcal{F}_{\boldsymbol{i}_k}$ is said to be BSG_δ in the set $E_{\boldsymbol{i}_k}$. In symbols, $\Re := \sum_{x_{\boldsymbol{i}_k} \in E_{\boldsymbol{i}_k}} \mathbb{I}_k d\mathcal{F}_{\boldsymbol{i}_k}(x_{\boldsymbol{i}_k})$.

Definition 7. *Let for an arbitrary multivector field F the oriented differential $(k-1)$-form $\mathbb{I}_{\boldsymbol{i}_k - 1} F d\sigma_{\boldsymbol{i}_k}$ be totally $\mathcal{KH}$-directly integrable with respect to $\partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$. Then, the limit $\mathbb{I}_k d\mathcal{F}_{\boldsymbol{i}_k}$ of $\mathcal{KH}\text{−}vt \int^{\circlearrowleft}_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k - 1} F d\sigma_{\boldsymbol{i}_k}$ is said to be the residual field denoted by $\mathcal{R}$ of F. In symbols, $\mathcal{R} := \mathbb{I}_k d\mathcal{F}_{\boldsymbol{i}_k}$.*

Comparing the two previous definitions with Definition 4 we may conclude that the sum of residues of F in $[a,b]_{\boldsymbol{i}_k}$ is the total Kurzweil–Henstock directed integral of $\mathbb{I}_{\hat{N}} d^{\hat{N}} \mathcal{F}$ over $[a,b]_{\boldsymbol{i}_k}$, as follows

$$\mathcal{KH}\text{−}vt \int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k d\mathcal{F}_{\boldsymbol{i}_k} = \sum_{x_{\boldsymbol{i}_k} \in [a,b]_{\boldsymbol{i}_k}} \mathcal{R}(x_{\boldsymbol{i}_k}).$$

Let $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ be a set of Lebesgue outer measure zero at whose points the spacial (integral) derivative f of a multivector field F, as the limit of $\mathcal{F}$

defined by (3.2), can take the infinite values or not be defined at all. Since $\sum_{x_{\boldsymbol{i}_k}\in E_{\boldsymbol{i}_k}} \mathbb{I}_k D_{ex} f dx_{\boldsymbol{i}_k} = 0$, this further implies that if the oriented differential k-form $\mathbb{I}_k f dx_{\boldsymbol{i}_k}$ on $[a,b]_{\boldsymbol{i}_k}$ is $\mathcal{KH}$-directly integrable on $[a,b]_{\boldsymbol{i}_k}$ and hence

$$\mathcal{KH}\!-\!\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k f dx_{\boldsymbol{i}_k} = \sum_{x_{\boldsymbol{i}_k}\in [a,b]_{\boldsymbol{i}_k}\setminus E_{\boldsymbol{i}_k}} \mathcal{R}\left(x_{\boldsymbol{i}_k}\right),$$

then

$$\mathcal{KH}\!-\!vt\oint_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k} = \mathcal{KH}\!-\!vt\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k d\mathcal{F}_{\boldsymbol{i}_k}$$

$$= \mathcal{KH}\!-\!\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k f dx_{\boldsymbol{i}_k} + \sum_{x_{\boldsymbol{i}_k}\in E_{\boldsymbol{i}_k}} \mathcal{R}\left(x_{\boldsymbol{i}_k}\right).$$

In what follows we shall formulate the previous result as a theorem and prove it explicitly.

Theorem 2. *For any $\boldsymbol{i}_k \in \mathfrak{I}(\hat{N},k)$ let $E_{\boldsymbol{i}_k} \subset [a,b]_{\boldsymbol{i}_k}$ be a set of Lebesgue outer measure zero at whose points the spacial (integral) derivative f of a multivector field F, as the limit on $[a,b]_{\boldsymbol{i}_k}\setminus E_{\boldsymbol{i}_k}$ of $\mathcal{F}$ defined by (3.2), can take the infinite values or not be defined at all. If $d\mathcal{F}_{\boldsymbol{i}_k}$ is basically summable (BS_δ) in $E_{\boldsymbol{i}_k}$ to the sum $\Re$, then $\mathbb{I}_k f dx_{\boldsymbol{i}_k}$ is $\mathcal{KH}$-directly integrable on $[a,b]_{\boldsymbol{i}_k}$ and*

$$\mathcal{KH}\!-\!\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k f dx_{\boldsymbol{i}_k} + \Re = \mathcal{KH}\!-\!vt\oint_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k}. \tag{3.4}$$

Proof. Let the oriented differential $(k-1)$-form $\mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k}$ be totally $\mathcal{KH}$-directly integrable with respect to $\partial\mathcal{I}([a,b]_{\boldsymbol{i}_k})$. Since $d\mathcal{F}_{\boldsymbol{i}_k}$ is BS_δ in the set $E_{\boldsymbol{i}_k}$ to the sum $\Re$, it follows from Definition 6 that for every $\varepsilon > 0$ there exists a gauge δ in $[a,b]_{\boldsymbol{i}_k}$ such that

$$\left\|\Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}}) - \Re\right\| < \varepsilon,$$

whenever $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}_\delta([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}}$. In addition, $D_{ex} f \equiv 0$ in $E_{\boldsymbol{i}_k}$ and

$$\Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k})) = \mathcal{KH}\!-\!vt\oint_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_{\boldsymbol{i}_k-1} F d\sigma_{\boldsymbol{i}_k},$$

whenever $P\,[a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}}$. Taking (3.1) into consideration it is readily seen that

$$\left\|\Xi_{\mathbb{I}D_{ex}f\triangle x}(P([a,b]_{\boldsymbol{i}_k})) - [\mathcal{KH}\text{--}vt\int_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}}^{\circlearrowleft} \mathbb{I}_{\boldsymbol{i}_k-1}Fd\sigma_{\boldsymbol{i}_k} - \Re]\right\|$$

$$\leq \left\|\Xi_{\mathbb{I}D_{ex}f\triangle x}(P([a,b]_{\boldsymbol{i}_k})\backslash P([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}}) - \Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k})\backslash P([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}})\right\|$$

$$+\left\|\Xi_{\mathbb{I}\triangle\mathcal{F}}(P([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}}) - \Re\right\| < \varepsilon(\left|[a,b]_{\boldsymbol{i}_k}\right| + 1),$$

whenever $P([a,b]_{\boldsymbol{i}_k}) \in \mathcal{P}_\delta([a,b]_{\boldsymbol{i}_k})\Big|_{E_{\boldsymbol{i}_k}}$. Hence, $\mathbb{I}_k f dx_{\boldsymbol{i}_k}$ is $\mathcal{KH}$-directly integrable over $[a,b]_{\boldsymbol{i}_k}$ and

$$\mathcal{KH}\text{--}\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k f dx_{\boldsymbol{i}_k} + \Re = \mathcal{KH}\text{--}vt\int_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}}^{\circlearrowleft} \mathbb{I}_{\boldsymbol{i}_k-1}Fd\sigma_{\boldsymbol{i}_k}. \qquad \square$$

By Definition 7 and (3.3), the result (3.4) of Theorem 2 can be rewritten as

$$\mathcal{KH}\text{--}vt\int_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}}^{\circlearrowleft} \mathbb{I}_{\boldsymbol{i}_k-1}Fd\sigma_{\boldsymbol{i}_k} = \mathcal{KH}\text{--}vt\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k f dx_{\boldsymbol{i}_k} \tag{3.5}$$

$$= \mathcal{KH}\text{--}\int_{[a,b]_{\boldsymbol{i}_k}} \mathbb{I}_k f dx_{\boldsymbol{i}_k} + \sum_{x_{\boldsymbol{i}_k}\in E_{\boldsymbol{i}_k}} \mathcal{R}\left(x_{\boldsymbol{i}_k}\right).$$

If the spatial (integral) derivative f of F vanishes identically on $[a,b]_{\boldsymbol{i}_k}\backslash E_{\boldsymbol{i}_k}$, then it follows from (3.5) that

$$\mathcal{KH}\text{--}vt\int_{\boldsymbol{\partial}[a,b]_{\boldsymbol{i}_k}}^{\circlearrowleft} \mathbb{I}_{\boldsymbol{i}_k-1}Fd\sigma_{\boldsymbol{i}_k} = \sum_{x_{\boldsymbol{i}_k}\in E_{\boldsymbol{i}_k}} \mathcal{R}\left(x_{\boldsymbol{i}_k}\right).$$

The obtained result provides an extension of Cauchy's integral formula from the calculus of residues in $\mathbb{R}^{\hat{N}}$ (compare with results in [6, 12]).

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