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ZIP PROPERTY OF SKEW HURWITZ SERIES RINGS AND MODULES

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Abstract. Carl Faith [9] called a ring R to be right zip if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, for a left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. In this article, we study the behavior of zip property of the skew Hurwitz series rings and modules for the non-commutative ring. In particular, we prove the following results:

(1) Let R be a ring and ω be an endomorphism of R. If R is skew Hurwitz series-wise Armendariz and ω-compatible, then R is a right zip ring if and only if the skew Hurwitz series ring (HR, ω) is a right zip ring.

(2) Let M_R be a module and ω be an endomorphism of R. If M_R is ω-Armendariz of skew Hurwitz series type and ω-compatible, then M_R is a zip right R-module if and only if the skew Hurwitz power series module $HM_{(HR,\omega)}$ is a zip right (HR, ω)-module.

(3) Let R be a ring, I a Σ-compatible semi prime ideal of R and $\mathrm{char}(R/I) = 0$, then R is a Σ_I-zip ring if and only if the skew Hurwitz power series ring (HR, ω) is a $\Sigma_{(HI,\omega)}$-zip ring.

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1. Introduction. Throughout this article, R and M_R denote an associative ring with identity and a right R-module, respectively. For any subset X of R, $r_R(X)$ indicate the right annihilator of X in R. Faith [9] called a ring R to be right zip if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, if a left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. R is said to be simply zip ring if it is both a right as well as a left zip ring. The concept of zip rings was initiated by Zelmanowitz [39]. It appeared in various papers [5, 7, 8, 9, 10], and references therein. Zelmanowitz proved that any ring satisfying the descending chain conditions on right annihilators is a right zip ring. But the converse does not hold. Beachy and Blair [5] studied rings that satisfy the condition that every faithful right ideal I of a ring R (a right ideal I of a ring R is faithful if $r_R(I) = 0$) is cofaithful (a right ideal I of a ring R is cofaithful if there exists a finite subset $I_1 \subseteq I$ such that $r_R(I_1) = 0$). They showed that right zip rings have this property and conversely for commutative ring R.

Extensions of zip rings were studied by several authors. Beachy and Blair [5] showed that if R is a commutative zip ring, then polynomial ring $R[x]$ over R is a zip ring. Afterward, Cedo [7] proved that if R is a commutative zip ring, then the $n \times n$ full matrix ring $M_n(R)$ over R is zip; moreover, he settled negatively the following questions which were posed by Faith [9]: Does R being a right zip ring imply $R[x]$ to be right zip?; Does R being a right zip imply $M_n(R)$ to be right zip?; Does R being a right zip ring imply the group ring $R[G]$ to be right zip if G is a finite group? Faith [10] also raised the following questions: When does R being a right zip ring imply $R[x]$ to be right zip?; Characterize a ring R such that $M_n(R)$ is right zip; When does R being a right zip ring imply the group ring $R[G]$ to be a right zip ring if G is a finite group? He also proved that if R is a commutative ring and G is a finite abelian group, then the group ring $R[G]$ of G over R is zip.

In [16], Hong et al. studied above stated questions. They proved that R is a right zip ring if and only if $R[x]$ is a right zip ring when R is an Armendariz ring. They also demonstrated that if R is a commutative ring and G a unique product monoid containing an infinite cyclic submonoid, then R is a zip ring if and only if $R[G]$ is a zip ring. Further, Cortes [8] studied the relationship between right (left) zip property of R and skew polynomial (skew power series) extensions over R by using the skew versions of Armendariz rings. After that, Singh and Dixit [37] generalized the results of Cortes [8] and investigated a relationship between zip property of a ring R and the skew generalized power series ring $R[[S,\omega]]$, where S is a monoid and $\omega : S \to \mathrm{End}(R)$ is monoid homomorphism.

They proved that if R is (R,ω)-Armendariz and S-compatible, then the skew generalized power series ring $R[[S,\omega]]$ is right zip if and only if R is right zip. Ahmadi et al. [1, Theorem 2.19] studied same property of zip ring to the skew Hurwitz series ring (HR,ω) for a commutative ring R and proved that if R is skew Hurwitz Armendariz (SHA)-ring and ω is an endomorphism of R, then R is a right zip ring if and only if (HR,ω) is a right zip ring. Another generalization of zip rings to zip modules was studied by Zhang and Chen [40]. They observed that if M_R is an Armendariz right R-module, then M_R is a right zip R-module if and only if $M[x]$ is a zip right $R[x]$-module, and if M_R is an Armendariz right R-module of power series type, then M_R is a right zip R-module if and only if $M[[x]]$ is a right zip $R[[x]]$-module. Furthermore, Ouyang et al. [30] extended the results of Zhang and Chen [40] and proved that if $(S,\leq)$ is a cancellative torsion-free strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism, M_R an ω-compatible and skew S-Armendariz right R-module, then skew generalized power series module $M[[S^{\leq}]]$ is a right $R[[S,\omega]]$-module if and only if M_R is a zip right R-module. They also examined the same for a Malcev-Neumann module $M * ((G))$, when G is an ordered monoid. Recently, in 2017, Ouyang et al. [30] introduced the notion of a $\sum$-zip ring which is a common generalization of zip rings and weak zip rings, and studied the basic properties of a $\sum$-zip ring. They proved that if $(S,\leq)$ is a strictly totally ordered monoid, and I a $\sum$-compatible semi-prime ideal of R, then R is $\sum_I$-zip if and only if the skew generalized power series ring $[[R^{S,\leq},\omega]]$ is $\sum_{[[I^{S,\leq},\omega]]}$-zip.

Motivated by various extensions of zip rings and modules, in this article, we study behavior zip property over skew Hurwitz series rings and modules by considering R to be a non-commutative ring and prove the following main results:

(1) Let R be a ring and ω be an endomorphism of R. If R is skew Hurwitz series-wise Armendariz and ω-compatible, then R is a right zip ring if and only if the skew Hurwitz series ring (HR,ω) is a right zip ring.

(2) Let M_R be a module and ω be an endomorphism of R. If M_R is ω-Armendariz of skew Hurwitz series type and ω-compatible, then M_R is a zip right R-module if and only if the skew Hurwitz power series module $HM_{(HR,\omega)}$ is a zip right (HR,ω)-module.

(3) Let R be a ring, I a Σ-compatible semi prime ideal of R and $\mathrm{char}(R/I) = 0$, then R is a Σ_I-zip ring if and only if the skew Hurwitz power series ring (HR,ω) is a $\Sigma_{(HI,\omega)}$-zip ring.

2. Construction of skew Hirwitz series rings and modules. The concept of Hurwitz series ring was proposed by Keigher [18], as a variant of the ring of formal power series. He also studied some of its properties especially the categorical properties. He and later with Pritchard in [17, 19] demonstrated that Hurwitz series has many interesting applications in differential algebra and in the discussion of weak normalization. The elements of Hurwitz series HR are sequences of the form $a = (a_n) = (a_0, a_1, a_2, a_3, \ldots) \in R$, for each $n \in \mathbb{Z}^+$ or $f(n) = a_n$, where $f : \mathbb{Z}^+ \to R$ and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ is a set of non negative integers. Addition in HR is pointwise and the product of two sequences using binomial coefficients. This was studied by Fliess [11] and Taft [38].

Number of authors see [1, 6, 13, 18, 19, 26, 32, 33, 34, 36], studied the properties of abstract ring structures in Herwitz series HR. Motivated by the construction of Hurwitz series, Hassenin [12] extended the construction to skew Hurwitz series rings (HR, ω), where $\omega : R \to R$ is an automorphism of R. And defined as follows: The elements of (HR, ω) are functions $\omega : \mathbb{Z}^+ \to R$, where $\mathbb{Z}^+$ is the set of positive integers with zero. The operation of addition in (HR, ω) is component wise and the multiplication is defined for every $f, g \in (HR, \omega)$, by

$$fg(p) = \sum_{k=0}^{p} C_k^p f(k) \omega^k (g(p-k))$$

for all $p, k \in \mathbb{Z}^+$, where $C_k^p = \dfrac{p\,!}{k\,!(p-k)\,!}$. Define the mappings $h_n : \mathbb{Z}^+ \to R$ via

$$h_n(p) = \begin{cases} 1 & \text{if } p = n-1 \\ 0 & \text{if } p \neq n-1 \end{cases},$$

where $p, n \in \mathbb{Z}^+$. And for each $r \in R$, $\grave{h_r} : \mathbb{Z}^+ \to R$ via

$$\grave{h_r}(p) = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \geq 1 \end{cases}.$$

Thus (HR, ω) is a ring with identity h_1, defined by

$$h_1(p) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

for each $n \in \mathbb{Z}^+$. The ring R can be canonically embedded as a subring of (HR, ω) via $r \to \grave{h_r} \in (HR, \omega)$. Further, Kamal [32, 33] gave the construction of a skew

Hurwitz series ring by considering $\omega : R \to R$ as an endomorphism of R instead of $\omega : R \to R$ an automorphism of R.

Example 2.1. We get following extensions as special cases of skew Hurwitz series rings (HR, ω).

(1) If ω is an identity endomorphism of R, then $(HR, \omega) \cong (HR, I_\omega)$. Thus Hurwitz series ring is a special case of skew Hurwitz series ring.
(2) If $f(n) = a_n$ for all $n \in \mathbb{Z}^+$, the addition component-wise and multiplication of two elements as defined in a skew Hurwitz series ring except binomial coefficients, then $(HR, \omega) \cong R[[x; \omega]]$. Thus skew formal power series ring is a special case of skew Hurwitz series ring.

For any function $f \in (HR, \omega)$, $\text{supp}(f) = \left\{n \in \mathbb{Z}^+ | f(n) \neq 0\right\}$ denotes the support of f, and by $\pi(f)$ denote the minimal element of supp(f). For any nonempty subset X of R, we denote:

$$(HX, \omega) = \left\{f \in (HR, \omega) | f(n) \in X \cup \{0\}, n \in \mathbb{Z}^+\right\}$$

the subset of (HR, ω). Further, motivated by this study, in [2017], Kamal [34] gave a new direction to study of the skew Hurwitz series ring and proposed the concept of the skew Hurwitz series modules. He proved some properties of simple and semisimple modules of the skew Hurwitz series module $HM_{(HR,\omega)}$. Let M_R be a right R-module and HM be the set of all maps $\phi : \mathbb{Z}^+ \to M$. With pointwise addition, HM is an abelian additive group. It becomes a module over the skew Hurwitz series ring (HR, ω), under the scalar multiplication for each $\phi \in HM$ and $g \in (HR, \omega)$ is defined by:

$$\phi g(p) = \sum_{k=0}^{p} C_k^p \phi(k) \omega^k (g(p-k))$$

for every $p \in \mathbb{Z}^+$. For any $m \in M$ and any $n \in \mathbb{Z}^+$, we define $h_m \in HM$ by

$$h_m(n) = \begin{cases} m & \text{if } \ n = 0 \\ 0 & \text{if } \ n \geq 0 \end{cases}.$$

Clearly the map $m \to h_m$ is a module embedding of M into HM.

Example 2.2. We get following extensions as special cases of skew Hurwitz series module $HM_{(HR,\omega)}$.

(1) If ω is an identity endomorphism of R, then $HM_{(HR,\omega)}$ is isomorphic onto $(HM_{(HR,I_\omega)}$. Thus Hurwitz series module is a special case of the skew Hurwitz series module.

(2) If $f(n) = a_n$ for all $n \in \mathbb{Z}^+$, the addition as component wise and multiplication of two elements as in a skew Hurwitz series module except for binomial coefficients,then $HM_{(HR,\omega)} \cong M[[x]]_{R[[x;\omega]]}$. Thus the skew formal power series module is a special case of the skew Hurwitz series module.

3. Zip property of skew Hurwitz series rings. This section deals with zip property of skew Hurwitz series rings. We establish an equivalence relation between a right zip ring R and its skew Hurwitz series ring if a ring R is a non-commutative skew Hurwitz series-wise Armendariz. In order prove our main results we shall need following definitions and results. Due to Krempa [22], a monomorphism ω of a ring R is said to be rigid, if $a\omega(a) = 0$ implies $a = 0$, for all $a \in R$. A ring R is called ω-rigid if there exists a rigid endomorphism ω of R. Annin [1] called a ring R to be ω-compatible if for every $a, b \in R$, $ab = 0$ if and only if $a\omega(b) = 0$. Hashemi and Moussavi [12], gave some examples of nonrigid ω-compatible rings. They proved the following Lemma.

Lemma 3.1. *Let ω be an endomorphism of a ring R. Then*

(1) *if ω is compatible, then ω is injective*

(2) *ω is compatible if and only if for all $a, b \in R$, $\omega(a)b = 0 \Leftrightarrow ab = 0$*

(3) *the following conditions are equivalent:*

  (a) *ω is rigid*

  (b) *ω is compatible and R is reduced*

  (c) *for every $a \in R$, $\omega(a)a = 0$ implies that $a = 0$.*

Ahmadi et al. [1] introduced the concept of a skew Hurwitz series wise Armendariz ring. He defined it as follows:

Definition 3.2. *Let R be a commutative ring, and $\omega : R \to R$ be an endomorphism of R. The ring R is said to be skew Hurwitz series wise Armendariz (SHA), if for every skew Hurwitz series $f, g \in (HR, \omega)$, the product $fg = 0$ if and only if $f(n)g(m) = 0$ for all $n, m \in \mathbb{Z}^+$.*

We extend the concept of skew Hurwitz series wise Armendariz in case of a non-commutative ring, as follows:

Definition 3.3. *Let R be a ring and $\omega : R \to R$ be an endomorphism of R. The ring R is said to be skew Hurwitz series wise Armendariz, if for every skew Hurwitz series $f, g \in (HR, \omega)$, $fg = 0$ implies $f(n)\omega^n g(m) = 0$ for all $n, m \in \mathbb{Z}^+$.*

We begin with some interesting observations.

Lemma 3.4. *Let R be a ring and $\omega : R \to R$ be an endomorphism of R. If R is skew Hurwitz series-wise Armendariz and ω-compatible then R is a torsion-free $\mathbb{Z}$-module.*

Proof. Suppose R is not torsion-free as a $\mathbb{Z}$-module so there exists an integer $p > 0$ such that and $pa = 0$ for any $0 \neq a \in R$. For $a \in R$, define the mapping $\alpha : \mathbb{Z}^+ \to R$ by

$$\alpha(n) = \begin{cases} a & \text{if } \ n = 1 \\ 0 & \text{if } \ n > 1. \end{cases}$$

Hence $\alpha \in (HR, \omega)$. Thus

$$\alpha h_p(p) = \alpha h_p(1 + p - 1) = \sum_{k=0}^{p} C_k^p \alpha(k)\omega^k(h_p(p-k)) = pa\omega(h_p(p-1)) = 0.$$

Since R is skew Hurwitz series wise Armendariz and ω-compatible we get $a(h_p(p-1)) = 0$. But then $a = 0$ a contradict. Hence R is a torsion free $\mathbb{Z}$-module. □

In the following Theorem we show that every reduced ring (a ring with no nonzero nilpotent elements) is skew Hurwitz series wise Armendariz under some additional conditions.

Theorem 3.5. *Let R be a ring and ω be an endomorphism of R. If R is reduced, ω-compatible and torsion-free as a $\mathbb{Z}$-module, then R is skew Hurwitz series wise Armendariz.*

Proof. Suppose $f, g \in (HR, \omega)$, such that $fg = 0$ with $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$. To prove the result we need to show $f(u)g(v) = 0$ for every $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$. Suppose $s = \pi(f)$ and $t = \pi(g)$. Now,

$$fg(s+t) = \sum_{(u,v) \in X_{s+t}(f,g)} C_u^{u+v} f(u)\omega^u(g(v)) = C_s^{s+t} f(s)\omega^s(g(t)) = 0.$$

Therefore $f(s)g(t) = 0$ as R is a torsion-free $\mathbb{Z}$-module and ω-compatible. Let $k \in \mathbb{Z}^+$ be such that $f(u)g(v) = 0$ for every $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$ with $u + v < k$. By induction on k, we prove that $f(u)g(v) = 0$ for all $u + v = k$. Let $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$ be such that $u + v = k$. We have,

$$fg(u+v) = \sum_{(p,q) \in X_{u+v=k}(f,g)} C_p^{p+q} f(p)\omega^p(g(q)),$$

where $X_{u+v=k}(f,g) = \{(p,q)|p+q=k, p \in \text{supp}(f), q \in \text{supp}(g)\}$. Now, without loss of generality, we can assume that

$$\{(p_i,q_i)|p_i+q_i=k, i=1,2,3,\ldots,m\} = X_{p+q=k}(f,g).$$

We obtain,

$$fg(p+q) = \sum_{i=1}^{m} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i)) = 0. \tag{3.1}$$

Since $p_1+q_i < p_i+q_i = k$ for each $i \geq 2$, we get $C_{p_1}^{p_1+q_i} f(p_1)\omega^{p_1}(g(q_i)) = 0$ for every $i \geq 2$. Hence, $f(p_1)g(q_i) = 0$ since R is a torsion-free $\mathbb{Z}$-module and ω-compatible. It follows that $\omega^{p_i}(g(q_i))f(p_1) = 0$ as R is ω-compatible and reduced. Multiplying (3.1) by $f(p_1)$ from right, we obtain, $f(p_1)\omega^{p_1}(g(q_1))f(p_1) = 0$. Since R is reduced and ω-compatible, therefore $f(p_1)g(q_1) = 0$ for every $p_1 \in \text{supp}(f)$ and $q_1 \in \text{supp}(g)$. Now from (3.1), we get

$$\sum_{i=2}^{m} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i)) = 0. \tag{3.2}$$

Similarly multiplying (3.2) by $f(p_2)$, we get $f(p_2)g(q_2) = 0$. This reduces (3.2) to,

$$\sum_{i=3}^{m} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i)) = 0. \tag{3.3}$$

Infact, continuing same way we can prove that $f(p_i)g(q_i) = 0$ for every $p_i \in \text{supp}(f)$, $q_i \in \text{supp}(g)$ with $p_i+q_i = k$. Thus $f(u)g(v) = 0$ for all $u \in \text{supp}(f)$, $v \in \text{supp}(g)$ which implies $f(u)\omega^u(g(v)) = 0$. Hence, R is a skew Hurwitz series wise Armendariz ring. □

Now, we prove our main result of this section.

Theorem 3.6. *Let R be a ring and ω be an endomorphism of R. If R is skew Hurwitz series-wise Armendariz and ω-compatible. The following statements are equivalent:*

(1) *R is right zip*

(2) *(HR,ω) is right zip*

Proof. We first prove that (2) implies (1). Suppose (HR,ω) is a right zip ring. We show that R is a right zip ring. For this consider $Y \subseteq R$

with $r_R(Y) = 0$. Since $Y \subseteq R$, hence Y is also a subset of (HR, ω). We prove $r_{(HR,\omega)}(Y) = 0$. Let $f \in r_{(HR,\omega)}(Y)$ recall $\text{supp}(f) = \{n \in \mathbb{Z}^+ | f(n) \neq 0\}$. It follows that $yf = 0$ for each $y \in Y$, and also that

$$0 = (h_y f)(n) = \sum_{k=0}^{n} C_k^n h_y(k)\omega^k (f(n-k)) = yf(n).$$

Therefore, $f(n) \in r_R(Y) = 0$ this implies $f = 0$. Hence $r_{(HR,\omega)}(Y) = 0$. Since (HR, ω) is a right zip ring, there exists a finite subset Y_0 of Y such that $r_{(HR,\omega)}(Y_0) = 0$. This makes $r_R(Y_0) = r_{(HR,\omega)}(Y_0) \cap R = 0$, and hence R is right zip.

Conversely, suppose R is a right zip ring. U be a subset of (HR, ω) with $r_{(HR,\omega)}(U) = 0$. If $C_U = \cup_{f \in U}\{f(n) | f \in U \text{ and } n \in \text{supp}(f)\}$. Then C_U is a nonempty subset of R. We show that $r_R(C_U) = 0$. Let $a \in r_R(C_U)$ then $f(n)a = 0$ for any $n \in \text{supp}(f)$. This gives $0 = f(n)a = f(n)h_a(0) = f(n)\omega^n(h_a(0))$ since R is ω-compatible. It follows that $fh_a = 0$. Hence $h_a = 0$ this implies that $a = 0$. Therefore $r_R(C_U) = 0$. Now R is right zip. Hence there exists a nonempty finite subset X of C_U such that $r_R(X) = 0$. Let $X = \{a_1, a_2, \ldots, a_k\}$ be a subset of C_U. Then, for every $i = 1, 2, 3 \ldots, k$ there exists f_i such that $f_i(n) = a_i$ for some $n \in \mathbb{Z}^+$. Suppose U_0 is a minimal subset of U such that $f_i \in U_0$ for every $a_i \in X$ this implies that $X \subseteq C_{U_0}$. Hence $r_R(C_{U_0}) \subseteq r_R(X) = 0$. Now we prove that $r_{(HR,\omega)}(U_0) = 0$. Suppose $g \in r_{(HR,\omega)}(U_0)$. Let $f_i \in U_0$, then $f_i g = 0$. Since R is skew Hurwitz series wise Armendariz and ω-compatible, we get $g = 0$. This proves that (HR, ω) is right zip. □

As a consequence of the above theorem, we obtain the following corollary.

Corollary 3.7. *Let R be a ring, and $\omega : R \to R$ identity endomorphism of R. If R is skew Hurwitz series-wise Armendariz. The following statements are equivalent:*

(1) *R is right zip*

(2) *HR is right zip*

We wish to highlight the importance of Theorem 3.5. Many results proved by several authors, [1, 8, 30, 32, 33, 40] can be proved as a consequence of this Theorem. We begin with the following result was proved by Cortes [8].

Definition 3.8 ([8, Definition 2.2(ii)]). *A ring R is said to satisfy $SA2'$ if for $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x; \omega]]$, and ω an endomorphism of R, the product $f(x)g(x) = 0$ implies that $a_i \omega^i(b_j) = 0$ for all i and j.*

The following result was proved by Cortes [8]. It follows here as a corollary.

Corollary 3.9 (Cortes [8, Theorem 2.8(ii)], Singh and Dixit [37, Corollary 3.8]). *Let ω be an automorphism of R and R satisfies $SA2'$. The following conditions are equivalent:*

(1) *R is right zip*

(2) *$R[[x;\omega]]$ is right zip*

Proof. From the Example 2.1 it is clear that $(HR,\omega) \cong R[[x;\omega]]$. The result follows from Theorem 3.6. □

A ring R is said to be an Armendariz ring of power series type if whenever $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$, the product $f(x)g(x) = 0$ holds if and only if $a_i b_j = 0$ for all $i, j \geq 0$.

The following result appears in [30] and [40].

Corollary 3.10 (Zhang and Chen [40, Theorem 2.3], Ouyang et al. [30, Corollary 1.7]). *Let R be an Armendariz ring of power series type. Then R is right zip if and only if $R[[x]]$ is right zip.*

Proof. If ω is an identity endomorphism of R. Then $R[[x;\omega]] \cong R[[x]]$. The result follows from corollary 3.9. □

Corollary 3.11 (Paykan [32, Corollary 2.15]). *Let R be a ring such that R is torsion-free as a $\mathbb{Z}$-module and ω be an endomorphism of R. If R is ω-rigid, then R is zip if and only if (HR,ω) is zip.*

Corollary 3.12 (Paykan [33, Corollary 2.16]). *Let R be a reduced ring and let R be torsion-free as a $\mathbb{Z}$-module. Then R is zip if and only if (HR) is zip.*

The following result was proved by Ahmadi et al. [1, Theorem 2.19] for commutative ring.

Corollary 3.13 (Ahmadi et al.[1, Theorem 2.19]). *Let R be an SHA-ring and ω be an endomorphism of R. Then R is right zip if and only if (HR,ω) is right zip.*

4. Zip property of skew Hurwitz series modules.

This section is devoted to the study of zip property of skew Hurwitz series modules. Zhang and Chen [40], generalized the class of zip rings, and introduced the concept of zip modules. A right R-module M_R is called zip if for any subset X of M_R, such that $r_R(X) = 0$ implies that there exits a nonempty finite subset Y of X such that $r_R(Y) = 0$. It is clear that R is a right zip ring if and only if R_R is a zip right R-module. They proved that if M_R is an Armendariz right R-module of power series type, then M_R is a zip right R-module if and only if $M[[x]]$ is a zip right $R[[x]]$-module. Luqnun et al. [30] extended this result and displayed that: (i) If $(S, \leq)$ is cancellative torsion-free strictly ordered monoid and M_R is an S-Armendariz module, then the generalized power series module $M[[S^{\leq}]]$ is a zip right $R[[S^{\leq}]]$-module if and only if M_R is a zip right R-module. (ii) If $(S, \leq)$ be a cancellative torsion-free strictly ordered monoid, $\omega : S \to End(R)$ a monoid homomorphism, M_R a ω-compatible and skew S-Armendariz right R-module then the skew generalized power series module $M[[S^{\leq}]]$ is a zip right $R[[S, \omega]]$-module if and only if M_R is a zip right R-module.

In this section, we study zip property of the skew Hurwitz series modules and generalize some earlier results [40, Thoerem 2.3], [30, Corollary 1.6], and [30, Corollary 1.9(i)]. We shall need some definitions.

Due to Annin [2] called a module M_R to be ω-compatible if for any $m \in M_R$ and $p \in R$, then $mp = 0$ if and only if $m\omega(p) = 0$, where $\omega : R \to R$ is an endomorphism of R. It follows that, if M_R is ω-compatible, then $mp = 0$ if and only if $m\omega^k(p) = 0$ for every k. In [23], Lee-Zhou introduced the concept of ω-Armendariz of power seris type as follows:

Definition 4.1. *A module M_R is said to be ω-Armendariz of power seris type if the following conditions are satisfied:*

(1) *For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\omega]]$ and $f(x) = \sum_{i=0}^{\infty} a_j x^j \in R[[x;\omega]]$, $m(x)f(x) = 0$ implies $m_i\omega^i(a_j)$, for all $i, j \geq 0$.*

(2) *For every $m \in M_R$ and $p \in R$, $mp = 0$ if and only if $m\omega(p) = 0$.*

We introduce the concept of ω-Armendariz of skew Hurwitz series type as a generalization of ω-Armendariz of power series type:

Definition 4.2. *Let M_R be a module and $\omega : R \to R$ be an endomorphism of R. The ring M_R is said to be ω-Armendariz of Hurwitz series type if the following conditions are satisfied:*

(1) *For every skew Hurwitz series $\phi \in HM_{(HR,\omega)}$ and $g \in (HR, \omega)$, $\phi g = 0$ implies $\phi(p)\omega^p g(q) = 0$ for all $p, q \in \mathbb{Z}^+$, and*

(2) *For any $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\omega(a) = 0$.*

Lee and Zhou [23] called a module M_R to be ω-reduced, if for every $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$ and ω-compatible, where $\omega : R \to R$ is a ring of endomorphism with $\omega(1) = 1$. They also demonstrated the following Lemma:

Lemma 4.3. *The following are equivalent for a module M_R:*

(1) *M_R is ω-reduced*

(2) *The following conditions hold: For any $m \in M$ and $a \in R$*

  (a) *$ma = 0$ implies $mRa = mR\omega(a) = 0$*
  (b) *$ma\omega(a) = 0$ implies $ma = 0$*
  (c) *$ma^2 = 0$ implies $ma = 0$.*

Lemma 4.4. *Let M_R be a module and ω be an endomorphism of R. If R is ω-Armendariz of skew Hurwitz series type then M_R is torsion-free as a $\mathbb{Z}$-module.*

P r o o f. Proof is similar to Lemma 3.4. □

In the following theorem we show that every ω-reduced module is ω-Armendariz of skew Hurwitz series type under some additional conditions.

Theorem 4.5. *Let M_R be a module and ω be an endomorphism of R. If M_R is ω-reduced and torsion-free as a $\mathbb{Z}$-module, then M_R is ω-Armendariz of skew Hurwitz series type.*

P r o o f. Let for every $\phi, g \in (HR, \omega)$, such that $\phi g = 0$ with $u \in \operatorname{supp}(\phi)$ and $v \in \operatorname{supp}(g)$. In order to prove the result we need to show $\phi(u)g(v) = 0$ for every $u \in \operatorname{supp}(\phi)$ and $v \in \operatorname{supp}(g)$. Suppose $s = \pi(f)$ and $t = \pi(g)$. We get,

$$\phi g(s+t) = \sum_{(u,v)\in X_{s+t}(\phi,g)} C_u^{u+v}\phi(u)\omega^u(g(v)) = C_s^{s+t}\phi(s)\omega^s(g(t)) = 0.$$

Hence $\phi(s)g(t) = 0$ as R is torsion-free as a $\mathbb{Z}$-module and ω-compatible. Let $k \in \mathbb{Z}^+$ such that for every $u \in \operatorname{supp}(\phi)$ and $v \in \operatorname{supp}(g)$ with $u + v < k$, $\phi(u)g(v) = 0$. By induction we prove that $\phi(u)g(v) = 0$ for all $u + v = k$. We check for every $u \in \operatorname{supp}(\phi)$, $v \in \operatorname{supp}(g)$ such that $u + v = k$. We have,

$$\phi g(u+v) = \sum_{(p,q)\in X_{u+v=k}(\phi,g)} C_p^{p+q}\phi(p)\omega^p(g(q)),$$

where $X_{u+v=k}(\phi, g) = \{(p, q) | p + q = k, p \in \operatorname{supp}(\phi), q \in \operatorname{supp}(g)\}$. Now, without loss of generality, we can assume that

$\{(p_i, q_i)|p_i + q_i = k, i = 1, 2, \ldots, m\} = X_{p+q=k}(\phi, g)$. We obtain,

$$\phi g(p+q) = \sum_{i=1}^{m} C_{p_i}^{p_i+q_i} \phi(p_i)\omega^{p_i}(g(q_i)) = 0. \tag{4.1}$$

Since $p_i + q_1 < p_i + q_i = k$ for each $i \geq 2$, so

$$C_{p_i}^{p_i+q_1} \phi(p_i)\omega^{p_i}(g(q_1)) = 0$$

for each $i \geq 2$. Thus $\phi(p_i)g(q_1) = 0$ since M_R is torsion-free as a $\mathbb{Z}$-module and ω-compatible. It follows that $\phi(p_i)Rg(q_1) = 0$ from Lemma 4.3. Multiplying (4.1) by $g(q_1)$, we get, $\phi(p_1)\omega^{p_1}(g(q_1))g(q_1) = 0$. It follows that $\phi(p_1)g(q_1)^2 = 0$ as M_R is ω-reduced. Thus $\phi(p_1)g(q_1) = 0$ for every $p_1 \in \mathrm{supp}(\phi)$ and $q_1 \in \mathrm{supp}(g)$ from Lemma 4.3. Therefore from (4.1), we have

$$\sum_{i=2}^{m} C_{p_i}^{p_i+q_i} \phi(p_i)\omega^{p_i}(g(q_i)) = 0. \tag{4.2}$$

Similarly multiplying (4.2) by $g(p_2)$, we get $\phi(p_2)g(q_2) = 0$. Thus from (4.2),

$$\sum_{i=3}^{m} C_{p_i}^{p_i+q_i} \phi(p_i)\omega^{p_i}(g(q_i)) = 0. \tag{4.3}$$

Continuing this way we can prove that, $\phi(p_i)g(q_i) = 0$ for all $p_i \in \mathrm{supp}(\phi)$, $q_i \in \mathrm{supp}(g)$ with $p_i + q_i = k$. Thus $\phi(u)g(v) = 0$ for every $u \in \mathrm{supp}(\phi)$ and $v \in \mathrm{supp}(g)$. Hence, M_R is ω-Armendariz of skew Hurwitz series type. □

Theorem 4.6. *Let M_R be a module and ω be an endomorphism of R. If M_R is ω-Armendariz of skew Hurwitz series type and ω-compatible. The following conditions are equivalent:*

(1) *M_R is a zip right R-module*

(2) *$HM_{(HR,\omega)}$ is a zip right (HR,ω)-module*

Proof. We first prove that (2) implies (1). Suppose that $HM_{(HR,\omega)}$ is a zip right (HR,ω)-module. We show that M_R is a zip right R-module. For this consider $Y \subseteq M_R$ with $r_R(Y) = 0$. Since $Y \subseteq M_R$, Y is also subset of $HM_{(HR,\omega)}$. Now, we prove $r_{(HR,\omega)}(Y) = 0$. Let $f \in r_{(HR,\omega)}(Y)$ with $\mathrm{supp}(f) = \{n \in \mathbb{Z}^+ | f(n) \neq 0\}$. It follows that $mf = 0$ for all $m \in Y$, $0 = h_m f(n) = \sum_{k=0}^{n} C_k^n h_m(k)\omega^k\left(f(n-k)\right) = mf(n)$. Thus, $f(n) \in r_R(Y) = 0$. This implies

$f = 0$. It follows that $r_{(HR,\omega)}(Y) = 0$. Since $HM_{(HR,\omega)}$ is a zip right (HR,ω)-module, there exists a finite subset Y_0 of Y such that $r_{(HR,\omega)}(Y_0) = 0$. Thus $r_R(Y_0) = r_{(HR,\omega)}(Y_0) \cap R = 0$. Hence M_R is a zip right R-module.

Conversely, assume that M_R is a zip right R-module and $U \subseteq HM_{(HR,\omega)}$ with $r_{(HR,\omega)}(U) = 0$. Let $C_U = \cup_{\phi \in U} C_\phi$, where $C_\phi = \{\phi(n) | \phi \in U$ and $n \in \text{supp}(\phi)\}$.Then C_U is a nonempty subset of M_R. Now, we show $r_R(C_U) = 0$. Let $a \in r_R(C_U)$, $\phi(n)a = 0$ for all $n \in \text{supp}(\Phi)$. Which gives $0 = \phi(n)a = \phi(n)h_a(0) = \phi(n)\omega^n(h_a(0))$ as M_R is ω-compatible. It follows that $\phi h_a = 0$, $\phi \in r_{(HR,\omega)}(U)$. Thus $h_a = 0$ this implies $a = 0$. Therefore $r_R(C_U) = 0$. Since M_R is a zip right R-module, so there exists a nonempty finite subset X of C_U such that $r_R(X) = 0$. Consider $X = \{m_1, m_2, \ldots, m_k\}$ subset of C_U. Now, for each m_i there exists f_{m_i} for each $i = 1, 2 \ldots, k$ such that $f_{m_i}(n) = m_i$ for some $n \in \mathbb{Z}^+$. Suppose U_0 is a minimal subset of U such that $f_{m_i} \in U_0$ for each $m_i \in X$. Which implies that $X \subseteq C_{U_0}$. Thus $r_R(C_{U_0}) \subseteq r_R(X) = 0$. Now we demonstrate that $r_{(HR,\omega)}(U_0) = 0$. Suppose that $g \in r_{(HR,\omega)}(U_0)$ and $f \in U_0$. Thus $fg = 0$. Since M_R is ω-Armendariz of skew Hurwitz series type, we get $f(m)g(n) = 0$ for every $m \in \text{supp}(f)$ and $n \in \text{supp}(g)$. It follows that $g(n) \in r_R(C_{U_0}) = 0$ for all $n \in \text{supp}(g)$. This implies that $g = 0$. Hence $r_{(HR,\omega)}(U_0) = 0$. Thus $HM_{(HR,\omega)}$ is a zip right (HR,ω)-module. □

Here, we obtain the following result as special case of the Theorem 4.6.

Corollary 4.7. *Let R be a ring, M_R a right R-module and ω an identity endomorphism of R. If M_R is ω-Armendariz Hurwitz series type. The following conditions are equivalent:*

(1) *M_R is a zip right R-module*

(2) *HM_{HR} is a zip right HR-module*

Proof. Let ω be an identity endomorphism of R. So from Example 2.2 we have $HM_{(HR,\omega)} \cong HM_{HR}$. The result follows from Theorem 4.6. □

The following corollary was proved by Ouyang [30], the same can also be obtained as a corollary of the Theorem 4.6.

Corollary 4.8 (Ouyang et al. [30, Corollary 1.9])**.** *Let R be a ring, M_R a right R-module and ω an endomorphism of R. If M_R is ω-compatible and skew Armendariz power series type module. The following conditions are equivalent:*

(1) *M_R is a zip right R-module*

(2) *$M[[x]]_{R[[x;\omega]]}$ is a zip right $R[[x;\omega]]$-module*

P r o o f. From Example 2.2, $R[[x;\omega]] \cong (HR,\omega)$. Result from the Theorem 4.6. □

The following corollary appears in [40, Theorem 2.3], and in [30, Corollary 1.6].

Corollary 4.9 (Zhang and Chen [40, Theorem 2.3], Ouyang et al. [30, Corollary 1.6])**.** *Let R be a ring and M_R be a right R-module. If M_R is Armendariz power series type module. The following conditions are equivalent:*

(1) *M_R is a zip right R-module*

(2) *$M[[x]]_{R[[x]]}$ is a zip right $R[[x]]$-module*

5. Σ- Zip property of skew Hurwitz series rings. Ouyang et al.[31] defined $I : X = \{x \in R | Xx \subseteq I\}$ for two nonempty subsets I and X of a ring R. If X contains only single element say y, then $I : X$ is simply denoted by $I : y$. If I and X are two right ideals of R, then $I : X$ is an ideal of R and called quotient of I by X. Using this Ouyang et al. [31] generalized zip rings to Σ-zip rings as follows:

Definition 5.1. *A ring R is called Σ_I-zip if for any subset X of R where X is not a subset of an ideal I, if $I : X = I$, then there exists a finite subset $Y \subseteq X$ such that $I : Y = I$.*

From this definition it is clear that if $I = 0$, then for any subset X of R, we have $I : X = r_R(X) = 0$, and so R is Σ_0-zip if and only if R is right zip.

Recall Hashemi et al. [12], called an ideal I to be Σ-compatible if for each $a, b \in R$, $ab \in I \Leftrightarrow a\omega(b) \in I$. They proved the following lemma:

Lemma 5.2. *Let I be a Σ-compatible ideal of R. For every $a, b \in R$, $ab \in I$ if and only if $a\omega^k(b) \in I$ for every positive integer k.*

Lemma 5.3. *Let I be a Σ-compatible semi prime ideal of R, and* char(R/I)=0, *and $f, g \in (HR,\omega)$. Then $fg \in (HI,\omega)$ if and only if $f(u)g(v) \in I$ for every $u, v \in \mathbb{Z}^+$.*

P r o o f. Let $f, g \in (HR,\omega)$ be such that $fg = 0$ with $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$. Suppose $s = \pi(f)$ and $t = \pi(g)$. So, we have

$$fg(s+t) = \sum_{(u,v)\in X_{s+t}(f,g)} C_u^{u+v} f(u)\omega^u(g(v)) = C_s^{s+t} f(s)\omega^s(g(t)) \in I.$$

Since $\text{char}(R/I) = 0$ and I is Σ-compatible we get $f(s)g(t) \in I$. Let $k \in \mathbb{Z}^+$ be such that for every $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$ with $u + v < k$, $f(u)g(v) \in I$.

By induction we prove that $f(u)g(v) \in I$ for all $u+v=k$. Hence we check this for every $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$ such that $u+v=k$. Now,

$$fg(u+v) = \sum_{(p,q)\in X_{u+v=k}(f,g)} C_u^{u+v} f(u)\omega^u(g(v)),$$

where $X_{u+v=k}(f,g) = \{(p,q)|p+q=k, p \in \text{supp}(f), q \in \text{supp}(g)\}$. We can assume that
$\{(p_i,q_i)|p_i+q_i=k, i=1,2,\ldots,m\} = X_{p+q=k}(f,g)$. We obtain,

$$fg(p+q) = \sum_{i=0}^{m} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i)) \in I. \tag{5.1}$$

Since $p_1+q_i < p_i+q_i = k$, we get $C_{p_1}^{p_1+q_i} f(p_1)\omega^{p_i}(g(q_i)) \in I$ for each $i \geq 2$. Thus $f(p_1)g(q_i) \in I$ since $\text{char}(R/I)=0$ and I is Σ-compatible. Hence $\omega^{p_1}(g(q_i))f(p_1) \in I$ since I is Σ-compatible and semiprime. Multiplying (5.1) by $f(p_1)$ on the right, we obtain, $f(p_1)\omega^{p_1}(g(q_1))f(p_1) \in I$. Thus

$$f(p_1)\omega^{p_1}(g(q_1))f(p_1) \in I.$$

Since $\text{char}(R/I)=0$, and I is Σ-compatible and semiprime, hence $f(p_1)g(q_1) \in I$ for each $p_1 \in \text{supp}(f)$ and $q_1 \in \text{supp}(g)$. Therefore from (5.1), we get

$$\sum_{i=2}^{m} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i)) \in I. \tag{5.2}$$

Similarly multiplying (5.2) by $f(p_2)$, we get $f(p_2)g(q_2) \in I$. Again from (5.2), we get

$$\sum_{i=3}^{m} C_{p_i}^{p_i+q_i} f(p_i)\omega^{p_i}(g(q_i)) \in I. \tag{5.3}$$

Continuing, we get, $f(p_i)g(q_i) \in I$ for every $p_i \in \text{supp}(f)$ and $q_i \in \text{supp}(g)$ with $p_i+q_i=k$. Thus $f(u)g(v) \in I$ for every $u \in \text{supp}(f)$ and $v \in \text{supp}(g)$. The converse is obvious. □

Theorem 5.4. *Let I be a Σ-compatible semi prime ideal of R and* $\text{char}(R/I)=0$. *The following statements are equivalent:*

(1) *R is Σ_I-zip*

(2) *(HR,ω) is $\Sigma_{(HI,\omega)}$-zip*

P r o o f. Suppose R is Σ_I-zip and I a proper ideal of R. Then (HI, ω) is an ideal of (HR, ω). Let Y be a nonempty subset of (HR, ω) such that Y is not a subset of (HI, ω) and $(HI, \omega) : Y = (HI, \omega)$. For any subset Y, define $C_Y = \cup_{\phi \in I} C_\phi$, where $C_\phi = \{\phi(n) | \phi \in Y \text{ and } n \in \text{supp}(\phi)\}$. Then C_ϕ is a nonempty subset of R. We manifest that $I : C_Y = I$. For, suppose $a \in I : C_Y$ and $f \in Y$ with $f(n) \in C_Y$. Hence,

$$fh_a(n) = \sum_{k=0}^{n} C_k^n f(n)\omega^n(h_a(n-k)) = f(n)\omega^n(h_a(n-k)) = f(n)a \in I.$$

Since I is Σ_I-compatible, $h_a \in (HI, \omega) : Y = (HI, \omega)$. It follows that $a \in I$. Thus $I : C_Y = I$. Since R is Σ_I-zip, there exists a finite subset $X = \{a_1, a_2, \ldots, a_k\}$ of C_Y such that $I : X = I$. For every $a_i \in X$ there exists $f_{a_i} \in Y$ such that $f_{a_i}(n) = a_i$ for some $n \in \mathbb{Z}^+$. Suppose Y_0 is a minimal subset of Y such that for each $a_i \in X$, $f_{a_i} \in Y_0$. Thus $X \subseteq C_{Y_0}$. It follows that $I{:}C_{Y_0} \subseteq I{:}X = I$. Next we show $(HI, \omega) : Y_0 = (HI, \omega)$. To prove this it enough to show that $(HI, \omega) : Y_0 \subseteq (HI, \omega)$. Now for any $f \in (HI, \omega) : Y_0$ with $\text{supp}(f) = \{f(n) | f(n) \neq 0, n \in \mathbb{Z}^+\}$ and $g \in Y_0$ with $\text{supp}(g) = \{g(m) | f(m) \neq 0, m \in \mathbb{Z}^+\}$. We get $g(m) \in C_{Y_0}$, and $gf \in (HI, \omega)$. Thus $g(m)f(n) \in I$ for every $m \in \text{supp}(g)$ and $n \in \text{supp}(f)$ from Lemma 5.2. We have $f(n) \in I : C_{Y_0} = I$ for every $n \in \text{supp}(f)$. Thus $f \in (HI, \omega)$. Hence (HR, ω) is $\Sigma_{(HI,\omega)}$-zip. Conversely, suppose (HR, ω) is $\Sigma_{(HI,\omega)}$-zip, and $X \subseteq R$ where X is not a subset of I but $I : X = I$. Since $X \subseteq R$, hence $X \subseteq (HR, \omega)$. Now we show $(HI, \omega) : X = (HI : \omega)$. For this let $f \in (HI, \omega) : X$ with $n \in \text{supp}(f)$ and any $y \in X$, $(h_y f)(n) = yf(n) \in I$. Thus $f(n) \in I : X = I$ this implies that $f \in (HI, \omega)$. It follows that $(HI, \omega) : X = (HI : \omega)$. Since (HR, ω) is $\Sigma_{(HI,\omega)}$-zip, hence there exists a nonempty finite subset X_0 of X such that $(HI, \omega) : X_0 = (HI : \omega)$. Thus $I : X_0 = R \cap (HI, \omega) : X_0 = (HI, \omega)$. Hence R is Σ_I-zip. □

We get the following result proved by Ouyang et al.[31, Corollary 4.5], as a direct consequence of theorem 5.3.

Corollary 5.5 (Ouyang et al. [31, Corollary 4.5])**.** *Let I be a Σ-compatible semiprime ideal of R. Then R is Σ_I-zip if and only if the skew power series ring $R[[x;\omega]]$ is $\Sigma_{I[[x;\omega]]}$-zip.*

P r o o f. If $f(n) = a_n$ for every $n \in \mathbb{Z}^+$ then $(HR, \omega) \cong R[[x;\omega]]$. Thus from Theorem 5.3, we procured the result. □

If ω is an identity endomorphism of R. Then $(HR, \omega) \cong (HR, 1_R)$. The following corollaries are immediate.

Corollary 5.6. *Let I be a Σ-compatible semiprime ideal of R and* $\mathrm{char}(R/I) = 0$. *The following statements are equivalent:*

(1) *R is Σ_I-zip*

(2) *HR is Σ_{HI}-zip*

Corollary 5.7. *Let I be a semiprime ideal of R. Then R is Σ_I-zip if and only if the power series ring $R[[x]]$ is $\Sigma_{I[[x]]}$-zip.*

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